GEPNER: INTRODUCTION TO SPECTRAL ALGEBRAIC GEOMETRY

AARON MAZEL-GEE

ABSTRACT. These lectures will consist of an introduction to the theory of spectral algebraic geometry, with the goal of stating (and, time permitting, sketching a proof of) Lurie's generalization of the Artin representability theorem. We will assume familiarity with basic homotopy theory and higher category theory and begin with an account of structured ∞-topoi. We will then go on to define spectral schemes and Deligne–Mumford stacks as locally ringed ∞-topoi satisfying certain conditions, and discuss how they embed into the big étale topos of the sphere via their functor of points. We will then turn to quasicoherent sheaves, the cotangent complex, and deformation theory, leading up to the statement of the representability theorem. The remaining time (if any) will be devoted to applications.

Contents

1. INTRODUCTION

We'll hope to get to the proof of the following theorem.

Theorem 1 (Artin–Lurie representability theorem)**.** *A functor*

$$X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$$

is represented by a (connective) spectral (Deligne–Mumford) stack if it satisfies some nice conditions.

Importantly, CAlg means $\mathbb{E}_\infty$ ring spectra (not just dga's or scr's) – this is because many things (such as algebraic K-theory, or excitingly topological modular forms) don't live over $\mathbb{Z}$. We'd like a statement for non-connective spectra, but a lot of subtlety arises in the interplay between connective and nonconnective spectra.

Over the course of these lectures, we'll first try to give some indication of what these conditions are (by introducing spectral stacks and observing some properties of their represented functors); if there's time, we'll try to explain the converse.

Recall that there are two approaches to scheme theory. One can define a scheme to be a locally ringed space $(X, \mathcal{O}_X)$ which is locally of the form $\mathrm{Spec}\, R$. But, after Grothendieck, it's a powerful idea to instead consider the subcategory $\mathrm{CAlg}^\heartsuit \hookrightarrow \mathrm{CAlg}$ of discrete commutative rings, and a scheme is classically a particular sort of functor $\mathrm{CAlg}^\heartsuit \to \mathcal{S}\mathrm{et}$. Of course, $\mathrm{CAlg}^\heartsuit \hookrightarrow \mathrm{CAlg}$ and $\mathcal{S}\mathrm{et} \hookrightarrow \mathcal{S}$, and so a ***derived scheme*** will be a sort of functor $\mathrm{CAlg} \to \mathcal{S}$. (Throughout, we'll be careful about the difference between $\mathrm{CAlg}^\heartsuit$, $\mathrm{CAlg}^{\mathrm{cn}}$, and CAlg.)

In fact, it gives a lot more control to use the first technique. On the other hand, there's not going to be an "underlying topological space" of a spectral stack – only an "underlying ringed ∞-topos". So we'll start with those.

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2. RINGED ∞-TOPOI

Whatever an ∞-topos is, the canonical example is sheaves on a topological space (or a site).

Definition 2. Say $\mathcal{X}$ is an ∞***-topos*** if there's a small ∞-category $\mathcal{C}$ such that $\mathcal{X}$ is a(n accessible) left exact localization of $\mathcal{P}(\mathcal{C})$.

Here, the localization functor runs

$$\mathcal{P}(\mathcal{C}) \xleftarrow{L} \mathcal{X};$$

left exactness means that it preserves finite limits. The accessibility condition says that this admits a fully faithful right adjoint $R : \mathcal{X} \to \mathcal{P}(\mathcal{C})$.

This definition is quick to state, but there are other very useful characterizations.

First, suppose that $\mathcal{X}$ is an ∞-category with finite limits. Consider the target map

$$t = \mathrm{ev}_t : \mathrm{Fun}([1], \mathcal{X}) \to \mathcal{X}.$$

The fact that $\mathcal{X}$ admits finite limits implies that t is both a cartesian and a cocartesian fibration. To see this, observe that the fiber over $X \in \mathcal{X}$ is the slice $\mathcal{X}_{/X}$. Given any $Y \to X$, we get our pullback $f^* : \mathcal{X}_{/X} \to \mathcal{X}_{/Y}$, which participates in an adjunction

$$f_! : \mathcal{X}_{/Y} \rightleftarrows \mathcal{X}_{/X} : f^*$$

(where the left adjoint is just postcomposition).

Now, in all these geometric contexts, we also have a further right adjoint f_*. Such settings abstractify as follows.

Definition 3. Let $\mathcal{X}$ be presentable (i.e. leave out "left exact" in the previous definition – or equivalently, for a cardinal κ we have $\mathcal{X} \simeq \mathrm{Ind}_\kappa(\mathcal{X}^0)$ (adjoining κ-filtered colimits) where $\mathcal{X}^0$ has κ-small colimits). We say that $\mathcal{X}$ is ***locally cartesian closed*** if every functor

$$f^* : \mathcal{X}_{/X} \to \mathcal{X}_{/Y}$$

has a right adjoint

$$\mathcal{X}_{/X} \leftarrow \mathcal{X}_{/Y} : f_*.$$

For instance, if $X = \mathrm{pt}_\mathcal{X}$, then $f^* = Y \times -$, and this has adjoint $\underline{\mathrm{hom}}_\mathcal{X}(Y, -)$ (this is the cartesian closure). The "locally" just means that we can do this "over any object X".

