Formal moduli spaces in equal characteristic

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0 Introduction – Michael Rapoport

The aim of this seminar is to understand the analog of "Rapoport-Zink theory" (as given in the book *Period spaces for p-divisible groups*) in the equal characteristic case.

0.1 A brief summary of the unequal characteristic case

For an expanded summary, one should consult the notes from Rapoport's 1994 ICM talk, or Kottwitz's in-depth review of the original source.

Let $k = \overline{k}$ be an algebraically closed field of characteristic p, let $W = W(k)$ be its ring of Witt vectors, σ denote the Frobenius on both k and W, and $\texttt{Nilp}$ be the category of W-schemes S such that $p \cdot \mathcal{O}_S$ is a locally nilpotent ideal sheaf. If $S \in \texttt{Nilp}$, write $\overline{S} = S \times_{\text{Spec } W} \text{Spec } k$. Fix a p-divisible group X/k, and consider the functor $\texttt{Nilp} \to \texttt{Sets}$ given by

$$S \mapsto \{(X, \rho) : X/S \text{ a } p\text{-divisible group, } \rho : X \times_S \overline{S} \to X \times_{\text{Spec } k} \overline{S} \text{ a quasi-isogeny}\}/\text{iso}.$$

The main theorem is the following.

Theorem 1. *This functor is representable by a formal scheme $\mathcal{M}$ which is separated and locally of finite type over* Spf W.

From here, one can take the theory in a number of directions.

1. Completions of local rings $\hat{\mathcal{O}}_{\mathcal{M},x}$ can be understood in terms of the theory of *local models*.
2. We have a *period map*, a morphism $\mathcal{M}^{rig} \to \mathcal{F}^{rig}$ of rigid spaces (where the target is a Grassmannian), defined via Grothendieck-Messing theory.
3. We can study variants of p-divisible groups that have additional structure (e.g. polarizations).
4. There is a tower of coverings of $\mathcal{M}^{rig}$ via level structures on the Tate module of the universal p-divisible group.
5. One can use the formal schemes here to uniformize moduli spaces of abelian varieties along the "basic" locus in the mod p reduction.

The equal characteristic case might look superficially similar, but it is actually quite different. This seminar will look at the equal characteristic analogs of items 1-4.

0.2 A summary of the equal characteristic case

We begin by observing that the prime p enters in two ways in the above theory: as a uniformizer of $\mathbb{Z}_p$, and as an element of $\mathcal{O}_S$ (for S a $\mathbb{Z}_p$-scheme). In the equal characteristic case, we unlink these two elements by taking z to be the uniformizer of $\mathbb{F}_q[[z]]$ (the analog of $\mathbb{Z}_p$) and ζ to be the image of z in $\mathcal{O}_S$ (where now S is an $\mathbb{F}_q[[z]]$-scheme). Then, the appropriate analog of a p-divisible group is the following.

Definition 1. Let S be an $\mathbb{F}_q[[z]]$-scheme, $\zeta \in \mathcal{O}_S$ be the image of z, and $d, h \geq 0$. Then a *divisible Anderson module* of *height* h and *dimension* d over S is an inductive system

$$\underline{H} = (H_1 \overset{i_1}{\hookrightarrow} H_2 \overset{i_2}{\hookrightarrow} H_3 \overset{i_3}{\hookrightarrow} \cdots)$$

of finite $\mathbb{F}_q[[z]]$-schemes over S, such that:

1. As $\mathbb{F}_q$-module schemes, H_n is *strict* (a technical condition which is trivial if $p = q$) and can be embedded into an $\mathbb{F}_q$-vector group scheme (i.e. a power of $\mathbb{G}_a$), and rk $H_n = q^{hn}$.
2. For all n, we have an exact sequence

$$0 \to H_n \xrightarrow{i_n} H_{n+1} \xrightarrow{\cdot z} H_{n+1}.$$

3. For all n, $(z-\zeta)^d|_{\text{Lie } H_n} = 0$.
4. For $n \gg 0$, $\dim_{\kappa(s)}(\text{Lie } H_n) \otimes \kappa(s) = d$ for all $s \in S$.

Note that we do *not* impose that $(z-\zeta)|_{\text{Lie } H_n} = 0$ (i.e. that z acts as ζ).

As p-divisible groups arise from abelian varieties, these divisible Anderson modules will also arise from global objects, namely *Drinfeld modules* and *abelian t-modules*.

Now, Dieudonné theory plays a big role in the theory of p-divisible groups. We recall in the simpler case that k is a perfect field, a *Dieudonné crystal* over k is a pair (M, F_M), where M is a finitely generated free $W(k)$-module and $F_M : M \to M$ is a σ-linear endomorphism of M such that $p \cdot M \subset F_M(M) \subset M$; equivalently, we could talk $F_M^\# : \sigma^*(M) \to M$ to be an injective $W(k)$-linear homomorphism such that $\text{coker}(F_M^\#)$ is annihilated by p. Then, Dieudonné theory gives an equivalence

$$\{p\text{-divisible groups over } k\}^{op} \xrightarrow{\sim} \{\text{Dieudonné crystals over } k\},$$

wherein Lie $H \simeq \text{coker}(F_M^\#)^\vee$.

In the theory of divisible Anderson modules, we have the following analog.

Definition 2. If S is an $\mathbb{F}_q[[z]]$-scheme, then a *local shtuka* of rank n over S is a pair (M, F_M), where M is a sheaf of $\mathcal{O}_S[[z]]$-modules on S which is locally on S a free $\mathcal{O}_S[[z]]$-module of rank n, and $F_M : \sigma^* M[\frac{1}{z-\zeta}] \xrightarrow{\sim} M[\frac{1}{z-\zeta}]$ is an isomorphism of $\mathcal{O}_S[[z]]$-modules, such that there exists some $e \in \mathbb{N}$ such that $F_M(\sigma^*(M)) \subset (z-\zeta)^{-e} \cdot M$, with cokernel a locally free $\mathcal{O}_S$-module of finite rank on which $(z-\zeta)$ is nilpotent. The local shtuka (M, F_M) is called *effective* if we don't need to introduce denominators (i.e. we can take $e = 0$), and it is called *effective miniscule* if additionally $M/\text{im}(F_M)$ is annihilated by $(z-\zeta)$.

There is the following "miracle theorem".

Theorem 2. *Let S be an $\mathbb{F}_q[[z]]$-scheme such that $\zeta \cdot \mathcal{O}_S$ is locally nilpotent. Then there is an equivalence of categories*

$$\{\text{divisible Anderson modules}\}^{op} \simeq \{\text{effective miniscule local shtukas}\}.$$

This is much better than in the unequal characteristic case! There, one constantly uses the Dieudonné-Manin classification of isocrystals over $k = \overline{k}$ (i.e. pairs (N, F_N), where N is a finite dimensional $\text{Frac}(W(k))$-vector space and $F_N : N \to N$ is a σ-linear bijection). Dieudonné proved that that category of isocrystals is *semisimple*, and gave an explicit description of the simple objects. In the equal characteristic case, one simply replaces $\text{Frac}(W(k))$ by $\mathbb{F}_q((z))$.

Classically, p-divisible groups over a perfect field k are classified by their Dieudonné modules, and a lift to characteristic 0 (say, over $W(k)$) is equivalent to a lift of a certain filtration.

Definition 3. In the unequal characteristic case, a *filtered isocrystal* is a triple $(N, F_n, \text{Fil}^\bullet)$, where (N, F_N) is an isocrystal over k and $\text{Fil}^\bullet$ is a descending, separated, exhaustive filtration of the K-vector space $N \otimes_{K_0} K$, where K is some field extension of $K_0 = \text{Frac}(W)$.

We need to introduce certain lattices, for which we use the black box of *Fontaine theory.* Let $\mathfrak{P}_D = N_K \otimes_K B_{DR}^+$ be a B_{DR}^+-lattice in $N_K \otimes_K B_{DR}$, and let $\mathfrak{Q}_D = \text{Fil}^0(N_K \otimes B_{DR})$. The latter gives us another B_{DR}^+-lattice, and we obtain a bijection

$$(\text{filtrations Fil}^\bullet \text{ on } N_K) \cong (\text{Gal}(\overline{K}/K)\text{-stable lattices } \mathfrak{Q} \text{ in } \mathfrak{P}_D \otimes_{B_{DR}^+} B_{DR}).$$

This suggests the analog in the equal characteristic case, where filtered isocrystals are replaced by *Hodge-Pink structures* (i.e. the first step of the filtration). However, the analog of the tensor product theorem only holds for *weakly admissible* Hodge-Pink structures. Moreover, the period map now becomes a map into a *jet variety* over the Grassmannian.

1 z-divisible Anderson modules and local shtukas, part 1 – Michael Rapoport

Throughout this talk we fix a prime number p, and we write $q = p^v$.

1.1 Finite flat group schemes in characteristic p

Let S denote an arbitrary $\mathbb{F}_p$-scheme (i.e. a scheme such that p annihilates the structure sheaf). We will consider flat (commutative) affine group schemes G/S (of finite type).

Let us recall the *relative Frobenius morphism* $F : G \to G^{(p)}$. Here, $G^{(p)} = G \times_{S,\sigma} S$ for σ the (absolute) Frobenius on S. Then, F is defined locally by declaring that if $S = \mathrm{Spec}(R)$ and $A(G)$ denotes the ring of regular functions on G, then F is induced by $A(G^{(p)}) = A(G) \otimes_{R,\sigma} R \to A(G)$ via $a \otimes r \mapsto a^p \cdot r$.

This admits a "quasi-inverse", the *Verschiebung homomorphism* $V : G^{(p)} \to G$. This is harder to define in general. However, if G/R is a finite flat group scheme, then under Cartier duality we have that $V_G^\vee = F_{G^\vee} : G^\vee \to (G^p)^\vee = (G^\vee)^{(p)}$. Explicitly, suppose $A(G)$ is a free R-module (which always holds locally in this case) with generators $a_1, \dots, a_n$, write $\Delta : A(G) \to A(G) \otimes_R A(G)$ for the comultiplication, and let

$$\Delta^{(p)} = (\Delta \otimes \mathrm{id}^{\otimes(p-2)}) \circ \cdots \circ (\Delta \otimes \mathrm{id}) \circ \Delta : A(G) \to A(G)^{\otimes p}$$

be given by

$$\Delta^{(p)}(a_i) = \sum_{1 \le i_1, \dots, i_p \le n} \alpha_{i,i_1,\dots,i_p} \cdot a_{i_1} \otimes \cdots \otimes a_{i_p}$$

for coefficients $\alpha_{i,i_1,\dots,i_p} \in R$. Then the Verschiebung pulls back functions as

$$V^*(a_i) = \sum_{1 \le j \le n} a_j \otimes \alpha_{i,j,\dots,j} \in A(G^{(p)}).$$

In any case, we have the formulas $V \circ F = p \cdot \mathrm{id}_G$ and $F \circ V = p \cdot \mathrm{id}_{G^{(p)}}$ (hence the term "quasi-inverse").

Example 1. Let $\mathcal{E}$ be a locally free $\mathcal{O}_S$-module of finite rank. To this we can associate a *vector group*. This is the vector bundle $\mathbb{V}(\mathcal{E}) = \underline{\mathrm{Spec}}_S(Sym(\mathcal{E}))$, and the group structure is given locally by saying that if $\mathcal{E} = \mathcal{O}_S^n$, then $\mathbb{V}(\mathcal{E}) = \mathbb{G}_a^n$. In particular, $\mathbb{G}_a = \mathrm{Spec}(\mathcal{O}_S[T])$ with $\Delta(T) = T \otimes 1 + 1 \otimes T$. If $S = \mathrm{Spec}(R)$, then $\mathrm{End}(\mathbb{G}_a) = R[F]$, where $F \cdot \lambda = \lambda^p \cdot F$.

Let G be an arbitrary group scheme over R. Then we have

$$\mathscr{M}(G) = \mathrm{Hom}_{\mathtt{Gr}_R}(G, \mathbb{G}_a) = \{a \in A(G) : \Delta(a) = a \otimes 1 + 1 \otimes a\},$$

the *primitive elements* of the Hopf algebra $A(G)$. This is an $R[F]$-module of finite type, called the *Dieudonné module* of G. If G is finite, then this has finite type as an R-module.

Conversely, to every $R[F]$-module M of finite type we can associate a group scheme $\mathscr{G}(M)$ over R. We give the construction in the special case that as an R-module, M is free and of finite type. Fix an R-basis $m_1, \dots, m_n$ of M, and write $Fm_i = \sum_j \gamma_{ji} m_j$. Then we set $A(\mathscr{G}(M)) = R[X_1, \dots, X_n]/I$, where I is the ideal generated by $\{X_i^p - \sum_j \gamma_{ji} X_j : 1 \le i \le n\}$, with comultiplication $\Delta(X_i) = X_i \otimes 1 + 1 \otimes X_i$.

Theorem 3. *Let G/R be a flat affine group scheme of finite type over R. Then the following are equivalent:*

1. $V_G = 0$.
2. *G is isomorphic to a closed subgroup of a vector group.*
3. *G is of the form $G = G(M)$ for M a $R[F]$-module of finite type.*

Furthermore, the functors $\mathscr{M}$ and $\mathscr{G}$ induce an equivalence of categories

$$\{G/R \text{ a group scheme with } V_G = 0\}^{op} \simeq \{M \text{ an } R[F]\text{-module of finite type}\}.$$

If M is finite as an R-module, then $\mathrm{rk}(\mathscr{G}(M)) = p^{\mathrm{rk}_R(M)}$.

Any such group scheme G/R (i.e. one with $V_G = 0$) is called *unipotent*.

Remark 1. If $R = k = \overline{k}$ is an algebraically closed field, then this category is semisimple, with simple objects $\mathbb{G}_a$, $\mathbb{Z}/p$, and $\alpha_{p^r} = \ker(F^r : \mathbb{G}_a \to \mathbb{G}_a^{(p)})$.

1.2 Strict $\mathbb{F}_q$-modules

We would like to look at higher powers of p. (In the above story, everything is an $\mathbb{F}_p$-scheme.) In particular, we'd like to extend Dieudonné theory to such group schemes. (In his book, Laumont claims that the above story extends immediately, but this is not true, as we will see below.)

Let us write $\mathtt{DAug}_R$ for the category of surjective R-algebra homomorphisms $A^\flat \twoheadrightarrow A$, where A is an augmented R-algebra, such that there exists a surjection $R[\underline{X}] = R[X_1, \ldots, X_n] \twoheadrightarrow A^\flat$ such that:

1. $I = \ker(R[\underline{X}] \twoheadrightarrow A^\flat \twoheadrightarrow A)$ is generated by a regular sequence of length n in $R[\underline{X}]$.
2. $I \cdot I_{R[\underline{X}]} = \ker(R[\underline{X}] \twoheadrightarrow A^\flat)$, where $I_{R[\underline{X}]} = (X_1, \ldots, X_n)$.

Given an object $\mathcal{A} = (A^\flat \twoheadrightarrow A) \in \mathtt{DAug}_R$, we can associate the complex

$$\mathcal{L}_\bullet^{A/R} = (I/I \cdot I_{R[\underline{X}]} \to I_{R[\underline{X}]}/I^2_{R[\underline{X}]}),$$

concentrated in degrees -1 and 0. This is a model for the *cotangent complex* of A/R (which is only well-defined up to homotopy equivalence).

