CISINSKI: CONSTRUCTIBILITY AND CONTINUITY IN CATEGORIES OF MIXED MOTIVES

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ABSTRACT. These lectures will be an introduction to the theory of mixed motives, after Voevodsky. Although we will focus on motives over classical schemes, we will present the constructions and proofs in a way which can be adapted to more general settings (algebraic stacks, derived geometry, locally ringed topoi). We will mainly insist on various notions of constructibility, from a geometrical and a categorical points of view. Since we will work with motives locally with respect to the étale topology, this will have rather direct interpretations in terms of both intersection theory and étale cohomology.

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0. ÉTALE MOTIVES

Motives appear in different flavors; this is only one of them. I'll explain why we're looking at this particular type later. But motives are essentially determined by their behavior on classical schemes, so we'll stay there. In particular, we'll see why étale sheaves are motives themselves.

Let X be a scheme. This has an ***étale site***, denoted $X_{\text{ét}}$. Recall that a map $U \xrightarrow{f} X$ is called ***étale*** if it is (totally) of finite presentation, flat, and has $\Omega_{U/X} = 0$. These are the objects of $X_{\text{ét}}$. This comes with a Grothendieck pretopology, defined by those finite families $\{U_i \xrightarrow{f_i} X\}_{i \in I}$ such that the f_i are all étale and $\amalg U_i \to X$ is surjective. This completes to the étale topology.

Given a point $x \in X$, we can discuss its ***étale stalks***. Writing $\mathcal{O}_{X,x}$ as usual for the local ring, we write $\mathcal{O}^{sh}_{X,x}$ for its *strict henselization*, the colimit

$$\operatorname{colim}_{f: V \to X \text{ étale}, x \in f(x)} \Gamma(V, \mathcal{O}_V).$$

This has

$$\operatorname{Spec}(\mathcal{O}^{sh}_{X,x}) \xrightarrow{\overline{x}} X,$$

and given any presheaf

$$(X_{\text{ét}})^{op} \xrightarrow{F} \mathcal{C}$$

we define its stalk at $\overline{x}$ to be

$$F_{\overline{x}} = \operatorname{colim}_{(\text{same})} F(V).$$

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For any $X \xrightarrow{f} Y$ we get an induced commutative diagram

$$\begin{array}{ccc} Y_{\text{ét}} & \xrightarrow{f^*} & X_{\text{ét}} \\ \downarrow & & \downarrow \\ \text{Sh}(Y_{\text{ét}}) & \xrightarrow[f^*]{} & \text{Sh}(X_{\text{ét}}) \end{array}$$

which has that

$$\Gamma(\text{Spec}(\mathcal{O}^{sh}_{X,x}, \overline{x}^*(F))) = F_{\overline{x}}.$$

Now, let R be a commutative ring of coefficients. This defines a derived category $\mathscr{D}(X_{\text{ét}}, R)$, the unbounded derived category of the abelian category of étale sheaves of R-modules; by definition, this is obtained by taking

$$\text{Ch}(\text{Fun}(X^{op}_{\text{ét}}, \text{Mod}_R))$$

and freely inverting those maps $F \to G$ for which, for all $V \xrightarrow{f} X$ étale and all $x \in V$, the map $F_{\overline{x}} \to G_{\overline{x}}$ is a quasi-isomorphism.

A map $X \xrightarrow{f} Y$ induces a functor

$$\mathscr{D}(X_{\text{ét}}, R) \xrightarrow{f^*} \mathscr{D}(Y_{\text{ét}}, R),$$

which admits a (derived) right adjoint $(Rf)_*$, a sort of "relative cohomology": if we take $Y = \text{Spec}\, k$ (for k algebraically closed), then we obtain $H^*(Rf)_*(F) = H^i(X_{\text{ét}}, F)$.

Now, classically (i.e. in SGA4), one studies derived categories $\mathscr{D}^+(X_{\text{ét}}, R)$, the bounded-below subcategory. This no longer has all co/limits, which is unfortunate. In a way, we'd really like to work with the full derived category; this requires us to extend all the technology of SGA4 to the unbounded case. Thus, *from now on, we'll assume that our schemes are (at least locally) Noetherian.* In essence, we'll define a big category of motives, but we'll want to work with ones of bounded dimension in some sense.

Definition 1. We say that X is ***of cohomological dimension*** $\leq n$ if for any sheaf F of R-modules on $X_{\text{ét}}$, we have $H^i(X_{\text{ét}}, F) = 0$ for all $i > n$. For the smallest such n, we write $\text{cd}_R(X) = n$ for the cohomological dimension.

In general, this need not be finite, but it is in some cases. For instance, if $R \supset \mathbb{Q}$ and X has finite Krull dimension d, then for any $i > d$ we have $H^i(X_{\text{ét}}, F) = 0$; in particular, $\text{cd}_R(X) \leq \dim X$. So, rationally (for noetherian schemes), there are no problems.

But with torsion coefficients, things can go much wronger. In this case we have the following.

Theorem 2 (Gabber)**.** *If X is strictly local (i.e. $X = \text{Spec}(\mathcal{O})$ for a strict henselization $\mathcal{O}$) and of dimension d, then for any étale morphism $U \to X$ of finite type, we have $\text{cd}_R(U) \leq 2d - 1$.*

Relatedly, we also have the following.

Theorem 3 (Gabber)**.** *For S a strictly local and X/S of finite type, all residue fields are of finite cohomological dimension.*

Corollary 4. *In this case, for all R, we have $\text{cd}_R(X) < \infty$.*

Now, to talk about unbounded complexes, we can write them as colimits of bounded ones, and then we can appeal to the following abstract result.

Proposition 5. *Let $\mathcal{A}$ be an abelian category (e.g. $\text{Fun}(X^{op}_{\text{ét}}, \text{Mod}_R)$). Let $F : \mathcal{A} \to \text{Mod}_{\mathbb{Z}}$ be a left-exact functor. Then we obtain a right-derived functor $RF : \mathscr{D}(\mathcal{A}) \to \mathscr{D}(\text{Mod}_{\mathbb{Z}})$. This gives us a functor*

$$R^i F = H^i \circ RF : \mathcal{A} \to \text{Mod}_{\mathbb{Z}};$$

if this commutes with filtered colimits, then TFAE:

(1) the functor $\text{Ch}(\mathcal{A}) \to \text{Mod}_{\mathbb{Z}}$ given by $K \mapsto H^0(RF(K))$ commutes with filtered colimits;
(2) the functor RF commutes with (small) sums (and hence all colimits);
(3) the functor RF commutes with countable sums;
(4) for any $K \in \text{Ch}(\mathcal{A})$ which is degreewise F-acyclic, then the canonical map $F(K) \to RF(K)$ is invertible.

This last condition gives us acyclic resolutions for *unbounded* complexes.