Of course, one naturally wonders: Can we always write the cotangent complex of A/R in this form? If so, how unique is it? The answers to these questions are given by the following result.

Proposition 1. *If A is a relative local complete intersection which is locally free of finite rank over R, then:*

1. *A always extends to an object $(A^\flat \twoheadrightarrow A) \in \mathtt{DAug}_R$.*
2. *Any homomorphism $A \to A'$ of augmented R-algebras lifts to $A^\flat \to A'^\flat$, and the set of lifts is a principal homogeneous space for* $\mathrm{Hom}(\mathcal{L}_0, \mathcal{L}'_{-1})$.

That is, a lift of $A \to A'$ determines a map $\mathcal{L}_\bullet^{A/R} \to \mathcal{L}_\bullet^{A'/R}$ on (this presentation of) the respective cotangent complexes, and then the set of lifts naturally becomes a homogeneous space for the set of maps $\mathcal{L}_\bullet^{A/R} \to \mathcal{L}_\bullet^{A'/R}$ that are homotopic to this one.

Then, this relates to the theory of group schemes as follows.

Proposition 2. *Suppose $A = A(G)$, where G/R is a finite flat group scheme.*

1. *If $(A^\flat \twoheadrightarrow A) \in \mathtt{DAug}_R$, then $A^\flat = A(G^\flat)$ for a unique finite flat group scheme $G^\flat/R$.*
2. *If $(A^\flat \twoheadrightarrow A) \to (A'^\flat \twoheadrightarrow A')$ is a morphism in $\mathtt{DAug}_R$ such that $A \to A'$ is induced by a group homomorphism $G' \to G$, then $A^\flat \to A'^\flat$ is induced by a group homomorphism $G'^\flat \to G^\flat$.*

We can summarize these facts as follows. Let $\mathtt{DGr}_R$ be the category of pairs $(G^\flat, G)$ of finite flat group schemes over R such that $(A(G^\flat) \twoheadrightarrow A(G)) \in \mathtt{DAug}_R$, and let $\mathtt{DGr}^*_R$ be the category with the same objects but with $\mathrm{Hom}((G^\flat, G), (G'^\flat, G')) = \mathrm{Hom}(G, G')$.

Corollary 1. *The forgetful functor $\mathtt{DGr}^*_R \to \mathtt{Gr}_R$ is an equivalence of categories.*

This motivates the following definition.

Definition 4. Let R be an $\mathbb{F}_q$-algebra. We write $\mathtt{DGr}(\mathbb{F}_q)_R$ for the subcategory of those $\mathbb{F}_q$-group schemes in $\mathtt{DGr}_R$ (i.e. $(G^\flat, G) \in \mathtt{DGr}_R$ with $\mathbb{F}_q \to \mathrm{End}((G^\flat, G))$) such that the induced action of each $\alpha \in \mathbb{F}_q^\times$ on $\mathcal{L}_\bullet$ is homotopic to the action induced by the inclusion $\alpha \in \mathbb{F}_q \hookrightarrow R$. Then, an $\mathbb{F}_q$-module group scheme G/R is called *strict* if it comes from an object in $\mathtt{DGr}(\mathbb{F}_q)_R$ under the forgetful functor $\mathtt{DGr}(\mathbb{F}_q)_R \to \mathtt{Gr}_R$ (i.e. G embeds into some $G^\flat$ such that the $\mathbb{F}_q$-action lifts).

This condition is trivial in the base case.

Proposition 3. *Let $p = q$, and let $(G^\flat, G) \in \mathtt{DGr}_R$ be such that $[p]_G = 0$. Then the following are equivalent:*

1. *$(G^\flat, G) \in \mathtt{DGr}(\mathbb{F}_q)_R$.*

2. $[p]_{G^\flat} = 0$.

3. $V_G = 0$.

However, if $q \neq p$, the condition of strictness is a real restriction.

Example 2. Let $G = \alpha_p = \mathrm{Spec}(R[X]/X^p)$, equipped with the $\mathbb{F}_q$-module structure where $\alpha \in \mathbb{F}_q^\times$ acts as $[\alpha](X) = \alpha X$. We claim that this is not strict. We compute the cotangent complex as follows. First, $I = (X^p)$ and $I_{R[X]} = (X)$, so $I \cdot I_{R[X]} = (X^{p+1})$. Then

$$\mathcal{L}_\bullet = (R \cdot X^p \xrightarrow{0} R \cdot dX),$$

and the action of $[\alpha]$ is given by (α^p, α) (on the two terms). This is not homotopic to (α, α). Since the lift of a morphism off G to a morphism off $G^\flat$ is unique up to unique homotopy, this proves that this $\mathbb{F}_q$-module cannot be strict.

Example 3. α_q is strict.

Example 4. $\mathbb{F}_q = \mathrm{Spec}(R[X]/(X^q - X))$ is strict.

The following result is the analog of the result in the previous section.

Theorem 4 (Zink)**.** *There is an equivalence of categories*

$$\mathscr{M} : \{\text{strict } \mathbb{F}_q\text{-modules over } R\}^{op} \simeq \left\{ \begin{array}{l} (M, F) : M \text{ a locally free } R\text{-module of finite rank,} \\ F : \sigma_q^*(M) \to M \text{ an } R\text{-linear homomorphism} \end{array} \right\} : \mathscr{G}.$$

Furthermore:

1. *Both functors are $\mathbb{F}_q$-linear and exact.*

2. *$\mathscr{G}(M, F)$ is étale iff F is an isomorphism.*

3. *$\mathscr{G}(M, F)$ is radicial (i.e. its points are the same as those of* $\mathrm{Spec}(R)$*) iff F is locally nilpotent (with respect to R).*

4. *$|\mathscr{G}(M, F)| = q^{\mathrm{rk}_R(M)}$.*

5. *There are natural isomorphisms $H_0(\mathcal{L}_\bullet) = \omega_{G/R} = \mathrm{coker}(F)$ and $H_{-1}(\mathcal{L}_\bullet) = \ker(F)$ of R-modules.*

Let us spell out what the two functors are above.

1. The leftward functor $\mathscr{G} = \mathscr{G}_q$ is given as follows. Suppose $\mathscr{G}(M, F) = G$. Then $A(G) = \mathrm{Sym}_R M/J$, where J is the ideal generated by $\{m^q - F(m \otimes 1) : m \in M\}$. This has $\Delta(m) = m \otimes 1 + 1 \otimes m$ and $[\alpha](m) = \alpha \cdot m$. Moreover, this determines $A(G^\flat) = \mathrm{Sym}_R M/J \cdot I_M$, where I_M is the ideal generated by $\{m : m \in M\}$.

2. The rightward functor $\mathscr{M} = \mathscr{M}_q$ is given by $\mathscr{M}(G) = \{a \in A(G) : \Delta(a) = a \otimes 1 + 1 \otimes a,\ [\alpha](a) = \alpha \cdot a \text{ for all } \alpha \in \mathbb{F}_q\}$, with F induced by the relative Frobenius.

So, we have now seen the analog of Dieudonné theory. However, we want to move to the analogs of *p-divisible groups*, where we take limits.

2 z-divisible Anderson modules and local shtukas, part 2 – Michael Rapoport

Fix an $\mathbb{F}_q[[\zeta]]$-algebra R such that $\zeta \in R$ is nilpotent. These form the opposite of the category `Nilp`. In this case, $z - \zeta$ is a non-zerodivisor in $R[[z]]$, and so we can invert it to obtain an inclusion $R[[z]] \hookrightarrow R[[z]][\frac{1}{z-\zeta}]$.

Our goal is to classify the analogs of p-divisible groups over a given object $S \in$ `Nilp`.

2.1 Local shtukas

We begin with the objects that will play the role of Diedonné modules in our equivalence of categories. We first make a few preliminary observations.

1. $U \mapsto \Gamma(U, \mathcal{O}_S)[[z]]$ is a sheaf on S in the Zariski topology, which we denote by $\mathcal{O}_S[[z]]$. This is because in general, $\Gamma(U, \prod \mathcal{F}_i) = \prod \Gamma(U, \mathcal{F}_i)$.
2. Similarly, whenever S is quasicompact, then $U \mapsto \Gamma(U, \mathcal{O}_S)[[z]][\frac{1}{z-\zeta}]$ defines the sheaf $\mathcal{O}_S[[z]][\frac{1}{z-\zeta}]$.
3. If M is a sheaf of $\mathcal{O}_S[[z]]$-modules on S, then we can define $\sigma^*(M) = M \otimes_{\mathcal{O}_S, \sigma} \mathcal{O}_S$ and $M[\frac{1}{z-\zeta}] = M \otimes_{\mathcal{O}_S[[z]]} \mathcal{O}_S[[z]][\frac{1}{z-\zeta}]$.

We can now give the main definition.

Definition 5. A *local shtuka* of *rank* n over S is a pair (M, F), where M is a sheaf of $\mathcal{O}_S[[z]]$-modules on S which is locally free of rank n, and $F : \sigma^*(M)[\frac{1}{z-\zeta}] \xrightarrow{\cong} M[\frac{1}{z-\zeta}]$. The local shtuka (M, F) is called *effective* if F is induced from $F^\# : \sigma^* M \to M$ (i.e. if $F(\sigma^*(M)) \subset M$), and is called *effective miniscule* if additionally $z - \zeta$ annihilates coker $F^\#$.

We can think of F as being a Frobenius-linear endomorphism of M "up to zeros and poles", and as usual effectivity means that we don't need any poles. We make this precise with the following result.

Lemma 1. *Let (M, F) be a local shtuka on S. The locally on S:*

1. *There exist nonnegative integers d and e such that*
$$(z - \zeta)^d M \subset F(\sigma^*(M)) \subset (z - \zeta)^{-e} M,$$
and $F|_{\sigma^*(M)} : \sigma^*(M) \to M[\frac{1}{z-\zeta}]$ is injective.
2. *There exists an integer N such that $z^N M \subset F(\sigma^*(M))$.*
3. *In the first statement, the cokernel $(z - \zeta)^{-e} M / F(\sigma^*(M))$ is a locally free $\mathcal{O}_S$-module of finite rank.*

Proof. For the first statement, looking in a neighborhood where M is free gives both containments, and the injectivity is obvious. For the second statement, locally on S we have that ζ is nilpotent, so if N is a p-power with $N \gg 0$, then $(z - \zeta)^N = z^N - \zeta^N = z^N$. For the third statement, suppose $S = \operatorname{Spec} R$ and $\mathfrak{m} \subset R$ a maximal ideal with residue field κ. Then we have the sexseq

$$0 \to \sigma^* M \to (z - \zeta)^{-e} M \to \operatorname{coker} \to 0$$

of $R[[z]]$-modules; applying $- \otimes_R \kappa$ gives us a lexseq

$$\operatorname{Tor}_1^R(\kappa, \operatorname{coker}) \to \sigma^* M \otimes_R \kappa \to (z - \zeta)^{-e} M \otimes_R \kappa \to \operatorname{coker} \otimes_R \kappa \to 0$$

of $\kappa[[z]]$-modules. So, we must show that the Tor term is 0. But applying $- \otimes_R \kappa$ is the same as base changing along $R[[z]] \to \kappa[[z]]$; from this we see that $\sigma^* M \otimes_R \kappa$ is a free $\kappa[[z]]$-module of the same rank, and coker $\otimes_R \kappa$ is a torsion $\kappa[[z]]$-module; thus $\sigma^* M \otimes_R \kappa \to (z - \zeta)^{-e} M \otimes_R \kappa$ must be injective. □

Remark 2. Suppose (M, F) is an effective local shtuka. Then we have $\overline{M} = (M/z^n M, \overline{F})$, and these are precisely the modules from the previous lecture. This appears in the diagram

$$\begin{array}{ccccccccc}
 & & z^n \sigma^*(M) & \longrightarrow & z^n M & \longrightarrow & z^n \operatorname{coker} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \sigma^* M & \xrightarrow{F} & M & \longrightarrow & \operatorname{coker} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \sigma^*(\overline{M}) & \xrightarrow{\overline{F}} & \overline{M} & \longrightarrow & \operatorname{coker} \overline{F} & \longrightarrow & 0,
\end{array}$$

and the snake lemma gives us a map $z^n\text{coker} \to \sigma^*(\overline{M})$ splicing the upper and lower rows into a long exact sequence. Therefore, this explains why $\overline{F}$ is not injective in general, and why coker $\overline{F}$ is not necessarily locally free. But if $n \gg N$, then coker $F =$ coker $\overline{F}$, and hence $\overline{F}$ is then injective. This all mimics the story with p-divisible groups, where the co-Lie algebra is not in general a free module.

2.2 z-divisible Anderson modules

We now turn to the objects of study themselves that we are attempting to classify.

Definition 6. A *z-divisible Anderson module* over S is an fppf-sheaf G of $\mathbb{F}_q[[z]]$-modules on S, such that:

- G is z-torsion, i.e. $G = \operatorname{colim} G(n)$ (where $G(n) = \ker(z^n : G \to G)$);
- G is z-divisible, i.e. $z : G \to G$ is an epimorphism;
- for all n, $G(n)$ is a strict finite $\mathbb{F}_q$-module scheme;
- if we write $\omega_{G/S} = \lim \omega_{G(n)/S}$, then locally on S there exists a positive integer d such that $(z-\zeta)^d|_{\omega_{G/S}} = 0$.

One can show that there is a locally constant integer-valued function h on S such that rk $G(n) = q^{nh}$ for all n. This h is called the *height* of G.

Remark 3. As we mentioned, these arise from certain global objects, but we will not discuss these in this seminar.

Remark 4. Take $G(n) = \alpha_q^n$, with z-action given by the nilpotent $n \times n$ matrix with 1s on the superdiagonal and 0s everywhere else, and let $G = \operatorname{colim} G(n)$. Then G is *not* a z-divisible Anderson module, since it doesn't satisfy the condition on uniformity of nilpotence.

We can now describe the functors

$$\mathscr{M} : \{z\text{-divisible Anderson modules over } S\} \leftrightarrows \{\text{effective local shtukas over } S\} : \mathscr{G}.$$

These are defined using the functors $\mathscr{M}$ and $\mathscr{G}$ from the previous lecture; namely, we set $\mathscr{M}(G) = \lim \mathscr{M}(G(n))$ and $\mathscr{G}(M) = \operatorname{colim} \mathscr{G}(M/z^nM)$ (so that $(\mathscr{G}(M))(n) = \mathscr{G}(M/z^nM)$).

Theorem 5. *Let $S \in$ `Nilp`. Then these functors give an equivalence of categories. Both are $\mathbb{F}_q[[z]]$-linear and exact. Moreover:*

- *G is étale iff $F : \sigma^*M \xrightarrow{\cong} M$.*
- *G is a formal $\mathcal{O}_S[[z]]$-module iff F is locally nilpotent.*
- *$\omega_{G/S} = \omega_{G(n)/S}$ for $n \gg 0$, and we have a canonical isomorphism $\omega_{G/S} \cong \operatorname{coker} F$.*

The main difficulty in the proof already appeared last time, in the equivalence of categories involving strict $\mathbb{F}_q$-modules. (The architect of this theory really just went through Messing's book and changed every instance of p to z.) However, it's nevertheless rather technical, so we'll end here.