Proposition 6. *A sufficient condition for the above equivalent conditions to hold is that F (or X?) be of finite cohomological dimension.*

Corollary 7. *If $V \to X$ is étale and $\mathrm{cd}_R(V) < \infty$, then $R(V) \in \mathscr{D}(X_{\text{ét}}, R)$ is a compact object.*

Proof. This comes from applying the proposition to $F = \Gamma(V, -)$. □

Remark 8. For any X, if $k \hookrightarrow R$ and there exists a point $\mathrm{Spec}(k) \to X$, then $\mathrm{cd}_R(X) = \infty$. (For instance, this happens if $(\mathbb{Z}/2^\nu\mathbb{Z}) \subset R$.)

Exercise 9. If X is of finite dimension and $K \in \mathscr{D}(X_{\text{ét}}, R)$, then the canonical map

$$H^i_{\text{ét}}(X, K) \otimes \mathbb{Q} \to H^i_{\text{ét}}(X, K \otimes \mathbb{Q})$$

is an isomorphism. (This is trivial if X is compact.)

Remark 10. In general, it might be that X isn't compact over k, but the derived category could still be compactly generated. We could take equalities $k \hookrightarrow R = k$ and $\mathrm{Spec}\, k \to X = \mathrm{Spec}\, k$; we just don't get the compact generators we wanted.

Let us now turn to some classical topics, beginning with ***proper base change***.

Proposition 11. *Suppose $\mathrm{char}(R) > 0$, and consider a pullback*

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow[u]{} & S \end{array}$$

of schemes, in which p is proper and of finite presentation. Then we get a canonical map

$$u^*(Rp)_* \to (Rp')_* v^*,$$

and this is invertible in $\mathscr{D}(S'_{\text{ét}}, R)$.

Note that we have no finiteness assumptions on the schemes here.

Corollary 12. *Suppose that $\mathrm{char}(R) > 0$, and suppose that we have $X \xrightarrow{p} S$ proper and of finite presentation, suppose we have a point $p \in S$, and write $X_{\overline{s}} = p^{-1}(\overline{s})$ for the geometric fiber (i.e. the fiber over the induced geometric point). Then*

$$(Rp)_*(F)_{\overline{s}} \simeq R\Gamma(X_{s,\text{ét}}, F|_{X_s}).$$

Corollary 13. *For p proper and of finite presentation and $\mathrm{char}(R) > 0$, $(Rp)_*$ commutes with sums; thus $(Rp)_*$ itself admits a right adjoint $p^! : \mathscr{D}(Y_{\text{ét}}, R) \to \mathscr{D}(X_{\text{ét}}, R)$.*

Generally, one needs to make sure the ambient characteristic and the characteristic of the coefficients don't coincide. In the previous proposition, we'll basically want to take p to be arbitrary but require u to be smooth. But then we seem to need to require noetherianness assumptions. This is all called ***smooth base change***.

Proposition 14. *Given a cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow[v]{} & S \end{array}$$

in which v is smooth, p is locally of finite presentation, S is locally noetherian (but this may seem unnecessary?), and $\mathrm{char}(R)$ is invertible in $\mathcal{O}_S$. Then, the natural map

$$v^*(Rp)_* \to (Rp')_* u^*$$

is invertible.

Theorem 15. *If $\mathrm{char}(R)$ is invertible in $\mathcal{O}_S$ (and S is locally noetherian) and $V \xrightarrow{p} S$ is a vector bundle, then*

$$p^* : \mathscr{D}(S_{\text{ét}}, R) \to \mathscr{D}(V_{\text{ét}}, R)$$

is fully faithful.

Proof idea. We reduce to the case of bounded complexes, and then use the techniques of SGA4, or more precisely the following consequence of Gabber's theorem: for $f : X \to Y$ locally of finite presentation and Y locally noetherian, then $(Rf)_*$ commutes with small sums. □

1. Proper descent and topological invariance of $\mathscr{D}(X_{\text{ét}}, R)$

Topological invariance states that if you have a map $f : X \to Y$ such that it's a homeomorphism at the level of underlying topological spaces *after any base change* (i.e. it holds "geometrically"), this is equivalent to saying that f is radicial, integral, and surjective.

Example 16. We can take $X_{\text{red}} \to X$ over a field k, or k'/k a purely inseparable field extension: then, this "derived category" construction cannot see the difference between them.

Remark 17. Let us explain further by connecting with David's lecture. Suppose $(X, \mathcal{O}_X)$ is a connective spectral scheme. We can consider its truncation, an ordinary scheme $X_{cl} = (|X|, \pi_0\mathcal{O}_X)$. This has a map $(|X|, \pi_0\mathcal{O}_X) \xrightarrow{i} (X, \mathcal{O}_X)$. One can define an appropriate notion of étale maps, and we obtain an equivalence

$$\mathcal{X}_{\text{ét}} \xrightarrow{i^*} X_{cl,\text{ét}}$$

is an equivalence. In other words, the étale topology can't see derived schemes at all. On the other hand, derived objects can be used to construct cohomology classes of interest (such as (virtual) fundamental classes, e.g. in Gromov–Witten theory).

Definition 18 (Rydh)**.** Let $f : X \to Y$ be a morphism of schemes. We say that f is ***subtrusive*** if:

(1) any closure-ordered pair of points $y \leq y'$ in Y lifts to a closure-ordered pair $x \leq x'$ in X;
(2) the continuous map
$$f^{\text{cons}} : |X|^{\text{cons}} \to |Y|^{\text{cons}}$$
is surjective, and identifies the target as as (topological) quotient of the source.

Here, the underlying set of $|X|^{\text{cons}}$ is the underlying set of X, but we endow it with the *constructible topology*: opens are ind-constructible subspaces of X (so they're locally unions of constructible subsets).

Remark 19. These two conditions imply that the map $|X| \to |Y|$ (now with the Zariski topology) is also a topological quotient.

Definition 20. The map $f : X \to Y$ is called ***universally subtrusive*** if for any map $Y' \to Y$, the induced map

$$X' = X \times_Y Y' \to Y'$$

is subtrusive.

Example 21. Examples of morphisms that are universally subtrusive maps are:

- proper surjective;
- integral surjective;
- faithfully flat quasicompact;
- faithfully flat and locally of finite presentation.

Definition 22. Let S be a scheme, and consider the category $\text{Sch}_{/S}$ of (separated) schemes over S of finite presentation. The ***h-topology*** on $\text{Sch}_{/S}$ is the Grothendieck topology associated to the pretopology whose coverings are finite families of the form

$$\{p_i : X_i \to X\}_{i \in I}$$

such that $\amalg X_i \to X$ is universally subtrusive. (This requires X to be qcqs, otherwise we need to talk about maps which are only *locally* universally subtrusive.)

We're interested in studying descent properties with respect to the h-topology. This will encompass descent for the various types of morphisms just listed.

Remark 23. The h-topology was originally defined by Voevodsky, but only for excellent noetherian schemes. The correct full generalization was found by Rydh.