3 Dieudonné-Manin classification of isocrystals over an algebraically closed field – Daniel Kirch

We begin with some notation. We denote by $k/\mathbb{F}_q$ some algebraic extension, $\mathcal{O} = \mathbb{F}_q[[z]]$, $\mathcal{O}_k = k[[z]]$, and we write the fraction fields as F and F_k respectively. The Frobenius is denoted $\sigma \in \operatorname{Gal}(F_k/F)$.

Definition 7. A *z-crystal* over k is a pair (M, f), where M is a finite free $\mathcal{O}_k$-module and $f : M \to M$ is a σ-linear map (i.e. $f \cdot a = \sigma(a) \cdot f$ for any $a \in \mathcal{O}_k$) with coker(f) of finite length (i.e. finite-dimensional over k). Then, a *z-isocrystal* over k is a pair (N, f) such that N is a finite-dimensional F_k-vector space and $f : N \to N$ is σ-linear an bijective.

Remark 5. A z-crystal is nothing more or less than an effective local shtuka over k (or more precisely, over Spec k). Given a z-crystal, we can get a z-isocrystal simply by tensoring with F: (M, f) gives rise to $(M \otimes_{\mathcal{O}} F, f \otimes_{\mathcal{O}} F)$.

In this talk, we will characterize the isocrystals over an algebraically closed field (that is, in the case where $k = \overline{\mathbb{F}_q}$).

Example 5. Let $r, s \in \mathbb{Z}$, $r > 0$, $(r, s) = 1$. Then we define $N_{r,s} = (F_k)^r$ and set $f_{r,s} = M\sigma$, where M is the $r \times r$ matrix with 0 along the diagonal, 1 along the subdiagonal, and z^s in the top-right. These define the isocrystal $(N_{r,s}, f_{r,s})$.

Theorem 6. *Assume $k = \overline{\mathbb{F}_q}$. Then the category of z-isocrystals over k is F-linear and semisimple, and its simple objects are precisely the isocrystals $(N_{r,s}, f_{r,s})$. Moreover, the endomorphism algebra $\Delta_{r,s} = \mathrm{End}_{f_{r,s}}(N_{r,s})$ is a central division algebra over F with invariant $-s/r$.*

Let us first describe $F_k[f]$: this is noncommutative, with $f \cdot a = \sigma(a) \cdot f$. If (N, f) is any isocrystal, then N is a finitely generated torsion $F_k[f]$-module. In particular, $N_{r,s} = F_k[f]/F_k[f] \cdot (f^r - z^s)$.

We define the *degree* of a polynomial $P(f) \in F_k[f]$ simply to be the degree of P; with respect to this, we have a division algorithm. More precisely, if $P_1(f), P_2(f) \in F_k[f]$ and $P_2(f) \neq 0$, then there are unique $Q(f), R(f) \in F_k[f]$ such that $P_1(f) = Q(f) \cdot P_2(f) + R(f)$, and $\deg R(f) < \deg P_2(f)$.

From this, we get our first lemma.

Lemma 2. *$F_k[f]/F_k[f] \cdot P(f)$ is torsion and has dimension equal to $\deg P(f)$. Moreover, every finitely generated torsion $F_k[f]$-module is a direct sum of ones of the form $F_k[f]/F_k[f] \cdot P(f)$.*

We continue as follows.

Lemma 3. *If $P(f) \in F_k[f]$ is monic of degree d, then there exists some $m \in \mathbb{N}$ such that we have a factorization $P(f) = \prod_{i=1}^{d}(f - x_i)$ with $x_i \in F_k[z^{1/m}]$.*

Remark 6. Note that $F_k[z^{1/m}] = k((z^{1/m})) \cong k((z)) = F_k$, since we can rewrite $z^{1/m}$ as a power series. So this is quite harmless.

Proof sketch. Of course, we really only need to split off one factor: we must write $P(f) = Q(f) \cdot (f - x)$. So, write $P(f) = f^d + \sum_{i=0}^{d-1} a_i f^i$. If we plug in x, then we would like to solve $a_0 + a_1 x + a_2 x\sigma(x) + \ldots + a_{d-1} x\sigma(x) \cdots \sigma^{d-2}(x) + x\sigma(x) + \cdots \sigma^{d-1}(x) = 0$. Now, we can assume $a_0 \neq 0$, otherwise we can just take $x = 0$. Then, we want to normalize the coefficients a_i such that $v_z(a_i) \geq 0$ and $v_z(a_i) = 0$ for at least one i; this is so that we can apply a mild generalization of Hensel's lemma. We can do this by substituting $z^{a/m} \cdot x$ for x (where we can take $1 \leq m \leq \deg P(f)$). Then we obtain a solution mod z and continually lift this. □

Let us consider the 1-dimensional case, where we're taking $N_a = F_k[f]/F_k[f] \cdot (f - a)$. Then N_a is necessarily simple since $\dim N_a = 1$. Moreover, if we have an isomorphism $u : N_a \xrightarrow{\cong} N_b$, then we can write $u(1) = c\dot{1}$, which is equivalent to saying that $a = \frac{\sigma(c)}{c} \cdot b$, which is equivalent to saying that $v_z(a) = v_z(b)$. Thus, $N_a \cong N_{1,v_z(a)} = F_k[f]/F_k[f] \cdot (f - z^{v_z(a)})$.

This leads us to our next fact.

Lemma 4. *Any exact sequence $0 \to N_a \to N \to N_b \to 0$ with $a, b \neq 0$ splits.*

Proof sketch. Since dimension is additive, then $\dim N = 2$. So we can choose two generators $e_1, e_2 \in N$ such that e_1 is the image of the generator of N_a and e_2 projects to the generator of N_b. Then $f(e_1) = ae_1$ and $f(e_2) = be_2 + a'e_1$. From this, we want to solve $f(xe_1 + e_2) = b(xe_1 + e_2)$ and $a\sigma(x) + a' = bx$. We use the same trick as in the previous proof. □

Corollary 2. *If $P(f) = \prod_i (f - x_i)$, then $F_k[f]/F_k[f] \cdot P(f) \cong \bigoplus_i N_{1,v_z(x_i)}$.*

Thus, we are done with the 1-dimensional case. So we now turn to the case that $P(f)$ does not completely split in $F_k[f]$. However, remember that we know that $P(f)$ splits in some sufficiently large finite extension.

Lemma 5. *$N_{r,s}$ is simple.*

Proof. We can compute directly that $F_k[z^{1/mr}] \otimes_{F_k[f]} N_{r,s} \cong (N_{1,ms}^{(1/mr)})^r$ (where the superscript on the right denotes that we've extended our coefficients: $N_{1,ms}^{(1/mr)} = F_k[z^{1/mr}][f]/F_k[z^{1/mr}][f] \cdot (f - (z^{1/mr})^{ms}) = F_k[z^{1/mr}][f]/F_k[z^{1/mr}][f] \cdot (f - z^{s/r})$). Now, suppose that $N_{r,s}$ is not simple, say $N \subset\neq N_{r,s}$. Then we can assume $N = F_k[f]/F_k[f] \cdot P(f)$, where $\deg P(f) < r$. By extension of scalars up to some $F_k[z^{1/m}]$ with $m < r$, we can assume that $P(f) = Q(f) \cdot (f - x)$. Then, by extending scalars further, we can split $N_{r,s}$ as above. Here, we again have $N_x^{(1/m)} = F_k[f][z^{1/m}]/F_k[f][z^{1/m}](f - x)$, and then as above we have $F_k[z^{1/mr}] \otimes_{F_k[z^{1/m}]} N_x^{(1/m)} \cong N_{1,ms}^{(1/m)}$. Looking at the natural generators, we have that $x = u \cdot z^{ms/mr} = u \cdot z^{s/r}$ for a unit u. But this is a contradiction because $x \notin F_k[z^{1/m}]$, as $r > m$. □

Finally, we reach the proof of the main theorem.

Proof. For the first statement, it remains to show that we can decompose any $N = F_k[f]/F_k[f] \cdot P(f)$ into $N = \bigoplus_i N_{r_i,s_i}$. To do this we extend scalars as $N \otimes_{F_k} F_k[z^{1/m}]$, where we choose m such that we get a factor of the form $N_{1,n}^{(1/m)}$ (i.e. a 1-dimensional summand). Then, take $\frac{s}{r} = \frac{n}{m}$ such that $(s, r) = 1$ and $r \geq 1$. Now let $x \in N \otimes_{F_k} F_k[z^{1/m}]$ be the image of the generator $1 \in N_{1,n}^{(1/m)}$. Then $(f - z^{s/r})x = 0$, and thus we get $(f^r - z^s)x = 0$. We have the componentwise decomposition $x = \sum_{j=0}^{n-1} z^{j/m} x_j$ with $x_j \in N$. But this implies that $(f^r - z^s)x_j = 0$ for all j. But there must be some $x_j \neq 0$, and this defines an injection $N_{r,s} \hookrightarrow N$ given by $1 \mapsto x_j$. So we must show that an exact sequence $0 \to N_{r,s} \to N \to N/N_{r,s} \to 0$ must split. This goes similarly to the 1-dimensional case.

For the second statement, it is clear that $\Delta_{r,s} = \mathrm{End}_{f_{r,s}}(N_{r,s})$ is a division algebra since it is the endomorphism algebra of a simple object, and it is clear that F is in the center. To find its invariant, we observe that

$$\Delta_{r,s} = \{f \mapsto P(f) : (f^r - z^s)P(f) \in F_k[f] \cdot (f^r - z^s)\}/\{P(f) \in F_k[f] \cdot (f^r - z^s)\}.$$

Then, the opposite algebra $\Delta_{r,s}^{opp}$ is given by the same generators $P(f)$ with the same relations. For any $P(f)$ we can write $P(f) = Q(f)(f^r - z^s) + R(f)$, and then $P(f) \in \Delta_{r,s}^{opp}$ iff $R(f) \in F_r$ (the unramified extension of degree r). In this case, we can write $\Delta_{r,s}^{opp} = F_r[f]/F_r[f] \cdot (f^r - z^s)$. Thus $\Delta_{r,s}^{opp}$ has invariant s/r, and thus $\Delta_{r,s}$ has invariant $-s/r$. □

4 The Grothendieck-Messing theorem for miniscule local shtukas – Siddarth Sankaran

In this talk, we will analyze the deformation theory for effective local shtukas. Suppose $\overline{S} \in \mathtt{Nilp}_{\mathbb{F}_q[[\zeta]]}$, $(\overline{M}, \overline{F})$ is an effective local shtuka over $\overline{S}$, and $j : \overline{S} \hookrightarrow S$ is a closed immersion. We would like to describe all lifts of $(\overline{M}, \overline{F})$ to S. We won't always be able to do this. However, we will have a complete classification in two particular cases (where I denotes the sheaf of ideals defining j):

1. $I^{(q)} = 0$;
2. $(\overline{M}, \overline{F})$ is *miniscule* and I has a nilpotent *divided power structure.*

Let us recall that $\mathtt{Nilp} = \mathtt{Nilp}_{\mathbb{F}_q[[\zeta]]}$ denotes the category of $\mathbb{F}_q[[\zeta]]$-schemes such that ζ is locally nilpotent. If $\overline{S} \in \mathtt{Nilp}$, recall that an *effective local shtuka* of *rank* r and *dimension* d over $\overline{S}$ is a pair $(\overline{M}, \overline{F})$, where $\overline{M}$ is a sheaf of $\mathcal{O}_{\overline{S}}[[z]]$-modules which is locally free of rank k, and $\overline{F} : \sigma^*\overline{M} \to \overline{M}$ is an injective map of modules (where $\sigma : \overline{S} \to \overline{S}$ denotes the q-power Frobenius), such that $\mathrm{coker}(\overline{F})$ is a locally free $\mathcal{O}_{\overline{S}}$-module which is annihilated by $(z - \zeta)^d$. This is called *miniscule* if we can take $d = 1$.

If $(\overline{M}, \overline{F})$ is an effective local shtuka over $\overline{S}$, we can define the *Verschiebung* $\overline{V} : \overline{M} \hookrightarrow \sigma^*\overline{M}$ by $\overline{V}(m) = \overline{F}^{-1}((z - \zeta)^d m)$. This satisfies $\overline{F} \circ \overline{V} = (z - \zeta)^d \cdot \mathrm{Id}_{\overline{M}}$ and $\overline{V} \circ \overline{F} = (z - \zeta)^d \cdot \mathrm{Id}_{\sigma^*\overline{M}}$. Then we have a short exact sequence

$$0 \to \mathrm{coker}(\overline{V}) \xrightarrow{\overline{F}} \overline{M}/(z - \zeta)^d \overline{M} \to \mathrm{coker}(\overline{F}) \to 0,$$

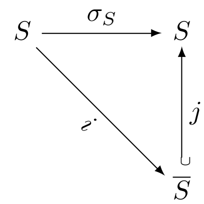
where $\mathrm{coker}(\overline{V})$ is a locally free $\mathcal{O}_{\overline{S}}$-module. From this we obtain the *Hodge exact sequence*

$$0 \to \mathrm{coker}(\overline{F}) \xrightarrow{\overline{V}} \sigma^*\overline{M}/(z - \zeta)^d \sigma^*\overline{M} \to \mathrm{coker}(\overline{V}) \to 0$$

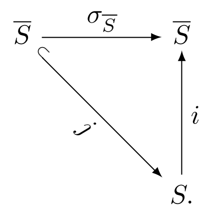
for $(\overline{M}, \overline{F})$. We will denote $\mathbb{D}(\overline{M})_{\overline{S}} = \sigma^*\overline{M}/(z - \zeta)^d \sigma^*\overline{M}$.

4.1 The first situation

Suppose that $j : \overline{S} \hookrightarrow S$ is a closed embedding in `Nilp` defined by a sheaf of ideals I such that $I^{(q)} = 0$. Then the two Frobenii factor as



and



Now, denote by $\mathbb{D}(\overline{M})_S = i^*\overline{M}/(z-\zeta)^d i^*\overline{M}$; this is a locally free sheaf of $\mathcal{O}_S$-modules. Note that $j^*\mathbb{D}(\overline{M})_S = \sigma^*\overline{M}/(z-\zeta)^d\sigma^*\overline{M} = \mathbb{D}(\overline{M})_{\overline{S}}$.

Our lifting theorem will relate lifts of the local shtuka downstairs to admissible submodules of $\mathbb{D}(\overline{M})_S$ upstairs.

Definition 8. Let $\mathcal{F} \subset \mathbb{D}(\overline{M})_S$ be an $\mathcal{O}_S[[z]]$-submodule. We say that $\mathcal{F}$ is *admissible* if:

1. $\mathbb{D}(\overline{M})_S/\mathcal{F}$ is flat as a sheaf of $\mathcal{O}_S$-modules;
2. $j^*\mathcal{F} = \operatorname{coker}(\overline{F})$.

Notice that there's one easy way to obtain admissible submodules: if we actually had a lift of our local shtuka, we can just take $\operatorname{coker}(F)$. That is, if (M, F) is an effective local shtuka over S, then (j^*M, j^*F) is an effective local shtuka on $\overline{S}$. Then we would have

$$\mathbb{D}(M)_S = \sigma_S^*M/(z-\zeta)^d\sigma_S^*M = i^*j^*M/(z-\zeta)^d i^*j^*M = i^*\overline{M}/(z-\zeta)^d i^*\overline{M} = \mathbb{D}(\overline{M})_S.$$

Of course, the theory is set up so that this will turn out to give all the examples.