Now, the h-topology is not subcanonical: the Yoneda embedding

$$\text{Sch}_{/S} \xrightarrow{\text{Yo}} \text{Shv}((\text{Sch}_{/S})_h)$$

is not fully faithful (i.e. really this is using an implicit sheafification). Thus, one can ask: which maps $X \xrightarrow{f} Y$ in $\text{Sch}_{/S}$ become equivalences? In fact, $\text{Yo}(f)$ is an equivalence iff f is universally a homeomorphism. It follows that for F an h-sheaf, $X \xrightarrow{f} Y$ implies that $f^* : F(X) \xrightarrow{\sim} F(Y)$. So,

Example 24. If $F(X) = \mathrm{Sh}(X_{\text{ét}})$, one of the main theorems of SGA1 (in other words, of course) is that this is a sheaf of categories. This implies that if $X \xrightarrow{f} Y$ is a universal homeomorphism, then

$$\mathrm{Shv}(X_{\text{ét}}) \xrightarrow{\sim} \mathrm{Shv}(Y_{\text{ét}}).$$

Theorem 25 (Voevodsky, Rydh). *Up to refinements, any h-covering takes the following form: take a set*

$$\{W_i \xrightarrow{q_i} V_i \xrightarrow{p_i} U_i \xrightarrow{j_i} X\}_{i \in I}$$

(where everything in sight is qcqs) for which $\{j_i\}_{i \in I}$ forms a Zariski covering, the p_i are proper surjective, and the q_i are étale surjective. In other words, such h-covers generate the h-topology.

In other words, "if you understand proper descent and étale descent, then you understand h-descent".

Recall that one of the major insights of SGA4 is that (proper) descent implies a base-change formula. To see this, let $Y \xrightarrow{p} X$, and take the Cech complex $\check{\mathrm{C}}(Y/X)_\bullet : \mathbf{\Delta}^{op} \to \mathrm{Sch}_{/X}$ given by $\check{\mathrm{C}}(Y/X)_n = Y^{\times_X n}$. Then, when you want to prove cohomological descent, proper base change gives you

$$R\Gamma_{\text{ét}}(X, F) \to R \lim_{\mathbf{\Delta}} R\Gamma_{\text{ét}}(Y^{\times_n X}, F).$$

Whenever p has a section, the Cech complex obtains an extra degeneracy. So if we have proper base change, we can work "more locally", and then these formulas actually hold immediately.

Returning to our small étale site, we get an embedding

$$X_{\text{ét}} \xrightarrow{\alpha} (\mathrm{Sch}_{/X})_h$$

of sites, which is "continuous" or whatever. We then get

$$\alpha^* : \mathrm{Shv}(X_{\text{ét}}, R) \rightleftarrows \mathrm{Shv}((\mathrm{Sch}_{/X})_h, R).$$

So this is literally nothing other than the restriction $\alpha_*(F) = F|_{X_{\text{ét}}}$. In fact this is equivalent to saying that α^* is fully faitfhul.

Now, on derived categories we get

$$\alpha^* : \mathscr{D}(X_{\text{ét}}, R) \rightleftarrows \mathscr{D}((\mathrm{Sch}_{/X})_h, R) : (R\alpha)_*.$$

Theorem 26. *The restriction of α^* to $\mathscr{D}^+(X_{\text{ét}}, R)$ is fully faithful.*

The main point of these lectures is to understand the essential image of this functor. It won't depend on the small étale site, and actually not on R either (assuming still that $\mathrm{char}(R) > 0$).

Now, for all $K \in \mathscr{D}^+(X_{\text{ét}}, R)$, we have $K \xrightarrow{\sim} (R\alpha)_* \alpha^*(K)$; equivalently, for any $U \to X$ étale, taking étale cohomology gives

$$H^i_{\text{ét}}(U, K) \xrightarrow{\sim} H^i_{\text{ét}}(U, (R\alpha)_* \alpha^* K) \underset{\text{adjn}}{\cong} H^i_h(U, \alpha^* K).$$

On the other hand, to drop the boundedness constraint, we'll have to work harder. First of all, we need the following result.

Theorem 27 (Goodwillie–Lichtenbaum). *Suppose that X is noetherian. Assume that all residue fields of X have finite cohomological dimension (for the étale topology). Then for any $Y \to X$ of finite type, Y has finite cohomological dimension for the h-topology.*

Combining this with Gabber's theorem that we stated previously, we get the following.

Corollary 28. *IF X is locally noetherian, then*

$$(R\alpha)_* : \mathscr{D}((\mathrm{Sch}_{/X})_h, R) \to \mathscr{D}(X_{\text{ét}}, R)$$

preserves small sums.

Then, we have our result.

Theorem 29. *If X is a locally noetherian scheme and* $\mathrm{char}(R) > 0$, *then*

$$\alpha^* : \mathscr{D}(X_{\text{ét}}, R) \to \mathscr{D}((\mathrm{Sch}_{/X})_h, R)$$

is fully faithful.

Now that we have full faithfulness, we would like to understand the essential image, ideally without reference to the functor α. Ideally we'd also remove the constraint that $\mathrm{char}(R) > 0$, and still have our various base change formulas (smooth, proper, etc.).

2. ÉTALE MOTIVES (AFTER VOEVODSKY)

Let's assume X is noetherian, char$(R) > 0$, and char(R) is invertible in $\mathcal{O}_X$. Then, we ask: what are the properties of the objects in the essential image of

$$\alpha^* : \mathscr{D}(X_{\text{ét}}, R) \to \mathscr{D}((\text{Sch}_{/X})_h, R)?$$

Let $K \in \mathscr{D}(X_{\text{ét}}, R)$. Then $\alpha^*(K)$ is defined by the formula

$$\alpha^*(K)(Y) = \Gamma(Y, f^*(K)).$$

Let's put $C = \alpha^*(K)$. Then, $H^i_h(Y, C) = H^i_{\text{ét}}(Y, f^*K)$.

This has ***homotopy invariance***: the map $Y \times \mathbb{A}^1 \xrightarrow{p} Y$ induces isomorphisms

$$H^i_h(Y \times \mathbb{A}^1, C) \underset{\sim}{\xleftarrow{p^*}} H^i(Y, C).$$

We would also like to express Poincaré duality. In order to do so, we need a Tate object. But for this, we can't work in a derived category anymore. To solve this, Voevodsky obtained a Tate object by explicitly enforcing homotopy invariance.

Definition 30. Let R be arbitrary. Write

$$\underline{DM}^{eff}_h(X, R) = \mathscr{D}((\text{Sch}_{/X})_h, R)/T$$

for the Verdier quotient, where we take T to be the smallest localizing subcategory (so, a full triangulated subcategory that's stable under small sums) to be the smallest one containing the complexes

$$\cdots \to 0 \to 0 \to R(Y \times \mathbb{A}^1) \to R(Y) \to 0 \to 0 \to \cdots$$

(where $R(X)$ denotes a representable object). This is the ***derived category of effective motives***. (This is chosen expressly so that the quotient map preserves sums.)

Moreover, the functor

$$\mathscr{D}((\text{Sch}_{/X})_h, R) \to \underline{DM}^{eff}_h(X, R)$$

admits a right adjoint, and the essential image of this is the full subcategory of complexes C such that for all $Y \in \text{Sch}_{/X}$,

$$H^i_h(Y, C) \xrightarrow{\sim} H^i_h(Y \times \mathbb{A}^1, C).$$

Corollary 31. *Assuming* char(R) *is invertible in* $\mathcal{O}_X$*, then*

$$\mathscr{D}(X_{\text{ét}}, R) \to \underline{DM}^{eff}_h(X, R)$$

is fully faithful.

3. DUALITY IN $\mathscr{D}(X_{\text{ét}}, R)$

Now, $\underline{DM}^{eff}_h(X, R)$ comes as a Verdier quotient

$$\mathscr{D}((\text{Sch}_{/X})_h, R) \to \underline{DM}^{eff}_h(X, R),$$

and also admits a map from $\mathscr{D}(X_{\text{ét}}, R)$. The first is always fully faithful, and the second is as well under an assumption of invertibility of char(R).

We would like to express duality in $\mathscr{D}(X_{\text{ét}}, R)$ for $n = \text{char}(R) > 0$ invertible in $\mathcal{O}_X$. For this, we need an (invertible) Tate object.

To simplify (and specialize to the main case of interest), let's take $R = \mathbb{Z}/n\mathbb{Z}$: then, the Tate object in $\mathscr{D}(X_{\text{ét}}, R)$ is $R(1) = \mu_n$, the n^{th} roots of unity. We'd like to understand its image under the functor

$$\mathscr{D}(X_{\text{ét}}, R) \to \underline{DM}^{eff}_h(X, R).$$

For arbitrary R, we define $R(1) \in \underline{DM}^{eff}_h(X, R)$ as follows. Let's consider the Verdier quotient

$$\mathscr{D}((\text{Sch}_{/X})_h, R) \to \underline{DM}^{eff}_h(X, R)$$

to be "the identity on objects". Then, we have a canonical map

$$R \to R(\mathbb{A}^1 - \{0\})$$

corresponding to the element $1 \in (\mathbb{A}^1 - \{0\})$, and we define

$$R(1) = \text{coker}\left(R \to R(\mathbb{A}^1 - \{0\})\right)[-1].$$

As the original map is a split monomorphism, we obtain a canonical decomposition

$$R(\mathbb{A}^1 - \{0\}) = R \oplus R(1)[1].$$

Considering $\mathbb{G}_m$ as a sheaf of abelian groups, we obtain a map of R-modules

$$R(\mathbb{A}^1 - \{0\}) \to \mathbb{G}_m \otimes_{\mathbb{Z}} R$$

(where the source is *free* as a sheaf, and hence quite different from $\mathbb{G}_m$). But really this should all be derived, so we're more interested in the map

$$R(\mathbb{A}^1 - \{0\}) \to \mathbb{G}_m \otimes^{\mathbb{L}}_{\mathbb{Z}} R.$$

But we'll suppress this notation.

Theorem 32 (Voevodsky)**.** *The map*

$$R(\mathbb{A}^1 - \{0\}) \to \mathbb{G}_m \otimes_{\mathbb{Z}} R$$

is an isomorphism in $\underline{DM}_h^{eff}(X, R)$, i.e. up to $\mathbb{A}^1$-equivalence.

This is a non-obvious statement, and involves cycles in a nontrivial way. Of course, it's false if you don't work up to $\mathbb{A}^1$-equivalence: Voevodsky's original proof describes the kernel explicitly

Corollary 33. *If n is invertible in $\mathcal{O}_X$, then we have an isomorphism*

$$(\mathbb{Z}/n\mathbb{Z})(1) \cong \mu_n$$

in $\underline{DM}_h^{eff}(X, \mathbb{Z}/n\mathbb{Z})$.

Proof idea. Kummer theory gives a short exact sequence

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{\cdot n} \mathbb{G}_m \to 0$$

in $(\mathrm{Sch}_{/X})_{\text{ét}}$. □

Remark 34. For X of characteristic $p > 0$, we have the Artin–Schreier short exact sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \xrightarrow{\text{Frob}} \mathbb{G}_a \to 0.$$

Given our characteristic p assumption, it follows that $\underline{DM}_h^{eff}(X, \mathbb{Z}/p\mathbb{Z}) = 0$. Thus, to understand torsion, it suffices to work coprimely to the characteristic.

Now, consider the decomposition

$$\mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{A}^1.$$

Applying $R(-)$, we get another pushout

$$\begin{array}{ccc} R(\mathbb{A}^1 - \{0\}) & \longrightarrow & R(\mathbb{A}^1) \\ \downarrow & & \downarrow \\ R(\mathbb{A}^1) & \longrightarrow & R(\mathbb{P}^1). \end{array}$$

The maps out of $R(\mathbb{A}^1 - \{0\})$ are monomorphisms, which in an abelian category of sheaves are cofibrations: thus this is also a homotopy pushout, and in $\underline{DM}_h^{eff}(X, R)$ we obtain

$$R(\mathbb{P}^1) \simeq R \oplus R(1)[2].$$

Remark 35. We have a map

$$H^1(X, \mathbb{G}_m) := \hom_{\mathscr{D}((\mathrm{Sch}_{/X})_h, \mathbb{Z})}(\mathbb{Z}, \mathbb{G}_m[1]) \xrightarrow{c_1} \hom_{\underline{DM}_h^{eff}(X,R)}(R, R(1)[2])$$

the *Chern character*. From here, one can obtain the ***projective bundle formula***: for $\mathcal{E}$ a locally free $\mathcal{O}_X$-module of rank $r+1$, we have an isomorphism

$$R(\mathbb{P}(\mathcal{E})) \cong \bigoplus_{i=0}^{r} R(i)[2i].$$

Now, to express Poincaré duality, we need this Tate object to be invertible.

Definition 36. Let's think of everything we've done so far as being in the world of stable ∞-categories. Then, we write

$$\underline{DM}_h(X,R) = \lim\left(\cdots \xrightarrow{\underline{\hom}(R(1),-)} \underline{DM}_h^{eff}(X,R) \xrightarrow{\underline{\hom}(R(1),-)} \underline{DM}_h^{eff}(X,R) \xrightarrow{\underline{\hom}(R(1),-)} \underline{DM}_h^{eff}(X,R)\right)$$

(taken in (presentable stable) ∞-categories). This is formally analogous to inverting the "loops" functor, which in homotopy theory brings us from the ∞-category of spaces to the stable ∞-category of spectra.

Remark 37. Thus, an object of $\underline{DM}_h(X,R)$ consists of a sequence

$$M = (M_n, \sigma_n)$$

where $M_n \in \underline{DM}_h^{eff}(X,R)$ and the σ_n are equivalences

$$\sigma_n : M_n \xrightarrow{\sim} \underline{\hom}(R(1), M_{n+1}).$$

Remark 38. The functor

$$\Sigma^\infty_{\mathrm{Tate}} : \underline{DM}_h^{eff}(X,R) \to \underline{DM}_h(X,R)$$

admits a right adjoint, which is simply given by taking $M = (M_n, \sigma_n)$ to M_0.