Theorem 7. *If $j : \overline{S} \hookrightarrow S$ is a closed embedding in* `Nilp` *defined by the ideal I with $I^{(q)} = 0$, then there is an equivalence of categories*

$$\{\text{effective local shtukas}/S\} \simeq \{\text{pairs } ((\overline{M}, \overline{F}), \mathcal{F}) \text{ of an effective local shtuka over } \overline{S} \text{ and an admissible submodule of } \mathbb{D}(\overline{M})_S\}.$$

The equivalence is given by $(M, F) \mapsto ((j^*M, j^*F), \operatorname{coker}(F))$.

Proof. We construct the inverse functor as follows. Given a pair $((\overline{M}, \overline{F}), \mathcal{F})$, we define $M = \ker(i^*\overline{M} \to \mathbb{D}(\overline{M})_S/\mathcal{F})$ and $V = M \hookrightarrow i^*\overline{M}$. To check that M lifts $\overline{M}$, we begin with the short exact sequence

$$0 \to M \xrightarrow{V} i^*\overline{M} \to \mathbb{D}(\overline{M})_S/\mathcal{F} \to 0;$$

we can apply j^* to get an other short exact sequence

$$0 \xrightarrow{j^*V} j^*M \to j^*i^*\overline{M} \to j^*(\mathbb{D}(\overline{M})_S/\mathcal{F} \to 0.$$

But we have $j^*i^*\overline{M} = \sigma_{\overline{S}}^*\overline{M}$, and by our flatness assumptions, $j^*(\mathbb{D}(\overline{M})_S/\mathcal{F}) = \operatorname{coker}(\overline{V})$. Thus, it must be that $j^*M = \overline{M}$, and $j^*V = \overline{V}$, and (by applying i^* to both sides) $\sigma_S^*M = i^*\overline{M}$.

On the other hand, note that $\operatorname{coker}(V)$ is annihilated by $(z-\zeta)^d$. This means that we have a unique $F : \sigma^*M \hookrightarrow M$, and we get a short exact sequence

$$0 \to \operatorname{coker}(F) \xrightarrow{V} \mathbb{D}(M)_S = \mathbb{D}(\overline{M})_S \to \mathbb{D}(\overline{M})_S/\mathcal{F} \to 0$$

which is a lift of the Hodge exact sequence. In particular, it must be that $\operatorname{coker}(F) = \mathcal{F}$. $\square$

4.2 The second situation

(Here, we will follow the reference Grenestier-Lafforgue, *Théorie de Fontaine en égales caractéristiques.* It is quite technical, so we'll only give a rough overview.)

We will give our definitions in the affine case first. Suppose that B is an $\mathbb{F}_q[[\zeta]]$-algebra where the image of ζ is nilpotent, and suppose we are given an ideal $I \subset B$. Loosely speaking, a divided power structure, which will be henceforth denote PD-structure (for "puissances divisées"), is supposed to allow us to write down exponential series in cases where we wouldn't otherwise be able to. Note that the definition here differs somewhat from the classical definition in unequal characteristic.

Definition 9. A *PD-structure* is a map $\gamma : I \to I$ which satisfies the following properties for all $x, y \in I$ and for all $b \in B$:

1. $\gamma(x + y) = \gamma(x) + \gamma(y)$;
2. $\zeta \cdot \gamma(x) = x^q$;
3. $\gamma(b \cdot x) = b^q \cdot \gamma(x)$.

We write $I^{[n]}$ for the ideal generated by elements of the form $\gamma^{a_1}(x_1) \cdots \gamma^{a_k}(x_k)$, where $x_i \in I$ and $\sum q^{a_i} \geq n$. We say that γ is *nilpotent* if $I^{[N]} = 0$ for some $N \gg 0$.

Note that the second condition is telling us that $x^{q^n} = \zeta^n \cdot \gamma^n(x)$; thus, $\gamma^n(x)$ functions as x^{q^n}/ζ^n. The third condition is also telling us that $\gamma(x)$ functions as some sort of q^{th} power operation. The first condition then reflects the fact that we're in characteristic p, where taking the q^{th} power is an additive operation.

Remark 7. Note that $\gamma(I^{[q^n]}) \subset I^{[q^{n+1}]}$. Moreover, if $\gamma(I) = 0$, then $x^q = 0$ for all $x \in I$, and hence $I^{(q)} = 0$. Conversely, if $I^{(q)} = 0$, then $\gamma = 0$ defines a PD-structure on I. Thus, this case should be seen as a generalization of the previous case.

Now, suppose $j : \overline{S} \hookrightarrow S$ is a closed immersion with nilpotent PD-structure on the defining ideal. If $(\overline{M}, \overline{F})$ is an effective local shtuka on $\overline{S}$, we'd like to define as before some object $\mathbb{D}(\overline{M})_S$ such that lifts of the effective local shtuka will be equivalent to certain subobjects of $\mathbb{D}(\overline{M})_S$. It turns out that it's easy to prove the equivalence, but hard to define the object itself. We will restrict to the miniscule case (following the given reference; it's unclear to us where exactly this assumption is used, however).

So, we sketch the definition of the object $\mathbb{D}(\overline{M})_S$. We return to the affine case for simplicity, i.e. we take $S = \operatorname{Spec} B$ and $\overline{S} = \operatorname{Spec} B/I$ (with a nilpotent PD-structure $\gamma : I \to I$), and we write $\overline{B} = B/I$. Then, there is a certain $B[[z]]$-algebra, denoted $D_n(B, I)$, which comes equipped with a map $\delta : \sigma^* D_n(B, I) \to B$. This has the property that the composition $B[[z]] \to \sigma^* D_n(B, I) \to B$ is given by $z \mapsto \zeta$. Of course, this all depends on γ. Now, the key fact is the following.

Proposition 4 (rigidity)**.** *Suppose that we have a map $f : (\overline{M}_1, \overline{F}_1) \to (\overline{M}_2, \overline{F}_2)$ of effective miniscule local shtukas over $\overline{B}$, and suppose that (M_i, F_i) is a lift of $(\overline{M}_i, \overline{F}_i)$ to B. Then there is a unique morphism $\tilde{f}$: $M_1 \otimes_{B[[z]]} \sigma^* D_n(B, I) \to M_2 \otimes_{B[[z]]} D_n(B, I)$ which lifts f (i.e. which recovers f when we tensor down to $\overline{B}$) and which satisfies $F_1 \circ \sigma^* \tilde{f} = \tilde{f} \circ F_2$. Moreover, this construction respects composition.*

Note that $D_n(B, I)$ is a $B[[z]]$-algebra, so we can twist or untwist along the Frobenius as we please. When we say that $\tilde{f}$ lifts f, we mean that we recover f when we tensor down to $\overline{B}$. Explicitly, note that $M \otimes_{B[[z]]} (B/I)[[z]] = \overline{M}$.

This implies that our crystal object is well-defined.

Corollary 3. *Suppose that $(\overline{M}, \overline{F})$ is an effective miniscule local shtuka over $\overline{B}$, and suppose it admits a lift (M, F) to B. Then $\mathbb{D}(\overline{M})_B = \sigma^* M/(z - \zeta)^d \sigma^* M$ is independent of the choice of lift: any two choices are uniquely isomorphic.*

Proof. Note that we can identify $\mathbb{D}(\overline{M})_B = \sigma^*(M \otimes D_n(B, I)) \otimes_\delta B$. Thus is we make the same construction with a different lift (M', F'), then we have a unique isomorphism between the constructions on the right side. ☐

Of course, this is assuming that we have a lift at all. However, the following fact saves us.

Proposition 5. *Locally on* Spec B*, any $(\overline{M}, \overline{F})$ admits a lift.*

Thus we can define $\mathbb{D}(\overline{M})_B$ locally, and then the gluing data downstairs lifts uniquely to gluing data upstairs. Of course, this all means that if we return from affine schemes to schemes, by unique gluing we can transport the whole setup without modification.

We list two important properties of this construction.

1. This is functorial in $\overline{M}$.

2. This is *crystalline*: if $f : (B', I', \gamma') \to (B, I, \gamma)$ is a morphism of PD-structures and $B'/I' = \overline{B} = B/I$, then there is a canonical isomorphism $\mathbb{D}(\overline{M})_{B'} \otimes_f B \cong \mathbb{D}(\overline{M})_B$.

Theorem 8. *If $j : \overline{S} \hookrightarrow S$ is a closed embedding defined by a sheaf I with a nilpotent PD structure $\gamma : I \to I$, then there is an equivalence of categories*

$$\{\text{effective miniscule local shtukas}/\overline{S}\} \simeq \left\{ \begin{array}{l} \text{pairs } ((\overline{M}, \overline{F}), \mathcal{F}) \text{ of an effective miniscule local shtuka} \\ \text{over } \overline{S} \text{ and an admissible } \mathcal{O}_S[[z]]\text{-submodule of } \mathbb{D}(\overline{M})_S \end{array} \right\}.$$

The functor is again given by $(M, F) \mapsto ((j^*M, j^*F), \operatorname{coker}(F))$.

(Note that we may consider $\operatorname{coker}(F) \subset \mathbb{D}(M)_S = \mathbb{D}(\overline{M})_S$, which makes this a reasonable definition.)

Proof. Again we simply construct the inverse functor. Suppose that $((\overline{M}, \overline{F}), \mathcal{F})$ is a pair on the right side. By rigidity we can work locally. Since we are assuming our PD-structure $\gamma : I \to I$ is nilpotent, then $I^{[q^N]} = 0$ for some $N \gg 0$. Then if we write $B_i = B/I^{[q^i]}$, we have that in the sequence $B = B_N \to B_{N-1} \to \cdots \to B_1 \to B_0 = B/I = \overline{B}$. Note that for each $B_i \twoheadrightarrow B_{i-1}$, we have that $(\ker)^{(q)} = 0$. (More precisely, γ descends to the trivial PD-structure on each of these maps.) This puts us in the situation of our previous theorem, which we can apply it inductively.

Given an admissible $\mathcal{F}$, if we tensor down to $B_1[[z]]$, then $\mathcal{F} \otimes B_1 \subset \mathbb{D}(\overline{M})_B \otimes B_1 = \mathbb{D}(\overline{M})_{B_1}$ by the crystalline property. Thus we have a shtuka on B_1, which gives us an M_1. Then, tensoring down to B_2 gives us an M_2, etc., until we obtain $M_N = M$. □

So, the entire crux of the proof is that we have these intermediate objects $\mathbb{D}(\overline{M})_B$ which satisfy the crystalline property.

5 Local G-shtukas – Stephen Kudla

Today, we replace the z with G: we pass from local z-shtukas to local G-shtukas. The basic idea is contained e.g. in the papers of Kottwitz and Rapoport of isocrystals with additional structure, where a reductive group now acts. Since it will appear everywhere, we'll drop the word "local".

5.1 z-shtukas and torsors

Recall the setup that we start out with $\mathbb{F}_q[[\zeta]]$, and we have the category $\texttt{Nilp} = \texttt{Nilp}_{\mathbb{F}_q[[\zeta]]}$ of schemes for which the image of ζ in the structure sheaf is locally nilpotent. Given $S \in \texttt{Nilp}$, we've also introduced the structure sheaves $\mathcal{O}_S[[z]] \subset \mathcal{O}_S[[z]][z^{-1}]$.

Then, recall that a z-shtuka over S is a pair (M, F_M), where M is a sheaf of $\mathcal{O}_S[[z]]$-modules which is locally free of rank r on S, and $F_M : \sigma^*(M)[\frac{1}{z-\zeta}] \xrightarrow{\sim} M[\frac{1}{z-\zeta}]$ is a linear isomorphism. Recall that locally on S, there is an integer $e \in \mathbb{Z}$ such that $F_M(\sigma^*(M)) \subset (z-\zeta)^{-e}M$, and we know that $\operatorname{coker}(F_M)$ is locally free of finite rank over $\mathcal{O}_S$. Finally, recall that (M, F_M) is *effective* if we can take $e \leq 0$, and is called *miniscule* if it is effective and $\operatorname{coker}(F_M)$ is killed by $z - \zeta$.

This is all just a reminder. From here, we would like to convert this data into a *torsor*, which notion we will be able to generalize. First, we think of M as similar to a vector bundle, and it is indeed equivalent to (a certain equivalence class of) a collection of transition functions. Namely, there is a covering $\{S'_\alpha \to S\}$ (in some topology – there are a lot of subtle issues in translating between the various topologies, and we won't address them in this talk) such that:

- $\Gamma(S'_\alpha, M) \simeq (\mathcal{O}_S[[z]](S'_\alpha))^r$, under which we identify s_α with some vector $(s_{\alpha,1}, \dots, s_{\alpha,r})^t$, and

- there are transition functions $f_{\beta,\alpha} \in GL_r(\mathcal{O}_S[[z]](S'_\alpha \cap S'_\beta)$ such that $s_\beta = f_{\beta,\alpha} s_\alpha$ (ignoring restrictions).

These of course satisfy a 1-cocycle condition.

To clean this up, we define a group scheme K over $\mathbb{F}_q$ by setting $K(R) = GL_r(R[[z]])$. Then our transition functions can be considered as $f_{\beta,\alpha} \in K(S'_\alpha \cap S'_\beta)$. Moreover, if we change the trivialization, this is equivalent to postcomposing with some element $k_\beta \in GL_r(K(S'_\beta))$, and so we obtain the cocycle relation $f_{\alpha,\beta} = k_\alpha f_{\alpha,\beta} k_\beta^{-1}$. Thus, we obtain that $\check{H}^1(S'_*, K)$ describes the isomorphism classes of locally free sheaves M. (It is proved in the paper we're following that the topology actually doesn't matter: we end up with the same answer in any case.)

Now, note that we have an action of $K(S'_\alpha)$ on $(\mathcal{O}_S[[z]], (S'_\alpha))^r$, which under our trivialization gives us an action on $\Gamma(S'_\alpha, M)$. Then on an overlap, these actions are related by $k_\alpha = f_{\alpha,\beta} k_\beta f_{\alpha,\beta}^{-1}$. Of course, there may not be a global action of the k_α. But by gluing together the k_α according to this rule, we get the structure of a torsor. Precisely, we get a *K-torsor* $\mathcal{G}$ over S given by $\mathcal{G}(S'_\alpha) \simeq K(S'_\alpha)$, with the actions of k_α and k_β required to satisfy $f_{\beta,\alpha} k_\alpha = k_\beta$ on the overlaps. These are parametrized by the same cohomology group, and indeed it is not hard to go back and forth between the two perspectives.

Now, we've figured out how to translate M into $\mathcal{G}$, and so we'd like to translate F_M into our new language too. The first thing to notice is that we still have a Frobenius action on our transition functions: we can consider $\sigma^*(f_{\alpha,\beta})$ by applying σ^* entry by entry. Then, $F_M : \sigma^*(M)[\frac{1}{z-\zeta}] \xrightarrow{\sim} M[\frac{1}{z-\zeta}]$ corresponds locally over S'_α to some $b_\alpha \in GL_r(\mathcal{O}_S[[z]][\frac{1}{z-\zeta}]$. (We observe that since ζ is locally nilpotent, then it is equivalent to invert z and to invert $z-\zeta$.) Then the map on sections will be given by $s^*_\alpha \mapsto b_\alpha \cdot \sigma^*(s^*_\alpha)$ (where the superscripts denote local coordinates on the base change). Of course, these must be related by $b_\alpha = f_{\alpha,\beta} \cdot b_\beta \cdot \sigma^*(f_{\alpha,\beta}^{-1})$.

We summarize as follows. Associated to M we have the K-torsor $\mathcal{G}$. Under this correspondence, related to $\mathcal{G}$ we have $\mathcal{LG}$, where we simply add z^{-1} to our coefficients, i.e. $\mathcal{LG} = \mathcal{G} \times_K LG$, and this is an LG-torsor over S, where $LG(R) = GL_r(R[[z]][\frac{1}{z}])$ is the *loop group* of GL_n. Then, F_M corresponds to $\varphi : \sigma^*(\mathcal{LG}) \xrightarrow{\sim} \mathcal{LG}$.

5.2 G-shtukas

This allows us to generalize as follows. Let G be an arbitrary connective reductive group over $\mathbb{F}_q$; we assume we have a Borel subgroup and and a maximal split torus, $T \subset B \subset G$. If M came equipped with extra structure, then the transition functions would lie in the group which preserves that extra structure. By working at the level of torsors, we sidestep the issue ever having to actually work with that extra structure: we simply work with the group that preserves it. Now we have the two groups $K(R) = G(R[[z]])$ and $LG(R) = G(R[[z]][z^{-1}])$, and we can follow the same story we have already mapped out above.

Definition 10. A *local G-shtuka* over S is a K-torsor $\mathcal{G}$ over S together with an isomorphism $\varphi : \sigma^*(\mathcal{LG}) \xrightarrow{\sim} \mathcal{LG}$ of LG-torsors over S.

Of course, what's missing here is the actual module M on which everything is realized. This will cause us some trouble in the following. The biggest consequence we'll grapple with is that we had the (locally-defined) integer e above telling us the extent to which F_M didn't really take $\sigma^*(M)$ into M. We need to figure out how to translate this into the language of torsors. We first study the pointwise case.

Remark 8. Suppose $S = \text{Spec } k$, where $k = \overline{k}$ and $\mathbb{F}_p \subset k$. Then any torsor $\mathcal{G}$ over k is trivial. Therefore, we can choose an isomorphism $\mathcal{G} \simeq K_k$ (where the latter is the base-change of K to k). Then, the map $\varphi : \sigma^*(\mathcal{LG}) \xrightarrow{\sim} \mathcal{LG}$ is given by a σ-linear isomorphism $b\sigma^* : LG_k \xrightarrow{\sim} LG_k$ for some $b \in LG(k)$. Given this, changing the choice of trivialization by $k \in K_k$ just changes b by a conjugation $k \cdot b \cdot \sigma(k^{-1})$, so in any case we get a well-defined double-coset $KbK = K(kb\sigma(k^{-1}))K$.

Now, the loop group $LG(k)$ has a *Cartan decomposition*

$$LG(k) = G(k[[z]][z^{-1}]) = \coprod_\mu K(k) z^\mu K(k),$$

where μ runs over the dominant *coweights* $X_*(T)_+$ of G. (Recall that for a coweight $\mu \in X_*(T) = \text{Hom}(\mathbb{G}_m, T)$ to be *dominant* means that $\langle \alpha, \mu \rangle \geq 0$ for all positive simple roots $\alpha \in X^*(T) = \text{Hom}(T, \mathbb{G}_m)$.) This should be thought of as something like a decomposition of $G(\mathbb{Q}_p)$ into things related to $G(\mathbb{Z}_p)$. Note that we have the equality of double-cosets $kb\sigma(k^{-1}) \in KbK = Kz^\mu K$, so these all correspond to the same torsor. We write this as $\mu = \mu_{\mathcal{G}} \in X_*(T)_+$.

Now, let us pass to a shtuka $\underline{\mathcal{G}}$ over a more general base S. (Here, the underline denotes that we have both the torsor $\mathcal{G}$ and the map φ.) Then over a geometric point $\overline{s} : \operatorname{Spec} k \to S$, we have $\overline{s}^*(\underline{\mathcal{G}})$; from the remark above, we hence obtain $\mu_{\underline{\mathcal{G}}}(s) \in X_*(T)_+$. (This depends only on the actual point s that is the image of $\overline{s}$, since this is a discrete invariant.) So, we get a function $\mu_{\mathcal{G}} : S \to X_*(T)_+$. Now, inside of the coweights $X_*(T)$, we have the sublattice $\Phi^\vee$ of coroots, and we have the *algebraic fundamental group* $\pi_1(G) = X_*(T)/\langle \Phi^\vee \rangle$. (This recovers the ordinary fundamental group of a Lie group, as a consequence of their general structure theory.) This gives us a composition

$$S \xrightarrow{\mu_{\mathcal{G}}} X_*(T) \xrightarrow{\mu \mapsto [\mu]} \pi_1(G),$$

which has the following important property.

Lemma 6. *The composite $[\mu_{\mathcal{G}}]$ is locally constant on S.*

5.3 Boundedness

With this in hand, we can discuss this issue of boundedness, which will correspond to our locally-defined integer e. We first recall some facts about the algebraic representation theory of G.

Let $\lambda \in X^*(T)_+$; recall that the $+$ denotes that $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all simple $\alpha^\vee \in \Phi^\vee$ (or more simply, $\langle \lambda, 2\rho^\vee \rangle \geq 0$). Now, attached to each such λ is the natural representations

$$V(\lambda) = (\operatorname{Ind}_B^G(-w_0\lambda))^*$$

where w_0 is the "long element" of the Weyl group and the superscript denotes contragradient. (Cf. the book of Jantzen; this is sometimes called the *Weyl module* for λ.) This module may not be irreducible, but it is the *universal* module for which λ is a heighest weight. (This is the analog of the *Verma module*.)

Now, given a (right) K-torsor $\mathcal{G}$ over S and some $\lambda \in X^*(T)_+$, we define $G_\lambda = \mathcal{G} \times_K (V(\lambda)[[z]])$. Then locally, this looks like

$$\mathcal{G}_\lambda(S'_\alpha) = \mathcal{G}(S'_\alpha) \times (V(\lambda) \otimes_{\mathbb{F}_q} \mathcal{O}_S[[z]])(S'_\alpha) = G(\mathcal{O}_S[[z]])(S'_\alpha) \times (\cdots) \simeq V(\lambda) \otimes_{\mathbb{F}_q} \mathcal{O}_S[[z]](S'_\alpha).$$

So, we should think of these as "formal vector bundles", which have coefficients in $\mathcal{O}_S[[z]]$.

We first give the definition of boundedness, and then we will try to motivate it. The key observation is that the above construction is functorial in the torsor. For instance, $\sigma^*(\mathcal{G}_\lambda) = \sigma^*(\mathcal{G})_\lambda$. So, the isomorphism $\varphi : \sigma^*(\mathcal{LG}) \xrightarrow{\sim} \mathcal{LG}$ gives rise to $\varphi_\lambda : \sigma^*(\mathcal{LG}_\lambda) \xrightarrow{\sim} \mathcal{LG}_\lambda$. Recall our setup

$$\begin{array}{ccc} \sigma^*(\mathcal{LG}_\lambda) & \hookleftarrow & \sigma^*(\mathcal{G}_\lambda) \\ \varphi_\lambda \downarrow \wr & & \\ \mathcal{LG}_\lambda & \hookleftarrow & \mathcal{G}_\lambda. \end{array}$$

Definition 11. A shtuka $\underline{\mathcal{G}} = (\mathcal{G}, \varphi)$ over S is said to be *bounded* by $\mu \in X_*(T)_+$ if:

1. for all $\lambda \in X^*(T)_+$,
$$\varphi_\lambda(\sigma^*(\mathcal{G}_\lambda)) \subset (z - \zeta)^{-\langle u w_0 \lambda, \mu \rangle} \mathcal{G}_\lambda,$$
and
2. $[\mu] = [\mu_{\mathcal{G}}(S)]$ (i.e. $[\mu]$ is constant).

Note that $w_0\lambda$ is the *lowest weight* of $V(\lambda)$. This should give some hint of why this all works out.

Let us examine what this all means when we return to the original case $G = GL_r$. Then we have the generators $\lambda_i = (0, \ldots, 1, 0, \ldots, 0)$ of the semigroup $X^*(T)_+$ (where λ_i has i 1's and $(r-i)$ 0's). Then, we recover $M = \mathcal{G}_{\lambda_1}$, and more generally we have $\mathcal{G}_{\lambda_i} = \bigwedge^i(M)$. In particular, $\mathcal{G}_{\lambda_r} = \det(M)$. Then, even more generally, if $\mu = (d_1, \ldots, d_r)$ is a (weakly) decreasing sequence of integers, then we get $F_M(\bigwedge^i \sigma^*(M)) \subset (z - \zeta)^{d_{r-i+1} + \cdots + d_r}$; taking $i = 1$ then recovers $F_M(\sigma^*(M)) \subset (z - \zeta)^{d_r}$.

6 Local G-shtukas, 2 – Stephen Kudla

6.1 Recollections

We begin by recalling where we left off. We have G a connective reductive group over $\mathbb{F}_q$; we assume it is split. We have $G \supset B \supset T$, and the (co)characters $X_*(T)_+$ and $X^*(T)_+$. We work in the category $\texttt{Nilp} = \texttt{Nilp}_{\mathbb{F}_q[[\zeta]]}$. We have the functors $K(R) = G(R[[z]])$ and the loop group $LG(R) = G(R[[z]][z^{-1}])$ (for $\mathrm{Spec}(R) \in \texttt{Nilp}$). Then, a (local) G-shtuka over S is a pair $\underline{\mathcal{G}} = (\mathcal{G}, \varphi)$ such that $\mathcal{G}$ is a K-torsor over S and $\varphi :: \sigma^*(\mathcal{LG}) \xrightarrow{\sim} \mathcal{LG}$. The torsor $\mathcal{G}$ corresponds to a Cech cocycle in $\check{H}^1(S, K)$, and then φ takes place in $\check{H}^1(S, LG)$.

Recall that as G no longer has a canonical representation, we deal with all representations at once. If $\lambda \in X^*(T)_+$ is a dominant weight, we have the Weyl module $V(\lambda)$, which is a finite dimensional $\mathbb{F}_q$-vector space with an algebraic representation $\pi(\lambda) : G \to GL(V(\lambda))$.

Now, there is a *loop version*, which we will denote by $V(\lambda)[[z]][z^{-1}]$. Namely, on R this takes the value $V(\lambda) \otimes_{\mathbb{F}_q} R[[z]][z^{-1}]$. There is also a *lattice version*, which takes the value $V(\lambda) \otimes_{\mathbb{F}_q} R[[z]]$. The point here is that the algebraic representation $\pi(\lambda)$ gives rise to an action (i.e. a representation) of either $LG(R)$ or $K(R)$ (respectively) on the corresponding modules.

Let us write $\mathcal{G}_\lambda = \mathcal{G} \times_K (\lambda)[[z]]$ and $\mathcal{LG}_\lambda = \mathcal{LG} \times_{LG} V(\lambda)[[z]][z^{-1}]$; these should be seen as vector bundles given by contracting the torsor via the Borel construction.

6.2 Boundedness, revisited

Recall that we have the notion of a *Hodge polygon*: if k is a field containing $\mathbb{F}_q$ (which we'll sometimes implicitly assume is algebraically closed), then there is the Cartan decomposition $LG(k) = \coprod_\mu K(k) z^\mu K(k)$, where μ runs over the dominant cocharacters. Then we have a trivialization $\mathcal{G} \simeq K_k$, and then the isomorphism $\varphi : \sigma^*(\mathcal{LG}) \xrightarrow{\sim} \mathcal{LG}$ is given via this chosen isomorphism as the isomorphism $\sigma^*(LG_k) \to LG_k$ given by $b\sigma^*$ for some $b \in LG(k)$. Of course, changing the choice of trivialization $\mathcal{G} \simeq K_k$ makes our choice of b only well-defined within the coset $b \in K(k) z^{\mu_0} K(k)$ for some $\mu_0 \in X_*(T)_+$. (The name *polygon* comes from the case of GL_n, where the cocharacters are just maps of the ground ring into the diagonal matrices, which just comes as $\mathbb{Z}$ in each slot (determining the power), so in this case we can choose our elements to be weakly decreasing and then we get the usual polygon picture.)

Now, let's compare the lattices $\varphi(\sigma^*(\mathcal{G}_\lambda))$ and $\mathcal{G}_\lambda$ as we vary λ; this will motivate the definition of boundedness. Now, recall that $\mathcal{G}_\lambda = V(\lambda) \otimes_{\mathbb{F}_q} k[[z]]$. The σ acts only on the second factor, and we're assuming that k is algebraically closed so it acts via isomorphisms. Thus $\sigma^*(\mathcal{G}_\lambda) \cong V(\lambda) \otimes_{\mathbb{F}_q} k[[z]]$. Finally, applying φ gives us $\varphi(\sigma^*(\mathcal{G}_\lambda)) = \pi(\lambda)(b) \cdot (V(\lambda) \otimes_{\mathbb{F}_q} k[[z]]) \subset V(\lambda) \otimes_{\mathbb{F}_q} R[[z]][z^{-1}]$; that is, φ just acts as $\pi(\lambda)(b)$. (Returning to $\mathcal{LG}_\lambda$, we see that this was the whole point of the Borel construction.) Note that we're considering $b \in K(k) z^{\mu_0} K(k)$. So let's apply these term by term. Of course, the $K(k)$ are negligible as far as comaring the lattices is concerned, so we may as well assume that $b = z^{\mu_0}$. That is, it suffices to calculate $\pi(\lambda)(z^{\mu_0}) \cdot (V(\lambda) \otimes_{\mathbb{F}_q} k[[z]])$. But now, recall that we can write the Weyl module $V(\lambda)$ as $V(\lambda) = \bigoplus_{\lambda'} V(\lambda)_{\lambda'}$, where $\lambda' \in X^*(T)$ and $V(\lambda)_{\lambda'}$ denotes the λ'-weight space. This is telling us how the torus is acting. But z^{μ_0} is in the torus, so we can write

$$\pi(\lambda)(z^{\mu_0}) \cdot (V(\lambda) \otimes_{\mathbb{F}_q} k[[z]]) = \bigoplus_{\lambda'} z^{\langle \lambda', \mu_0 \rangle} \cdot V(\lambda)_{\lambda'} \otimes_{\mathbb{F}_q} k[[z]].$$

Now, we'd like to say that this is contained in some $z^e \cdot V(\lambda) \otimes_{\mathbb{F}_q} k[[z]]$ for some $e \in \mathbb{Z}$. But the bound e is exactly $\langle \lambda', \mu_0 \rangle$, so we can therefore take $e = \langle w_0 \lambda, \mu \rangle$ (for w_0 the long element). This motivates the following definition.

Definition 12. A local $\mathcal{G}$-shtuka $\underline{\mathcal{G}}$ over S is said to be *bounded* by $\mu \in X_*(T)_+$, if:

1. The locally constant function $[\mu_{\underline{\mathcal{G}}}(s)]$ on S is constant at $[\mu] \in X_*(T)/\langle \Phi^\vee \rangle = \pi_1(G)$.
2. $\varphi(\sigma^*(\mathcal{G}_\lambda)) \subset z^{-\langle -w_0\lambda, \mu \rangle} \mathcal{G}_\lambda$ for all dominant weights λ.