Remark 39. We will abuse notation by writing $R(Y) = \Sigma^\infty_{\mathrm{Tate}} R(Y)$, for Y/X of finite type.

This functor $\Sigma^\infty_{\mathrm{Tate}}$ carries a universal property (among presentably symmetric monoidal ∞-categories). This relies on the fact that the cyclic permutation of

$$R(1) \otimes R(1) \otimes R(1)$$

is the identity (which is only true up to $\mathbb{A}^1$ equivalence).

Theorem 40 (Robalo)**.** *The functor $\Sigma^\infty_{\mathrm{Tate}}$ is the universal cocontinuous symmetric monoidal functor into a stable ∞-category so that $R(1)$ becomes invertible (under the symmetric monoidal structure).*

Corollary 41. *If* $\mathrm{char}(R) > 0$*, then*

$$\Sigma^\infty_{\mathrm{Tate}} : \underline{DM}_h^{eff}(X,R) \to \underline{DM}_h(X,R)$$

is an equivalence. (In particular, the source is already stable.)

Remark 42. Suppose we are given a pullback square

$$\begin{array}{ccc} Z & \xrightarrow{u} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow[v]{} & Y' \end{array}$$

in which all maps are of finite type. For an arbitrary map of finite type a, write $a_\#$ for the left adjoint to a^*. Categorical manipulations give us an equivalence

$$v^* f_* \xrightarrow{\sim} g_* u^*,$$

using the fact that this is equivalent to

$$u_\# g^* \to f^* v_\#.$$

So we have proper base change for *everything*, which is actually in a way kind of a shame. For instance, let $i : Z \hookrightarrow X$ be a closed immerstion, with complement $U = X - Z$ with inclusion $j : U \hookrightarrow X$. Then in $\mathscr{D}(X_{\text{ét}}, R)$, we get a cofiber sequence

$$j_! j^*(F) \to F \to i_* i^* F.$$

We end up getting a splitting $\mathscr{D}(X_{\text{ét}}) \simeq \mathscr{D}(Z) \times \mathscr{D}(U)$, which is not really what we want.

This leads us to restrict to *smooth* schemes in the étale site. The inclusion $(\mathrm{Sm}_{/X})_{\text{ét}} \hookrightarrow (\mathrm{Sch}_{/X})_{\text{ét}}$ induces an adjunction

$$\mathrm{Shv}((\mathrm{Sm}_{/X})_{\text{ét}}) \rightleftarrows \mathrm{Shv}((\mathrm{Sch}_{/X})_{\text{ét}}),$$

in which the left adjoint is fully faithful. Thus, in order to define a sheaf on the smooth étale site, it suffices to define one on the full étale site and then apply the right adjoint.

Definition 43. We define

$$DM_h^{eff}(X,R) \subset \underline{DM}_h^{eff}(X,R)$$

to be the smallest cocomplete stable subcategory (or triangulatedly, "localizing subcategory ") containing the objects $R(Y)$ for Y/X smooth. We similarly define

$$DM_h(X,R) \subset \underline{DM}_h(X,R),$$

now containing $R(Y)(n)$ for Y/X smooth and $n \in \mathbb{Z}$.

Notation 44. To clarify, we are writing

$$M(n) = M \otimes_R R(n),$$

where

$$R(n) = \begin{cases} R(1)^{\otimes n}, & n \geq 0 \\ R(-1)^{\otimes n}, & n < 0 \end{cases}$$

where

$$R(-1) = \underline{\hom}_{\underline{DM}_h(X,R)}(R(1), R).$$

Remark 45. There's no way to say that something is "locally smooth for the h-topology", except in the following case. Let $X = \operatorname{Spec} k$ for k a perfect field. Over such X, everything is smooth by ***de Jong's alteration theorem***. That is, in this case we have

$$DM_h^{eff}(X,R) = \underline{DM}_h^{eff}(X,R)$$

and

$$DM_h(X,R) = \underline{DM}_h^{eff}(X,R).$$

Note that the perfection assumption on k isn't so terrible, so purely separable extensions become invertible, so replacing an arbitrary k by its inseparable closure it won't change the "underline" versions of these categories – but in fact, the non-underline versions don't change either (though this is less trivial).

Theorem 46 (rigidity)**.** *For X noetherian and $n = \operatorname{char}(R) > 0$ with n invertible in $\mathcal{O}_X$, the natural functor*

$$\mathscr{D}(X_{\text{ét}}, R) \to DM_h^{eff}(X,R) \simeq DM_h(X,R)$$

(which we already knew to be fully faithful) is an equivalence.

For $X = \operatorname{Spec} k$ with k a field, this is Suslin–Voevodsky; this is due to Cisinski–Deglise more generally. To prove this, one needs to go a bit deeper into the homotopy theory of schemes. We already know that the source is a localizing subcategory of the target, which was itself defined as a localizing subcategory. So it suffices to show that this inclusion is essentially surjective. We'll indicate more of the proof later.

Definition 47. We define ***h-motivic cohomology*** to be

$$H^i(X, R(n)) = H_h^i(X, R(n)) = \hom_{DM_h(X,R)}(R, R(n)[i])$$

and ***étale motivic cohomology*** to be

$$H^i_{\text{ét}}(X, R(n)) = \hom_{\mathscr{D}(X_{\text{ét}},R)}(R, R(n)[i])$$

Remark 48. If $\operatorname{char}(R) > 0$ is invertible in $\mathcal{O}_X$, then

$$H^i_{\text{ét}}(X, R(n)) \cong H^i(X, R(n))$$

by full faithfulness.

Proposition 49. *Taking $R = \mathbb{Z}/p^\nu\mathbb{Z}$ and $X[\frac{1}{p}]$ the complement of the solutions to $p = 0$ in X, we have*

$$DM_h^{(eff)}(X, \mathbb{Z}/p^\nu\mathbb{Z}) \xrightarrow{\sim} DM_h^{(eff)}(X[1/p], \mathbb{Z}/p^\nu\mathbb{Z}) \simeq \mathscr{D}(X[1/p]_{\text{ét}}, \mathbb{Z}/p^\nu\mathbb{Z}).$$

So away from characteristic p, we can really understand torsion.

To understand what happens with rational coefficients, we have the following.

Theorem 50. *Let X be noetherian and $R \supset \mathbb{Q}$. Then*

$$\bigoplus_{n\in\mathbb{Z}} H^{2n-i}(X, R(n)) \cong KH_i(X) \otimes R,$$

where KH is Weibel's homotopy-invariant K-theory.

Remark 51. We note that for X regular, we have $KH(X) \simeq K(X)$ (e.g. as defined by Thomason–Trobaugh).