Example 6. Suppose we have $\underline{\mathcal{G}}/\mathrm{Spec}(k)$ for $k = \overline{k}$, $k \supset \mathbb{F}_q$. This is bounded by μ iff $[\mu] = [\mu_{\underline{\mathcal{G}}}]$ and $\mu_{\mathcal{G}} \leq \mu$ (i.e. μ is obtained from $\mu_{\mathcal{G}}$ by adding positive coroots).

We have the following convenient facts.

1. It suffices to check the second condition for boundedness only for λ in a set of generators of the semigroup $X^*(T)_+$.

2. In the algebraically closed case, it suffices to check the inequality $\mu_{\mathcal{G}} \leq \mu$ (as in the example).

3. Suppose that S is quasicompact and connected. Then a locally constant function is automatically constant. So suppose we are given $\underline{\mathcal{G}}/S$. Then if either$S$ is reduced or G is semisimple, then there is some μ such that $\underline{\mathcal{G}}$ is bounded by μ.

Example 7. However, beware of the following counterexample. Take $G = GL_1$, and suppose we work over the unreduced base $\operatorname{Spec}(R)$ where $R = k[\varepsilon]/(\varepsilon^2)$. Let us take $\mathcal{G} = K_R$ be the trivial torsor. Given this trivialization, we describe φ as $b = 1 + \frac{\varepsilon}{z} \in LG(R)$. (Note that this is a unit, since $b^{-1} = (1 - \frac{\varepsilon}{z})$.)

We claim that this is not bounded by any $\mu \in X^*(T)_+$. Of course, $X^*(T)_+ = X_*(G)$ since the group is abelian and hence there are no roots (as $\pi_1(G) = X_*(T) \cong \mathbb{Z}$). Now, to look at the second condition for boundedness, we compute that $\varphi(\sigma^*(\mathcal{G}_\lambda)) = b^\lambda \cdot R[[z]]$ (since the Frobenius acts as an isomorphism). Now, if p doesn't divide λ, then $b^\lambda \cdot R[[z]] \not\subset R[[z]]$. Now, the inner product is bilinear, and in this case it's just multiplication, so for any λ we're not going to be able to have our inner product be nonegative for all μ.

6.3 The future

From here, we given an outline of where this is all going. Of course, the point of this paper is to define the analog of Rapoport-Zink space in this setting.

Let'study the affine Grassmannian $Gr = LG/K$. This can be considered as a quotient sheaf for the fppf topology, as well as an ind-scheme over $\mathbb{F}_q$. We should think of this as the "space of lattices". If $k \supset \mathbb{F}_q$ and $k = \overline{k}$, then $Gr(k) = G(k[[z]][z^{-1}])/G(k[[z]])$. (In the background, we can compare this with the computation that $GL_r(\mathbb{Q}_p)/GL_r(\mathbb{Z}_p)$ is isomorphic to the space of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^r$.) It turns out that Gr has a nice modular interpretation.

First, to make things especially easy let's assume that S/k is a *variety* (i.e. that it's reduced, connected, irreducible, etc. – this will ensure that the geometry of S doesn't get involved in this first example in any nontrivial way). Suppose $\mathcal{G}/S$ is a K-torsor with a trivialization $\delta : \mathcal{L}\mathcal{G} \xrightarrow{\sim} LG_S$ of the corresponding loop object. We claim that this data is equivalent to a morphism $g : S \to Gr$. Let us give this explicitly. If $s \in S(k)$ is a geometric point, then we can choose a trivialization $\mathcal{G}_s \simeq K_s$ of the fiber over s. But $\mathcal{G}_s \subset (\mathcal{L}\mathcal{G})_s \xrightarrow{\sim} LG_s$, so altogether, the image of $K_s \to LG_s$ is just some translate $g_s K_s$ for some $g_s \in LG(k)$. But our choice of g_s is only well-defined up to its coset, and this precisely gives a well-defined element of $Gr(k)$. Altogether, this collects into a map $g : S \to Gr_k$. When all is said and done, it turns out that Gr_k represents the functor which takes S to pairs $(\underline{\mathcal{G}}, \delta)$.

Now, suppose we have the local G-shtuka $\underline{\mathbb{G}} = (K_k, b\sigma^*)$ over k, for some $b \in G(k)$. Then, we consider the pairs $(\underline{\mathcal{G}}, \delta)$, where $\underline{\mathcal{G}}$ is a local G-shtuka over S and $\delta : \mathcal{L}\mathcal{G} \xrightarrow{\sim} LG_S$ is a trivialization as G-shtukas; that is, the diagram

$$\begin{array}{ccc} \mathcal{L}\mathcal{G} & \xrightarrow{\ \delta\ } & LG_S \\ \varphi \uparrow & & \uparrow b \cdot \sigma^* \\ \sigma^*(\mathcal{L}\mathcal{G}) & \xrightarrow[\sigma^*(\delta)]{} & \sigma^*(LG_S) \end{array}$$

commutes. Observe that δ actually determines φ, since this is a diagram of isomorphisms so we can just take recover φ as $\varphi = \delta^{-1} \circ b \cdot \sigma^* \circ \sigma^*(\delta)$. This tells us that isomorphism classes of pairs $(\underline{\mathcal{G}}, \delta)$ are equivalent to morphisms $g : S \to Gr_k$; given a map g, we fill in φ via the above diagram.

Note that the calculation above shows us that we can calculate the invariant $\mu_{\underline{\mathcal{G}}}(s)$ in terms of g_s, and in fact that we have $g_s^{-1} b \sigma^*(g_s) \in K(k) z^{\mu_{\underline{\mathcal{G}}}(s)} K(k)$. With this in hand, we pass to the "hat" version. Namely, we view $\operatorname{Spf} \mathbb{F}_q[[\zeta]] = \operatorname{colim} \operatorname{Spec}(\mathbb{F}_q[\zeta]/(\zeta^n))$; we hence view `Nilp` as a category of ind-schemes. Then in this category we have the fiber product diagram

$$\begin{array}{ccc} \widehat{Gr} & \longrightarrow & \operatorname{Spf} \mathbb{F}_p[[\zeta]] \\ \downarrow & & \downarrow \\ Gr & \longrightarrow & \operatorname{Spec} \mathbb{F}_q. \end{array}$$

(We could also define $\widehat{Gr}$ as a completion, but we won't concern ourselves with that.) Now that we have these definitions, we can give two important results.

Proposition 6 (H.V.). *The functor* $\texttt{Nilp}^{op} \to \texttt{Sets}$ *which takes S to isomorphism classes of pairs $(\mathcal{G}, \delta)$, where $\mathcal{G}/S$ is a K-torsor and $\delta : \mathcal{L}\mathcal{G} \xrightarrow{\sim} LG_S$ is a trivialization, is (pro)represented by $\widehat{Gr}$.*

Theorem 9 (H.V., Theorem 6.2). *Suppose we have $\mathbb{F}_q \subset k' \subset k$. This gives rise to $Gr_{k'}$ and $\widehat{Gr}_{k'}$. Suppose we fix a model object $\underline{\mathbb{G}} = (K_{k'}, b \cdot \sigma^*)$. Then the functor* $\texttt{Nilp}^{op} \to \texttt{Sets}$ *which takes S to isomorphism classes of pairs $(\underline{\mathcal{G}}, \overline{\delta})$, where $\underline{\mathcal{G}}/S$ is a local G-shtuka and $\overline{\delta} : \underline{\mathcal{G}}_{\overline{S}} \to \underline{\mathbb{G}}_{\overline{S}}$ is a quasi-isogeny (i.e. induces an isomorphism on corresponding loop objects), is (pro)represented by $\widehat{Gr}_{k'}$.*

Remark 9. For G-shtukas, we have rigidity. Since quasi-isogenies of G-shtukas are rigid, then a quasi-isogeny over $\overline{S}$ induces one over all of S. This shows that the second result reduces to the first when we take $k' = k$.

Remark 10. Let us contrast this situation with the unequal characteristic case. Note that here, there is a universal local G-shtuka over $\widehat{Gr}$. Note that this is an ind-scheme which is a colimit of quasicompact schemes $X_n \in \texttt{Nilp}$. This gives us maps $X_n \to \widehat{Gr}$, and of course this gives local G-shtukas over the X_n which fit together appropriately.

Remark 11. If we fix $\mu \in X_*(T)_+$, then the locus where any $\underline{\mathcal{G}}/S$ is bounded by μ is a closed subscheme $S_\mu \subset S$.

Finally, we have a statement of result analogous to that of Rapoport-Zink. We have the following setup. Let $\underline{\mathbb{G}} = (K_{k'}, b \cdot \sigma^*)$ be boudned by some μ, suppose $|k' : \mathbb{F}_q| < \infty$, and suppose b is "decent". Then we have a functor $\texttt{Nilp}^{op} \to \texttt{Sets}$ which sends S to the isomorphism classes of pairs $(\underline{\mathcal{G}}, \overline{\delta})$, where $\underline{\mathcal{G}}$ is a local G-shtuka over S which is bounded by μ and $\overline{\delta} : \underline{\mathcal{G}}_{\overline{S}} \to \underline{\mathbb{G}}_{\overline{S}}$ a quasi-isogeny. Write $\widehat{X}_{\leq\mu}(b)$ define the closed sub-ind-scheme of $\widehat{Gr}_{k'}$ where the universal local G-shtuka is bounded by μ. (This should look like the

Theorem 10 (H.V., Theorem 6.3). *The above functor, which we write as $\mathcal{M}_\mu(b)$, is prorepresented by $\widehat{X}_{\leq\mu}(b)$. This is a formal scheme over* Spf $k'[[\zeta]]$.

Note that $X_{\leq\mu}(b)$ is precisely the affine Deligne-Lusztig variety in $Gr_{k'}$. In the pullback diagram defining $\widehat{Gr}$, one might expect that we just take a further pullback of $\widehat{Gr}$ along the inclusion $X_{\leq\mu}(b) \subset Gr$. But this doesn't take into account the boundedness (by our chosen μ); rather, this condition cuts out the closed sub-ind-scheme $\widehat{X}_{\leq\mu}(b)$ whose reduction is precisely $X_{\leq\mu}(b)$.

This all mimics the Rapoport-Zink story as follows. They instead consider the category $\texttt{Nilp}_{\mathbb{W}}$, and their functor is isomorphism classes of p-divisible groups (possible with extra structure), along with a trivialization of the special fiber.

7 RZ-spaces and affine Deligne-Lusztig varieties – Timo Richarz

7.1 Affine Grassmannians

We begin by recalling some facts. Throughout the talk, G will be a split connective reductive group over $\mathbb{F}_q$, and we will be working in the fpqc-topology. Associated to this, we have the following functors:

- the *loop group* LG is the sheafification of $R \mapsto G(R((z)))$;
- the *positive loop group* L^+G is the sheafification of $R \mapsto G(R[[z]])$.

Lemma 7. *L^+G is representable by a reduced, geometrically connected, formally smooth group scheme over $\mathbb{F}_q$, which is not of finite type (if G is nontrivial).*

Proof. Consider the map to the restriction

$$L^+G \to Res_{(\mathbb{F}_q[z]/z^{i+1})/\mathbb{F}_q}(G \otimes_{\mathbb{F}_q} (\mathbb{F}_q[z]/z^{i+1})).$$

These targets form an inverse system, and it is easy to see that the map from L^+G to the inverse limit is an isomorphism of functors. The rest of the properties follow easily. ☐

In fact, LG is representable by an ind-scheme, but to set this up properly we some more terminology and facts.

Definition 13. The *affine Grassmannian* $G = Gr_G$ is the sheaf quotient $G = LG/L^+G$.

Proposition 7. *G is representable by an ind-projective strict ind-scheme over $\mathbb{F}_q$. (That is, there exists an inductive system of projective schemes over $\mathbb{F}_q$, and the maps are closed inclusions.)*

Proof sketch. Suppose we have $S/\mathbb{F}_q$. Then $G(S)$ is the set of isomorphism classes of pairs $(\mathcal{G}, \delta)$, where $\mathcal{G} \to S$ is an L^+G-torsor and $\delta : \mathcal{L}\mathcal{G} \xrightarrow{\cong} LG_S$. Now, the descent lemma of Beauville-Laszlo implies that the set of isomorphism classes of pairs $(\mathcal{H}, h)$, where $\mathcal{H} \to \mathcal{A}^1_S$ is a G-torsor and $h : \mathcal{H}|_{\mathbb{A}^1_S \backslash \{0\}} \xrightarrow{\cong} G \times_{\mathbb{F}_q} (\mathbb{A}^1_S \backslash \{0\})$, is isomorphic to $G(S)$ via $(\mathcal{H}, h) \mapsto (L^+(\mathcal{H}_{\widehat{\mathcal{O}}_{\mathbb{A}^1_S,\{0\}}}, Lh)$. (Note that $\widehat{\mathcal{O}}_{\mathbb{A}^1_S,\{0\}} \cong \mathcal{O}_S[[z]]$.) ☐

Example 8. Let's take $G = GL_n$. For $N \geq 0$, set

$$X^{(N)}(S) = \{\mathcal{J} \subset \mathcal{O}^n_{\mathbb{A}^1_S}(-N\{0\})/\mathcal{O}^n_{\mathbb{A}^1_S}(N\{0\}) : \mathcal{J} \text{ a coherent } \mathcal{O}_{\mathbb{A}^1_S}\text{-module such that } \mathcal{O}^n_{\mathbb{A}^1_S}(-N\{0\})/\mathcal{J} \text{ is } S\text{-flat}\}.$$

Then we have that $X^{(N)}$ is projective over $\mathbb{F}_q$, and for $N_1 \geq N_2$ we get a closed immersion $X^{(N_1)} \hookrightarrow X^{(N_2)}$. Hence we get $\mathrm{colim}_N X^{(N)} \xrightarrow{\cong} G$.

Example 9. We reduce the situation of an arbitrary reductive group G to the previous case. Choose a faithful representation $G \hookrightarrow GL_n$. This induces an *affine* quotient GL_n/G, and one can see that we get that $G_G \to G_{GL_n}$ is a closed immersion.

Now, fix a maximal torus $T \subset B \subset G$. Then recall that we have the *Cartan stratification*

$$G = \coprod_{\mu \in X_*(T)_+} L^+G \cdot z^\mu \cdot e_0 = \coprod_\mu G^\mu,$$

where $X_*(T)_+$ are the dominant coweights and $e_0 \in G$ is the basepoint. The left L^+G-action factors through one of the groups in the inverse system $\lim G_i \cong L^+G$. This will have G^μ a smooth variety over $\mathbb{F}_q$ of dimension $\langle 2\rho, \mu \rangle$, where $2\rho = \sum_{\alpha \in R^+} \alpha$. We also have the reduced closure of G^μ as

$$\overline{G}^\mu = \coprod_{\lambda \leq \mu} G^\lambda,$$

where the partial order is given by $\lambda \leq \mu$ iff $\mu - \lambda = \sum_{\alpha \in R^\vee_+} n_\alpha \alpha^\vee$ for some $n_\alpha \in \mathbb{Z}_{\geq 0}$.

Example 10. Take $G = GL_n$. Then the $\mathbb{F}_q$-points are given by

$$G(\mathbb{F}_q) = \{\Lambda \subset \mathbb{F}_q((z))^n : \Lambda \text{ an } \mathbb{F}_q[[z]]\text{-lattice}\}.$$

The sheaf quotient LG/L^+G in this situation is even Zariski-locally trivial. So we can stick to $\mathbb{F}_q$ points, instead of considering $\overline{\mathbb{F}}_q$-points. (This is because the first Galois cohomology of the appropriate module vanishes.)

In this situation we have B the upper-triangular matrices and T the diagonal matrices, and hence $X_*(T)_+ = \{\mu \in \mathbb{Z}^n : \mu_1 \geq \ldots \geq \mu_n\}$, with partial ordering given by $\lambda \leq \mu$ iff $\sum_{i=1}^k \lambda_i \leq \sum_{i=1}^k \mu_i$ for $k = 1, \ldots, n-1$ and $\sum_{i=1}^n \lambda_i = \sum_{i=1}^n \mu_i$.

From this, we can deduce that $G^\mu(\mathbb{F}_q) = \{\Lambda \in G(\mathbb{F}_q) : \mathrm{inv}(\Lambda_0, \Lambda) = \mu\}$ for $\Lambda_0 = \mathbb{F}_q[[z]]^n$ the standard lattice, where the *invariant* is given by

$$\mathrm{inv} : G(\mathbb{F}_q) \to GL_n(\mathbb{F}_q[[z]]) \backslash GL_n(\mathbb{F}_q((z)))/GL_n(\mathbb{F}_q[[z]])$$

as follows. Since $\mathbb{F}_q[[z]]$ is a valuation ring, we can put the matrices in the middle into diagonal form, and then we can identify this double-coset space with $(\mathbb{Z}^n)^{\geq}$ by the elementary divisor theorem (with the usual partial order).

In fact, we have the following result.

Theorem 11. *$\overline{G}^{\mu}$ is reduced, normal, Cohen-Macaulay, and moveover $(\overline{G}^{\mu})^{sm} = G^{\mu}$.*

7.2 Local shtukas

We change notation slightly. Let $K = L^{+}G$ over $\mathbb{F}_q$, let $\sigma = Frob_q$ be the q-Frobenius, let $S \in \mathtt{Nilp}_{\mathbb{F}_q[[\zeta]]}$. Recall that a *local G-shtuka* over S is a pair $(\mathcal{G}, \varphi)$, where $\mathcal{G} \to S$ is a K_S-torsor and $\varphi : \sigma^{*}\mathcal{L}\mathcal{G} \xrightarrow{\cong} \mathcal{L}\mathcal{G}$ is an isomorphism of LG_S-torsors.

Definition 14. A *quasi-isogeny* of local G-shtukas $(\mathcal{G}, \varphi) \to (\mathcal{G}', \varphi')$ over the same base S is an isomorphism $f : \mathcal{L}\mathcal{G} \to \mathcal{L}\mathcal{G}'$ which induces a commutative diagram of isomorphisms

$$\begin{array}{ccc} \mathcal{L}\mathcal{G} & \xrightarrow{f} & \mathcal{L}\mathcal{G}' \\ \Big\uparrow{\scriptstyle \varphi} & & \Big\uparrow{\scriptstyle \varphi'} \\ \sigma^{*}\mathcal{L}\mathcal{G} & \xrightarrow[\sigma^{*}(f)]{} & \sigma^{*}\mathcal{L}\mathcal{G}'. \end{array}$$

These are rigid, in the following sense.

Proposition 8. *Let $\underline{\mathcal{G}}$ and $\underline{\mathcal{G}}'$ be local G-shtukas, let $i : \overline{S} \hookrightarrow S$ be defined by a locally nilpotent ideal sheaf $\mathcal{I}$. Then*

$$i^{*} : \mathtt{QIsog}_{S}(\underline{\mathcal{G}}, \underline{\mathcal{G}}') \xrightarrow{\cong} \mathtt{QIsog}_{S'}(i^{*}\underline{\mathcal{G}}, \underline{\mathcal{G}}')$$

is a bijection.

Proof. The proof goes by induction on n in $\mathcal{O}_S/\mathcal{I}^{q^n}$, which reduces to the case $\mathcal{I}^q = (0)$. Then the Frobenius $\sigma : S \to S$ factors as $S \xrightarrow{j} \overline{S} \xrightarrow{i} S$, and hence $\sigma^{*} = j^{*} \circ i^{*}$. This tells us that in the diagram defining a quasi-isogeny f, the bottom row is $j^{*}(i^{*}f)$. So f is uniquely determined by $i^{*}f$, and vice versa. ☐

We introduce the following notation. Let $\widehat{G} = G \times_{\mathbb{F}_q} \mathrm{Spf}(\mathbb{F}_q[[\zeta]]) = ((G \times_{\mathbb{F}_q} \mathrm{Spec}(\mathbb{F}_q[[\zeta]]))/G)^{\wedge}$ (the completion against the zero-section copy of G defined by $\zeta = 0$). This is a formal ind-scheme.

Now, let k be a field containing $\mathbb{F}_q$. Then denote by $\widehat{G}_k$ the restriction of $\widehat{G}$ to the fpqc-site of $\mathtt{Nilp}_{k[[\zeta]]}$. We then define a local G-shtuka $\mathbb{G} = (K_k, b\sigma^{*})$, where $b \in LG(k)$. For any $S \in \mathtt{Nilp}_{\mathbb{F}_q[[\zeta]]}$, write $\overline{S} \hookrightarrow S$ for the closed inclusion defined by $\zeta = 0$.

Theorem 12. *$\widehat{Gr}_k$ pro-represents the functor $(\mathtt{Nilp}_{k[[\zeta]]})^{op} \to \mathtt{Sets}$ which takes S to the set of isomorphism classes of pairs $(\underline{\mathcal{G}}, \overline{\delta})$, where $\underline{\mathcal{G}}$ is a local G-shtuka over S and $\overline{\delta} : \underline{\mathcal{G}} \times_S \overline{S} \to \underline{\mathbb{G}} \times_k \overline{S}$ is a quasi-isogeny. Here, an isomorphism $(\underline{\mathcal{G}}, \overline{\delta}) \cong (\underline{\mathcal{G}}', \overline{\delta}')$ exists precisely when $\overline{\delta}^{-1} \circ \overline{\delta}'$ lifts to an isomorphism $\underline{\mathcal{G}}' \xrightarrow{\cong} \underline{\mathcal{G}}$ (which must respect the corresponding K-torsors).*

In the case of GL_n, $\overline{\delta}^{-1} \circ \overline{\delta}'$ is an isomorphism of vector spaces, and then we're asking for an isomorphism of lattices inside these vector spaces.

Proof. We first claim that $\underline{\mathbb{G}}$ lifts to a shtuka $\widehat{\underline{\mathbb{G}}}$ over $k[[\zeta]]$. This is because we can write $b = s_0 \cdot z^{\mu} \cdot t_0$, where $s_0, t_0 \in K(k)$. Then, let $s, t \in K(k[[\zeta]])$ be any lifts. Then we set $\widehat{\underline{\mathbb{G}}} = (K_{k[[\zeta]]}, \widehat{b} \cdot \sigma^{*})$, where $\widehat{b} = s(z - \zeta)^{\mu}t$. (This is compatible with the notion of *boundedness* that we've addressed earlier.)

Thus, it suffices to show that $\widehat{G}_k$ pro-represents the functor which takes S to the set of isomorphism classes of pairs $(\underline{\mathcal{G}}, \delta)$, where $\underline{\mathcal{G}}$ is a local G-shtuka over S and $\delta : \underline{\mathcal{G}} \to \widehat{\underline{\mathbb{G}}}_S$ a quasi-isomorphism; this follows from the uniqueness of lifts of quasi-isogenies that we saw above.

So, given $(\widetilde{\mathcal{G}}, \widetilde{\delta}) \in \widehat{Gr}_k(S)$ with $\widetilde{\mathcal{G}} \to S$ a K-torsor and $\widetilde{\delta} : \mathcal{L}\widetilde{\mathcal{G}} \xrightarrow{\cong} LG_S$ an isomorphism, we must construct the pair $(\underline{\mathcal{G}}, \delta)$ over S. This goes as follows. In fact, the quasi-isogeny δ is determined by the isomorphism $\widetilde{\delta}$, and this actually determines everything else. Indeed, we have the diagram

$$\begin{array}{ccc} \mathcal{L}\widetilde{\mathcal{G}} & \xrightarrow[\cong]{\widetilde{\delta}} & LG_S \\ \Big\uparrow{\scriptstyle \varphi} & & \Big\uparrow{\scriptstyle \cong\, \widehat{b}} \\ \sigma^*\mathcal{L}\widetilde{\mathcal{G}} & \xrightarrow[\sigma^*(\widetilde{\delta})]{\cong} & \sigma^* LG_S \end{array}$$

uniquely defining φ, and we set $\underline{\mathcal{G}} = (\widetilde{\mathcal{G}}, \varphi)$ and $\delta = \widetilde{\delta} : \underline{\mathcal{G}} \to \underline{\widehat{\mathbb{G}}}_S$. From here, it's not hard to see that this lines up with the indicated notion of isomorphism. ☐

7.3 RZ-spaces for shtukas

For $\lambda \in X^*(T)_+$, let us define the algebraic G-representation

$$V(\lambda) = (\mathrm{Ind}_B^G((-\lambda)_{dom}))^\vee.$$

(The signs arise because we already made some choices, e.g. we defined the affine Grassmannian to be the quotient LG/K on the right, whereas we could've equally well defined it as $K\backslash LG$.) Then, if $\underline{\mathcal{G}} = (\mathcal{G}, \varphi)$ is a local G-shtuka, we can *twist* it via this representation in the following sense. Define

$$\mathcal{G}_\lambda = (\mathcal{G} \times^K L^*GL(V(\lambda))) \times^{L^+GL(V(\lambda))} (V(\lambda) \otimes_{\mathbb{F}_q} \mathcal{O}_S[[z]])$$

(where $\times^K$ denotes pushout; recall that we have $K = L^+G \to L^+GL(V(\lambda)))$. This comes with

$$\varphi_\lambda : \sigma^*\mathcal{L}\mathcal{G}_\lambda = (\sigma^*\mathcal{L}\mathcal{G})_\lambda \xrightarrow{\cong} (\mathcal{L}\mathcal{G})_\lambda = \mathcal{L}\mathcal{G}_\lambda.$$

Now, we recall the following definition.

Definition 15. Let $\mu \in X_*(T)_+$. Then $\underline{\mathcal{G}}$ is *bounded* by μ if:

1. $\varphi_\lambda(\sigma^*\mathcal{G}_\lambda) \subset (z-\zeta)^{-N}\mathcal{G}_\lambda$ where $N = \langle (-\lambda)_{dom}, \mu\rangle$, for all $\lambda \in X^*(T)_+$, inside of $\mathcal{L}\mathcal{G}_\lambda$.

2. The map $[\mu_{\underline{\mathcal{G}}}] : S \xrightarrow{\mu_{\underline{\mathcal{G}}}} X_*(T)_+$ (given by the *Hodge polygon* on geometric points), when composed with $X_*(T)_+ \to \pi_1(G)$, is constant at $[\mu] : S \to \pi_1(G)$.

Remark 12. It is equivalent in the first condition to check only for $\lambda_1, \ldots, \lambda_n \in X^*(T)_+$, where the λ_i generate $X^*(T)_+$ as a monoid.

The key ingredient to proving this is to consider the closed immersion $\rho : G \to GL(\bigoplus_{i=1}^n V(\lambda_i)$, which is in particular a tensor-generator of $\mathrm{Rep}_{\mathbb{F}_q}(G)$.

Remark 13. The condition of being bounded is closed on S.

Remark 14. Let $G_{sc} \xrightarrow{\pi} G_{der} \subset G$ be the simply-connected cover (where $G_{der} = [G, G]$ is the derived group; there's no notion of simply-connected cover unless the center of the group is finite). Let $T_{sc} = \pi^{-1}(T \cap G_{der})^\circ$. Then $\pi_1(G) = X_*(T)/X_*(T_{sc})$ is an equivalent description of the algebraic fundamental group. Then, $\pi_0(Gr) = \pi_0(LG) = \pi_1(G)$. Thus, in some sense the second condition just fixes a connected component of the affine Grassmannian. The point is that there are infinitely many connected components, and we don't want bounded shtukas to be distributed over infinitely many connected components.

We can understand the simplest case of boundedness as follows.

Lemma 8. *Let $\underline{\mathbb{G}} = (K_k, b\sigma^*)$ for some $b \in LG(k)$, with $\mathbb{F}_q \subset k$. Then $\underline{\mathbb{G}}$ is bounded by μ iff $b \in K(k)z^{\mu'}K(k)$ with $\mu' \leq \mu$.*

Proof. Let $b = sz^{\mu'}t$ for some $s, t \in K(k)$. Fix some $\lambda \in X^*(T)_+$. Let λ' be a weight of $V(\lambda)$. Then the weight-character formula tells us that $-(-\lambda)_{dom} \leq \lambda' \leq \lambda$, and moreover that $-(-\lambda)_{dom}$ does indeed occur.

Now, suppose we have some $v_{\lambda'} \in V(\lambda)$ in the weight space of λ'. Then, consider $\varphi_\lambda(t^{-1}v_{\lambda'})$. By definition, φ_λ is given by multiplication by b, so $\varphi_\lambda(t^{-1}v_{\lambda'}) = z^{\langle \lambda', \mu' \rangle} s(v'_\lambda) \in z^{\langle \lambda', \mu' \rangle}\mathcal{G}_\lambda$. This implies that $\varphi_\lambda(\sigma^*\mathcal{G}_\lambda) \subset z^{-\langle (-\lambda)_{dom}, \mu' \rangle}\mathcal{G}_\lambda - z^{-n}\mathcal{G}_\lambda$ for all $n < \langle (-\lambda)_{dom}, \mu' \rangle$.

Hence, if $\underline{\mathbb{G}}$ is bounded by μ then $[\mu] = [\mu']$ in $\pi_1(G)$ and $\langle (-\lambda)_{dom}, \mu - \mu' \rangle \geq 0$ for all $\lambda \in X^*(T)_+$. But the other direction is also true by what we have just seen. Then, the latter condition is in turn equivalent to the condition that $[\mu] = [\mu']$ and $\langle \lambda, \mu - \mu' \rangle \geq 0$ for all $\lambda \in X^*(T)_+$. The second half of this condition is equivalent to saying that $\mu - \mu' = \sum_{\alpha^\vee \in R^\vee_+} n_\alpha \alpha^\vee$ for some $n_\alpha \in \mathbb{R}_{\geq 0}$, but these must in fact be integers. Thus, it is equivalent to demand that $\mu' \leq \mu$. □

We will finish this talk next time.