Remark 52. This decomposition corresponds to the Adams operations in K-theory.

4. THE SIX OPERATIONS

"An ∞-category of motivic sheaves $\mathscr{D}$" can be seen as a family of symmetric monoidal stable ∞-categories over the category Sch of schemes, i.e. the fiber over a scheme X is $\mathscr{D}(X)$. A morphism $X \xrightarrow{f} Y$ induces a map $f^* : \mathscr{D}(Y) \to \mathscr{D}(X)$ (so this is a cartesian fibration), but there's much more:

- ***completeness*** (or cocontinuity?): f^* has a right adjoint $f_* : \mathscr{D}(X) \to \mathscr{D}(Y)$;
- each $\mathscr{D}(X)$ is ***closed*** (i.e. has an internal hom $\underline{\hom}$);
- ***smooth base change***: if f is smooth (and of finite presentation), then f^* has a left adjoint $f_\#$ (so, "smooth schemes in Y should be representable over X"): for a cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \underset{\text{smooth}}{\xrightarrow{f}} & Y \end{array}$$

  we have an equivalence

$$q^* f_\# \xleftarrow{\sim} g_\# p^*.$$

- ***projection formula***: if f is smooth, we have a canonical equivalence

$$f_\#(M) \otimes N \xleftarrow{\sim} f_\#(M \otimes f^*(N)).$$

There are other axioms, but we'll nevertheless jump in.

Example 53. If we take $\mathscr{D} = DM_h(-, R)$, we get all of this for free. For instance, the inclusion

$$\nu^* : DM_h(X, R) \hookrightarrow \underline{DM}_h(X, R)$$

admits a right adjoint ν_* for formal reasons. So the six operations on $DM_h(X, R)$ come from restricting those of $\underline{DM}_h(X, R)$.

For instance, suppose we're given a map $X \xrightarrow{f} Y$. Then we get

$$\begin{array}{ccc} DM_h(X, R) & \overset{f_*}{\dashrightarrow} & DM_h(Y, R) \\ {\scriptstyle \nu^*}\downarrow & & \uparrow{\scriptstyle \nu_*} \\ \underline{DM}_h(X, R) & \underset{f_*}{\longrightarrow} & \underline{DM}_h(Y, R). \end{array}$$

Similarly, we can set

$$\underline{\hom}(M, N) = \nu_* \underline{\hom}(\nu^* M, \nu^* N).$$

Example 54. Let's consider $\mathscr{D}(X) = \mathscr{D}(X_{\text{ét}}, R)$ where char$R$ is invertible in $\mathcal{O}_X$. Then we have ***relative purity***: for f smooth and proper of relative dimension d, then

$$f_* : \mathscr{D}(X_{\text{ét}}, R) \to \mathscr{D}(Y_{\text{ét}}, R)$$

has a right adjoint $f^!$ (just due to properness – this doesn't require smoothness), which is given by

$$f^!(F) \cong f^*(F)(d)[2d].$$

In particular, this implies the existence of $f_\#$.

We continue with a sequel to the définition:

- ***homotopy invariance***: given $X \times \mathbb{A}^1 \xrightarrow{p} X$, we get that

$$p^* : \mathscr{D}(X) \to \mathscr{D}(X \times \mathbb{A}^1)$$

  is fully faithful;
- ***Tate stability***: the projection $(\mathbb{A}^1 - \{0\}) \times X \xrightarrow{p} X$ induces

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{1} & p_\#(\mathbb{1}) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{1}(1)[1] \end{array}$$

  and $\mathbb{1}(1)$ is $\otimes$-invertible;

- ***localization***: if $i : Z \hookrightarrow X$ is a closed immersion with open complement $j : U = (X - Z) \hookrightarrow X$, then:
  - $i^* j_\# = 0$,
  - both $j_\#$ and i_* are fully faithful;
  - for any $M \in \mathscr{D}(X)$ we have a commutative square

$$\begin{array}{ccc} j_\# j^*(M) & \longrightarrow & M \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & i_* i^*(M) \end{array}$$

    (whose commutativity follows from the first property) is a pushout.

This last fact is true *only* because we've restricted to the smooth site. In fact, this last property actually implies Nisnevich descent.

Remark 55. Morel–Voevodsky proved localization for SH (or really in the unstable case, which implies this); a variation on this proof gives the localization axiom for DM_h. They work with the smooth site; the key is that we can write everything as colimits of smooth schemes.

Theorem 56 (Ayoub)**.** *In such a theory of motivic sheaves $\mathscr{D}$, we have proper base change: whenever $p : X \to Y$ is proper, and given any pullback square*

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[v]{} & Y, \end{array}$$

the map

$$v^* p_* \to p'_* u^*$$

is an equivalence in $\mathscr{D}(Y')$.

Remark 57. Ayoub's proof goes through relative purity and Poincaré (or perhaps Atiyah) duality. Relative purity says that if f is smooth and proper, then

$$f_\# \mathrm{Th}(T_f)^{-1} \cong f_*$$

(where T_f is a relative tangent space); that is, for a vector bundle $V \downarrow X$, we have the complement of the zero section

$$\begin{array}{ccc} (V - X) & \xhookrightarrow{j} & V \\ & \searrow^{pj} & \downarrow{\scriptstyle p} \\ & & X \end{array}$$

and then, at the level of motives, we have a distinguished triangle

$$\begin{array}{ccc} (pj)_\#(\mathbb{1}) & \longrightarrow & p_\#(\mathbb{1}) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Th}(V). \end{array}$$

Now, $\mathrm{Th}(\mathbb{A}^n) = \mathbb{1}(1)^{\otimes n}$, and we have that $f_* f^*(\mathrm{Th}(T_f))$ is the dual of $f_\#(\mathbb{1})$, and the latter is *rigid*. Recall that an object M (in a symmetric monoidal ∞-category) is called ***rigid*** if for all N we have $M^\vee \otimes N \simeq \underline{\mathrm{hom}}(M, N)$ (where $M^\vee = \underline{\mathrm{hom}}(M, \mathbb{1})$). So, think "perfect complexes". Indeed, if R is noetherian, then in $\mathscr{D}(X_{\text{ét}}, R)$ the rigid objects are exactly those complexes C such that there exists and étale and surjective map $u : X' \to X$ such that $u^*(C)$ is equivalent to a perfect complex of $\mathcal{O}_{X'}$-modules – that is, it's "étale-locally constant".

Remark 58. In the presence of *Chern classes*, i.e. a map

$$\mathrm{Pic}(X) \xrightarrow{c_1} \mathrm{hom}_{\mathscr{D}(X)}(\mathbb{1}, \mathbb{1}(1)[2])$$

which is natural in X and induces a projective bundle formula, i.e. a rank-$(r+1)$ vector bundle $V \xrightarrow{p} X$ has projectivization $q : \mathbb{P}(V) \to X$ with

$$q_\#(\mathbb{1}) \xrightarrow{\sim} \bigoplus_{i=0}^{r} \mathbb{1}(i)[2i].$$

Hence $\mathrm{Th}(V) \cong \mathbb{1}(r+1)[2r+2]$. In other words, our duality "really *is* Poincaré duality": it's just given by a dimension shift.