8 RZ-spaces and affine Deligne-Lusztig varieties, 2 – Timo Richarz

8.1 Recollections

Recall that G will be a split connected reductive group over $\mathbb{F}_q$. We have the fpqc-sheaves $LG : R/\mathbb{F}_q \mapsto G(R((z))$ and $K : R/F_q \mapsto G(R[[t]])$. Then the *affine Grassmannian* $G = Gr = LG/K$ is ind-pro-representable. If we have $S \in \mathtt{Nilp}_{\mathbb{F}_q[[\zeta]]}$ and we denote by $\sigma = \sigma_q$ the q-Frobenius, then a *local G-shtuka* over S is a pair $(\mathcal{G}, \varphi)$ such that $\mathcal{G}/S$ is a K-torsor and $\varphi : \sigma^*\mathcal{LG} \xrightarrow{\cong} \mathcal{LG}$. Recall that a *quasi-isogeny* $(\mathcal{G}, \varphi) \to (\mathcal{G}', \varphi')$ is by definition a commutative diagram of isomorphisms

$$\begin{array}{ccc} \mathcal{LG} & \xrightarrow{\cong} & \mathcal{LG}' \\ {\scriptstyle \varphi}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\uparrow{\scriptstyle \varphi'} \\ \sigma^*\mathcal{LG} & \xrightarrow[\cong]{} & \sigma^*\mathcal{LG}'. \end{array}$$

Let $\mathbb{F}_q \subset k$. Then we can define $\hat{G}_k = G|_{\text{fppf-}\mathtt{Nilp}_{k[[\zeta]]}}$. Let $\underline{\mathbb{G}} = (K_k, b \cdot \sigma^*)$ for some $b \in LG(k)$. For $S \in \mathtt{Nilp}_{k[[\zeta]]}$, let $\overline{S} \hookrightarrow S$ be defined by $\zeta = 0$.

Theorem 13 (HV)**.** *$\hat{G}_k$ pro-represents the functor $(\mathtt{Nilp}_{k[[\zeta]]})^{op} \to \mathtt{Sets}$ given by taking S to the set of isomorphism classes of pairs $(\underline{\mathcal{G}}, \overline{\delta})$, where $\underline{\mathcal{G}}$ is a local G-shtuka over S and $\overline{\delta} : \underline{\mathcal{G}} \times_S \overline{S} \xrightarrow{\text{q-isog.}} \underline{\mathbb{G}} \times_k \overline{S}$, where isomorphism is given by the condition that $\overline{\delta}^{-1} \circ \overline{\delta}'$ lifts to an isomorphism $\underline{\mathcal{G}}' \xrightarrow{\cong} \underline{\mathcal{G}}$.*

Fix $T \subset B \subset G$. For $\lambda \in X^*(X)^+$, define $V(\lambda) = \mathrm{Ind}_B^G((-\lambda)_{dom})^\vee$, for $\underline{\mathcal{G}} = (\mathcal{G}, \varphi)$ a local G-shtuka. Then we write $\mathcal{G}_\lambda = \mathcal{G} \times^K (V(\lambda) \otimes_{\mathbb{F}_q} \mathcal{O}_S[[z]])$, a locally free $\mathcal{O}_S[[z]]s$-module, with $\varphi_\lambda : \sigma^*\mathcal{LG}_\lambda \xrightarrow{\cong} \mathcal{LG}_\lambda$. This gives $\underline{\mathcal{G}}_\lambda = (\mathcal{G}_\lambda, \varphi_\lambda)$.

Recall that $\underline{\mathcal{G}}$ is said to be *bounded* by $\mu \in X_*(T)_+$ if:

1. $\varphi_\lambda(\sigma^*\mathcal{G}_\lambda) \subset (z - \zeta)^{-\langle (-\lambda)_{dom}, \mu \rangle}\mathcal{G}_\lambda$ for all $\lambda \in X^*(T)^+$;

2. $[\mu_{\underline{\mathcal{G}}}] : S \xrightarrow{\mu_{\mathcal{G}}} X_*(T)_+ \twoheadrightarrow \pi_1(G)$ is constant (and denoted $[\mu]$).

We had the following lemma.

Lemma 9. *Let $\underline{\mathbb{G}} = (K_k, b \cdot \sigma^*)$ for some $b \in LG(k)$. Then $\underline{\mathbb{G}}$ is bounded by μ iff $b \in K(k)z^{\mu'}K(k)$ for some $\mu' \leq \mu$.*

8.2 RZ-spaces for shtukas, revisited

Definition 16. Let $\widehat{X}_{\leq \mu}(k)$ be the functor $(\mathtt{Nilp}_{k[[\zeta]]})^{op} \to \mathtt{Sets}$ defined by taking S to the set of isomorphism classes of pairs $(\underline{\mathcal{G}}, \overline{\delta}) \in \widehat{G}_k(S)$ such that $\underline{\mathcal{G}}$ is bounded by μ.

Remark 15. The inclusion $\widehat{X}_{\leq\mu}(b) \to \widehat{G}_k$ is representable by a closed immersion.

Remark 16. There is an identification $\pi_0(Gr) = \pi_0(LG) = \pi_1(G)$, and under ths correspondence we have that $\widehat{X}_{\leq\mu}(b) \subseteq (\widehat{G}_k)^{[\mu]}$, the connected component corresponding to $[\mu] \in \pi_1(G)$.

Theorem 14 (HV)**.** *The ind-scheme $\widehat{X}_{\leq\mu}(b)$ is a formal scheme over* Spf $k[[\zeta]]$ *which is locally formally of finite type, and*

$$\widehat{X}_k(b)(\overline{k}) = \{g \in G(\overline{k}) : g^{-1}b\sigma^*(g) \in K(\overline{k})z^{\mu'}K(\overline{k}) \text{ for some } \mu' \leq \mu\}.$$

That is, the underlying space of the scheme is the same as that of the Deligne-Lusztig variety.

Proof of identification of geometric points. Let $x \in \widehat{G}_k(\overline{k}) = G(\overline{k})$, and let $(\underline{\mathcal{G}}_x, \overline{\delta}_x)$ be the corresponding G-shtuka. Choose a trivialization $(\underline{\mathcal{G}}_x, \overline{\delta}_x) \cong (\underline{\widetilde{\mathbb{G}}}, g_x)$ (i.e. g_x is a lift of x under $LG(\overline{k}) \to G(\overline{k})$). Then $\underline{\widetilde{\mathbb{G}}} = (K_{\overline{k}}, g_x^{-1}b{\cdot}\sigma^*(g_x){\cdot}\sigma^*)$. Hence, $x \in \widehat{X}_{\leq\mu}(b)(\overline{k})$ iff $\underline{\mathcal{G}}_x$ is bounded by μ. By the lemma, this is equivalent to saying that $g_x^{-1}b\sigma^*(g_x) \in K(\overline{k})z^{\mu'}K(\overline{k})$ for some $\mu' \leq \mu$, which is precisely the right side of the claimed equality. ☐

8.3 Hermitian local shtukas

Let $n \geq 1$, $S \in \texttt{Nilp}_{k[[\zeta]]}$. Recall that $G = GL_n$, B is the set of upper triangular matrices, T is the set of diagonal matrices, $\lambda = (1, 0, \ldots, 0)^t$ is the miniscule weight. Then $V(\lambda) = \mathrm{Ind}_{\overline{B}}^{GL_n}((-\lambda)_{dom})^\vee$ is the standard representation.

Now, if $(\mathcal{G}, \varphi)$ is a local GL_n-shtuka, then $(M, \Phi) = (\mathcal{G}_\lambda, \varphi_\lambda)$ is a local shtuka of rank n (in the sense of the first talks), i.e. M is (Zariski-locally on S) a free $\mathcal{O}_S[[z]]$-module of rank n and $\Phi : M^\sigma[\frac{1}{z}] \xrightarrow{\cong} M[\frac{1}{z}]$ is an isomorphism of $\mathcal{O}_S((z))$-modules.

Lemma 10. *1. There is an equivalence of categories*

$$\{\text{local } GL_n\text{-shtukas}/S,\ \text{q-isog.}\} \xrightarrow{\cong} \{\text{local shtukas of rank } n/S,\ \text{q-isog.}\}$$

given by $\underline{\mathcal{G}} \mapsto \underline{M}$.

2. If $\mu = (\mu_1, \ldots, \mu_n)$ with $\mu_1 \geq \ldots \geq \mu_n$ is a dominant coweight, then $\underline{\mathcal{G}}$ is bounded by μ iff $\Phi(\bigwedge^i M^\sigma) \subseteq (z-\zeta)^{\mu_{n-i+1}+\ldots+\mu_n} \bigwedge^i M$ for $i = 1, \ldots, n$ with equality for $i = n$. In this case, $\mathrm{cok}(M^\sigma \xrightarrow{\Phi} (z-\zeta)^{\mu_n}M)$ *is locally of finite rank over S.*

Note that if $\lambda_i = (1, \ldots, 1, 0, \ldots, 0)$ (with 1 appearing i times) is a miniscule weight, then $V(\lambda_i) = \bigwedge^i V(\lambda)$.

Corollary 4. *Let $\underline{\mathcal{G}} \mapsto \underline{M}$ under the above equivalence, let $\mu_r = (1, \ldots, 1, 0, \ldots, 0)$ be considered as a miniscule coweight of GL_n. Then $\underline{M}$ is miniscule (in the sense of the earlier talks) iff $\underline{\mathcal{G}}$ is bounded by μ_r for some $r = 1, \ldots, n-1$. In this case, we have that $(z-\zeta)M \subseteq \Phi(M^\sigma) \subseteq M$, with the former and latter inclusions of rank $n-r$ and r respectively (and where the first one is just saying that $(z-\zeta)$ acts trivially on the Lie algebra).*

Now we specifically choose $k = \mathbb{F}_{q^2}/\mathbb{F}_q$, and let (V, h) be an n-dimensional Hermitian vector space over k. Then by Lang's lemma, the Hermitian form h is split, i.e. there exists a basis $e_1, \ldots, e_n$ of V under which h (i.e. $(h(e_i, e_j))_{ij}$) is given by the antidiagonal matrix with 1's on the antidiagonal and 0's elsewhere. Then, the unitary group $U(V, h)$ is given by $R/\mathbb{F}_q \mapsto \{f \in \mathrm{Aut}_{k \otimes_{\mathbb{F}_q} R}(V \otimes_{\mathbb{F}_q} R) : f^*h_R = h_R\}$. In a basis, we have $U_n = U(V, h)$, where $U_n(R) = \{A \in GL_n(k \otimes_{\mathbb{F}_q} R) : A = (\mathrm{ant})(((A^\sigma)^t)^{-1}(\mathrm{ant})\}$, where (ant) denotes the antidiagonal matrix described above.

Again B consists of the upper-triangular matrices and T consists of the diagonal matrices. This implies that the notions of a local U_n-shtuka and of boundedness by $\mu \in X_*(T)^+$ are given as follows.

Definition 17 (Yun)**.** A *Hermitian local shtuka* of *signature* (r, s) (for $r + s = n$ with $r \leq s$) over $S \in \texttt{Nilp}_{k[[\zeta]]}$ is a triple $(M = M_0 \oplus M_1, \langle -, - \rangle_M, \Phi = (\Phi_{10}, \Phi_{01}))$, where:

- M_i is (Zariski-locally on S) a free $\mathcal{O}_S[[z]]$-module;
- $\Phi_{10} : M_1^\sigma \to M_0$ and $\Phi_{01} M_0^\sigma \to M_1$ are $\mathcal{O}_S[[z]]$-linear such that

$$(z-\zeta)M_0 \subseteq \Phi_{10}(M_1^\sigma) \subseteq M_0$$

are inclusions of rank s and rank r, respectively, and

$$(z-\zeta)M_1 \subseteq \Phi_{01}(M_0^\sigma) \subseteq M_1$$

are inclusions of rank r and rank s, respectively;

- $\langle -,-\rangle_M : M_0 \otimes M_1 \to \mathcal{O}_S[[z]]$ is a perfect pairing, such that $\langle \Phi_{10}x, \Phi_{01}y\rangle = \langle x,y\rangle^\sigma$ for all $x \in M_1$ and $y \in M_0$.

Lemma 11. *Let $\mu_r = (1,\dots,1,0,\dots,0) \in X_*(T)_+$, and let $S \in \mathtt{Nilp}_{k[[\zeta]]}$. Then there is an equivalence of categories*

$$\{\text{local } U_n\text{-shtukas}/S \text{ bounded by } \mu_r,\ \text{q-isog.}\} \xrightarrow{\cong} \{\text{Hermitial local shtukas }/S \text{ of signature } (r,s),\ \text{q-isog.}\}$$

Proof. Let us explicitly choose the isomorphism $k \otimes_{\mathbb{F}_q} k \simeq k \times k$ be given by $a \otimes b \mapsto (ab, a^\sigma b)$. Then we have the restriction $\rho : U_{n,h} = \mathrm{Res}_{k/\mathbb{F}_q}(GL_n)_k^\sigma \hookrightarrow \mathrm{Res}_{k/\mathbb{F}_q}(GL_n)_k \simeq GL_{n,k} \times GL_{n,k}$ given by $A \mapsto (A, A^c)$ (where $A^c = (\mathrm{ant})((A^\sigma)^t)^{-1}(\mathrm{ant})$).

Now, let $(\mathcal{G},\varphi)$ be a local U_n-shtuka over S. Define $(M,\Phi) = (\mathcal{G}_\rho, \varphi_\rho)$ by the same definition given above (note that this applies to any representation, as we have used ρ here). Then M is a Hermitian $k[[z]]\widehat{\otimes}_{\mathbb{F}_q}\mathcal{O}_S$-module which is locally free of rank n, and $\Phi : M^\sigma[\frac{1}{z}] \xrightarrow{\cong} M[\frac{1}{z}]$ is an isomorphism of Hermitian $(k[[z]]\widehat{\otimes}_{\mathbb{F}_q}\mathcal{O}_S)[\frac{1}{z}]$-modules. That is, $\langle \Phi x, \Phi y\rangle = \langle x,y\rangle^\sigma$ for all $x,y \in M[\frac{1}{z}]$. Now, $k[[z]]\widehat{\otimes}_{\mathbb{F}_q}\mathcal{O}_S \cong \mathcal{O}_S[[z]] \times \mathcal{O}_S[[z]]$ under the same identification as at the beginning of the proof. This implies that we have a decomposition $M = M_0 \oplus M_1$ (where $\mathcal{O}_S$ acts normally on the first factor and through the Frobenius on the second factor), and $\Phi : M^\sigma = M_1^\sigma \oplus M_0^\sigma \to M_0 \oplus M_1$ decomposes as the matrix $\Phi = (\Phi_{10}, 0; 0, \Phi_{01})$. Moreover, the Hermitian form induces a perfect pairing $\langle -,-\rangle_M : M_0 \otimes_{\mathcal{O}_S[[z]]} M_1 \to \mathcal{O}_S[[z]]$ such that the indicated equality holds.

Now, assume that $(\mathcal{G},\varphi)$ is bounded by μ_r. Then $\rho : \mu_r \mapsto (\mu_r, (-\mu_r)_{dom})$, where $(-\mu_r)_{dom} = (0,\dots,0,-1,\dots,-1)^t$ (where 0 appears s times). Then the signature condition holds because of the corollary above. ☐

Remark 17. $\Phi_{10}(M_1^\sigma)^\perp = (z-\zeta)\cdot\Phi_{01}(M_0^\sigma)$.

⋆⋆⋆ Lots of G's in this last talk (or maybe the last two talks) might really be supposed to be Gr's. Hopefully this will be obvious from context.