Following Deligne, one can define an adjunction $f_! \dashv f^!$. For f proper, we define $f_! = f_*$ (one must check that $f^!$ exists); for f étale (in particular an open immersion), we put $f_! = f_\#$. If $f : X \to Y$ is séparated of finite présentation, i.e. we have a factorization

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle j}\searrow \quad \nearrow{\scriptstyle p} & \\ & \overline{X} & \end{array}$$

where j is an open immersion and p is proper, then we put $f_! = p_* j_\#$ an $f^! = j^* p^!$. For f smooth, we can now recognize Poincaré duality as an isomorphism

$$f_! f^!(\mathbb{1}) \cong f_\#(\mathbb{1}).$$

Note that

$$f_! f^!(\mathbb{1}) = f_!(\mathbb{1})(d)[2d].$$

Now, suppose $\mathrm{char}(R)$ is invertible in $\mathcal{O}_X$, and consider the functor

$$\mathscr{D}(X_{\text{ét}}, R) \to DM_h(X, R),$$

which we would like to be an equivalence. For $f : Y \to X$ smooth, we would like $f_\#(R)$ to be in the essential image of this functor. We also have pullbacks essentially for free, so we get a distinguished triangle

$$\begin{array}{ccc} j_! j^*(C) & \longrightarrow & C \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & i_* i^*(C) \end{array}$$

and this gives the claim.

5. ℓ-ADIC REALIZATION FUNCTORS

Consider

$$DM_h(X, \mathbb{Z}) \to DM_h(X, \mathbb{Z}/n\mathbb{Z}).$$

This is given by

$$M \mapsto \mathbb{Z}/n\mathbb{Z} \otimes^{\mathbb{L}} M.$$

If n is invertible in $\mathcal{O}_X$, then we have

$$DM_h(X, \mathbb{Z}/n\mathbb{Z}) \simeq \mathscr{D}(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})$$

by rigidity, and this gives us the realization functor above. In particular, we have

$$DM_h(X, \mathbb{Z}/p^\nu\mathbb{Z}) \simeq \mathscr{D}(X[1/p]_{\text{ét}}, \mathbb{Z}/p^\nu\mathbb{Z}).$$

From here, we can go to the ℓ-adic world. To be precise, we fix ℓ a prime invertible in $\mathcal{O}_X$. There are two ways to think about this: we can consider the (∞-categorical) limit

$$\lim_\nu \mathscr{D}(X_{\text{ét}}, \mathbb{Z}/\ell^\nu\mathbb{Z}).$$

But this is actually more simply a Verdier quotient

$$\mathscr{D}(X_{\text{ét}}, \mathbb{Z})/\mathscr{D}(X_{\text{ét}}, \mathbb{Z}[1/\ell]).$$

In particular, the latter description is nice because the canonical functor

$$\mathscr{D}(X_{\text{ét}}, \mathbb{Z}) \to \lim_\nu \mathscr{D}(X_{\text{ét}}, \mathbb{Z}/\ell^\nu\mathbb{Z})$$

given by

$$M \mapsto (M \otimes^{\mathbb{L}} \mathbb{Z}/\ell^\nu\mathbb{Z})_{\nu \geq 0}$$

is just the projection. Moreover, we get that it has a fully faithful right adjoint. So ℓ-adic sheaves are also a full subcategory of $\mathscr{D}(X_{\text{ét}}, \mathbb{Z})$.

Similarly, the ℓ-adic completion

$$DM_h(X,\mathbb{Z}) \to \lim_{\nu} DM(X,\mathbb{Z}/\ell^\nu\mathbb{Z}) \simeq \lim_{\nu} \mathscr{D}(X_{\text{ét}},\mathbb{Z}/\ell^\nu\mathbb{Z})$$

has a fully faithful right adjoint. This augmented functor is quite easy: it's just

$$M \to \hat{M} = \mathbb{R}\lim_{\nu}(M \otimes^{\mathbb{L}} \mathbb{Z}/\ell^\nu\mathbb{Z}),$$

and the essential image of this inclusion is precisely the subcategory of those maps for which this completion map is an isomorphism. Moreover, this is all so formal that it preserves the six operations for free.

Now, usually one puts some finiteness conditions on one's coefficients: one looks at *constructible* coefficients. We'll see several versions of finiteness conditions, and we'll see why the six operations preserve these finiteness conditions: some of the arguments will be "of a geometric nature", but then we'll give equivalent categorical conditions (e.g. rigidity).

6. Finiteness and continuity

We previously stated a notion of rigidity internally to the theory of étale sheaves. More generally, in $\mathscr{D}(X_{\text{ét}},R)$ we have several notions of finiteness.

For instance, there is the full subcategory

$$\mathscr{D}^b_c(X_{\text{ét}},R) \subset \mathrm{Ch}(\mathrm{Shv}(X_{\text{ét}},R))$$

on those complexes with *constructible cohomology* which moreover vanishes except in finitely many degrees. But one problem is that this notion isn't closed under tensor product in general, unless R is a field.

Thus, one sometimes passes to the full subcategory

$$\mathscr{D}^b_{ctf}(X_{\text{ét}},R) \subset \mathscr{D}^b_c(X_{\text{ét}},R)$$

on those objects with finite Tor dimension.

Theorem 59 (Gabber)**.** *The six operations preserve this notion as soon as X is quasi-excellent (an étale-local property: each strict henselization $\mathcal{O}^{\mathrm{sh}}_{X,x}$ is excellent). On the other hand, the four operations f^*, f_*, $f_!$, $f^!$ already preserve $\mathscr{D}^b_c$.*

Remark 60. There's a variant, called the *injective derived category*: first define the *strong* homotopy category $K(\mathrm{Shv}(X_{\text{ét}},R))$, and then we have the full subcategory

$$\mathscr{D}^{inj}(X_{\text{ét}},R) \subset K(\mathrm{Shv}(X_{\text{ét}},R))$$

on complexes of injectives. This is automatically compactly generated, and sits in a right localization adjunction

$$\mathscr{D}^{inj}(X_{\text{ét}},R) \rightleftarrows \mathscr{D}(X_{\text{ét}},R).$$

In fact, its subcategory of compact objects is precisely

$$\mathscr{D}^{+}(X_{\text{ét}},R) \subset \mathscr{D}^{inj}(X_{\text{ét}},R).$$

(So Gabber's finiteness condition applies that the six operations preserve this.)

Proposition 61. *If the cohomological dimension of the reside fields of X is uniformly bounded, then $\mathscr{D}(X_{\text{ét}},R)$ is compactly generated by $\mathscr{D}^b_{ctf}(X_{\text{ét}},R)$.*

This gives an obstruction for the injective derived category to be equivalent to the full one.

Now, in our category of motives, we don't have a t-structure (yet! and who knows when, if ever) so we can't talk about an analog of $\mathscr{D}^b_c$. Nor do we know what "injective" means there. On the other hand, we *can* talk about

A good supply of constructible objects are constant sheaves and their finite extensions and whatnot. More generally, give a perfect complex of R-modules, its corresponding constant sheaf will be constructible. As we saw previously, these will be exactly the rigid objects. In fact, constructible sheaves aren't local for the étale topology, but also for the *constructible* topology.

Take a closed immersion i with

$$i : Z \hookrightarrow X \hookleftarrow (X-Z) : j.$$

Then, $C \in \mathscr{D}(X_{\text{ét}},R)$ is ***constructible*** iff $C|_Z = i^*(C)$ and $C|_{X-Z} = j^*(C)$ are both constructible; this comes from the distinguished triangle

$$j_!(C|_{X-Z}) \to C \to i_*(C|_Z).$$

Now, to prove that the six operations preserve constructibility, it's useful to know that constructible objects can actually be *constructed* from elementary pieces.

Theorem 62. *An étale sheaf $C \in \mathscr{D}(X_{\text{ét}}, R)$ is constructible of finite tor dimension (i.e. it lies in $\mathscr{D}^b_{ctf}(X_{\text{ét}}, R)$) iff there exists a stratification $\{X_i\}$ of X by finitely many locally closed subschemes such that each restriction $C|_{X_i}$ is rigid.*

Thus, this is true iff C is *locally constant* in the étale topology. (Note that the notion of rigidity is local for the étale topology!) So this is a good candidate for the notion of constructibility.

Theorem 63. *Suppose X is noetherian. For an object $M \in DM_h(X, R)$, the following conditions are equivalent:*

(1) there exists a surjective étale map $f : X' \to X$ such that $f^*(M)$ belongs to the smallest thick subcategory of $DM_h(X', R)$ containing objects of the form $R(Y)(n)$ for Y/X' smooth of finite type and $n \in \mathbb{Z}$;
(2) as in previous item, but without the word "smooth" anywhere;
(3) there exists a stratification $\{X_i\}$ of X by locally closed subschemes such that each restriction $M|_{X_i}$ is rigid in $DM_h(X_i, R)$;
(4) there exists a finite stratification $\{X_i\}$ of X by locally closed subschemes, such that each $M|_{X_i}$ has the following property:
  - *there exists a surjective étale map $f : V \to X_i$ such that $N = f^*(M|_{X_i})$ belongs to the smallest thick subcategory generated by objects of the form $g_*(R)(n)$ with $n \in \mathbb{Z}$ and g of the following form:*
    - *the map $g : W \to V$ is "smooth and proper up to an inseparable extension": that is, it factors as*
$$W \xrightarrow{g'} V' \xrightarrow{p} V$$
such that g' is smooth and proper, and p is finite flat surjective and is purely inseparable (i.e. a "universal morphism" in the h-topology);

(5) there exists a(n étale) surjective map $f : X' \to X$ such that $f^*(M)$ belongs to the smallest thick subcategory generated by objects of the form $p_*(R)(n)$, where p is proper and $n \in \mathbb{Z}$.

The 4th condition tells us "why" objects are rigid: morally, it's due to Poincaré duality. But this doesn't strictly work unless we're over a field – we need to appeal also to h-descent (so that motives over a field and an inseparable extension will be equivalent). To be clear, the "Poincaré duality" is happening through the map g'.

This is where continuity arises, and is quite helpful: it allows us to pass from notions over a field to more general notions.

Definition 64. We write $DM_{h,c}(X, R) \subset DM_h(X, R)$ for the thick subcategory spanned by ***constructible*** objects, i.e. M for which there exist a stratification $\{X_i\}$ such that each $M|_{X_i}$ is rigid. We also write $DM_{h,gm}(X, R)$ for the thick subcategory spanned by ***geometric*** objects, i.e. those of the form $R(Y)(n)$ for Y/X smooth of finite type and $n \in \mathbb{Z}$.

Theorem 65. *Assume $X = \lim X_i$ (a cofiltered limit) where $\{X_i\}$ is a pro-scheme with affine transition morphisms, and assume that everything here is noetherian. Then the canonical functors*
$$\operatorname{colim}_i DM_{h,gm}(X_i, R) \to DM_{h,gm}(X, R)$$
and
$$\operatorname{colim}_i DM_{h,c}(X_i, R) \to DM_{h,c}(X, R)$$
are equivalences.

Thus, these presheaves $DM_{h,c}$ and $DM_{h,gm}$ are "morally" stacks (though not quite truly).

This notion of continuity also gives a local-to-global principle (from fields).

Theorem 66. *Suppose that the residue fields of X are of finite cohomological dimension. Then, $M \in DM_h(X, R)$ is in $DM_{h,gm}$ if and only if there exists $f : X' \to X$ étale surjective such that $f^*(M)$ is in $DM_{h,gm}$. Moreover, the subcategory $DM_{h,gm} \subset DM_h$ is exactly that of the compact objects.*

This last statement is nice, because compactness is local (in the étale topology). So, the geometric objects really are the compact objects. Previously, we only had this fact over a field: the proof starts over a perfect (algebraically closed) field.

Theorem 67. *Fix a quasi-excellent scheme S, and consider finite type schemes over S. Then the six operations f^*, f_*, $f_!$, $f^!$ (for f of finite type) and $\otimes$, $\underline{\hom}$ preserve both the subcategories*
$$DM_{h,c}(-, R) \subset DM_h(-, R)$$

of constructible motives and the subcategories

$$DM_{h,gm}(-,R) \subset DM_h(-,R)$$

of geometric motives.

The following extremely useful result asserts that "locally for the h-topology, everything is regular".

Theorem 68 (Gabber). *Let X be a reduced quasi-excellent scheme and let $j : U \hookrightarrow X$ be a dense open immersion. Fix a prime ideal $\mathfrak{p} \subset \mathbb{Z}$ such that the exponent characteristic of X (for all residue fields) belongs to $\mathbb{Z} - \mathfrak{p}$. Then there exists the following data:*

(1) a finite cover for the h-topology

$$\{f_i : Y_i \to X\}_i$$

such that, for all $i \in I$, the map f_i is separated of finite type, Y_i is regular, and $f_i^{-1}(U)$ is either Y_i itself or the complement of a strict normal crossing divisor;

(2) writing $Y = \amalg Y_i$ and $f = \amalg f_i$, a commutative diagram

$$\begin{array}{ccccc} X''' & & \xrightarrow{\quad g \quad} & & Y \\ {\scriptstyle q}\downarrow & & & & \downarrow{\scriptstyle f} \\ X'' & \xrightarrow[u]{} & X' & \xrightarrow[p]{} & X \end{array}$$

in which

- *p is proper birational,*
- *u is a Nisnevich cover,*
- *q is a flat finite surjective map of degree $d \notin \mathfrak{p}$.*