We can now re-characterize ∞-topoi.

Proposition 4. *A presentable, locally cartesian closed ∞-category $\mathcal{X}$ is an ∞-topos if the slice functor*

$$\mathcal{X}_{/-} : \mathcal{X}^{op} \to \mathcal{C}\mathrm{at}_\infty$$

(using pullback functoriality, and secretly really jumping into $\widehat{\mathcal{C}\mathrm{at}_\infty}$) is a sheaf, i.e. it preserves limits.

Of course, a limit in $\mathcal{X}^{op}$ is exactly a colimit in $\mathcal{X}$, so what this is saying in practice is that we have *descent*: if we have $X = \operatorname{colim} X_\alpha$, then $\mathcal{X}_{/X} \simeq \lim \mathcal{X}_{/X_\alpha}$. To be clear, this intuition uses the fact that

$$\mathrm{Shv}(\mathcal{X}) = \mathrm{Fun}^{\lim}(\mathcal{X}^{op}, \mathcal{S}) \simeq \mathcal{X}.$$

(Really this is already true when $\mathcal{X}$ is presentable, by the adjoint functor theorem.)

Definition 5. If $\mathcal{C}$ is a presentable ∞-category, we define $\mathcal{C}$***-valued sheaves on*** $\mathcal{X}$ to be

$$\mathrm{Shv}_\mathcal{C}(\mathcal{X}) = \mathrm{Fun}^{\lim}(\mathcal{X}^{op}, \mathcal{C}) \simeq \mathcal{X} \otimes \mathcal{C}$$

Certainly we're interested in $\mathcal{C} = \mathcal{S}\mathrm{p}$. Then, $\mathcal{X} \otimes \mathcal{S}\mathrm{p} = \mathrm{Stab}(\mathcal{X})$. But even more interestingly, we can take $\mathcal{C} = \mathrm{CAlg}$. Then we have

$$\mathcal{X} \otimes \mathrm{CAlg} \simeq \mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{X}) \simeq \mathrm{CAlg}(\mathrm{Shv}_{\mathcal{S}\mathrm{p}}(\mathcal{X})).$$

This is prototypical of what we'll want to consider: an ∞-topos equipped with a sheaf of commutative ring spectra.

Definition 6. A (***geometric***) ***morphism*** of ∞-topoi is a functor

$$f_* : \mathcal{X} \to \mathcal{Y}$$

which is a right adjoint, and whose left adjoint $f^* : \mathcal{Y} \to \mathcal{X}$ is left exact. (We'll write this in terms of its right adjoint, since this is the "geometric" direction (e.g. for sheaves on topological spaces).)

Let's now see what we've gained by passing from topological spaces to ∞-topoi. We've already stated our reason: for a stack, we don't expect its underlying ∞-topos to be sheaves on a space.

But better, there's (almost) an embedding $\mathcal{T}\mathrm{op} \to \mathcal{T}\mathrm{op}_\infty$ (from the 1-category of topological spaces to ∞-topoi) – actually we need to restrict to the *sober* topological spaces (i.e. every irreducible closed subset has a unique closed point – this is satisfied for any scheme or manifold, so it's no real restriction as far as derived algebraic geometry is concerned). This uses a functor

$$\Lambda : \mathcal{T}\mathrm{op} \to \mathrm{Loc}$$

into locales (i.e. a "pointless space"), taking a topological space to its lattice of open sets. This functor Λ is a left adjoint. Then, we have

$$\mathrm{Loc} \xrightarrow{\mathrm{Shv}} \mathcal{T}\mathrm{op}_\infty,$$

whose image is by definition the *0-localic ∞-topoi*, and this also has an adjoint: the *underlying locale* of an ∞-topos $\mathcal{X}$ is the lattice of equivalence classes of subobjects of the terminal object $1_\mathcal{X} \in \mathcal{X}$. (This is a locale since we can take intersections and unions.) So in the end we have a composite of fully faithful embeddings

$$\mathcal{T}\mathrm{op}^{\mathrm{sober}} \hookrightarrow \mathrm{Loc} \hookrightarrow \mathcal{T}\mathrm{op}_\infty.$$

Now, topological spaces are hard: this is why homotopy theory was invented in the first place! So in a way, $\mathcal{T}\mathrm{op}_\infty$ is even worse. But the advantage is that now, *any* type of geometry that you want to do can be defined in this setting.

Definition 7. A ***ringed ∞-topos*** is a pair $(\mathcal{X}, \mathcal{O})$ of an ∞-topos $\mathcal{X}$ equipped with a CAlg-valued sheaf $\mathcal{O} \in \mathrm{CAlg}(\mathrm{Shv}_{\mathcal{S}\mathrm{p}}(\mathcal{X}))$. We write

$$\mathcal{T}\mathrm{op}_\infty^{\mathrm{CAlg}}$$

for the ∞-category of ringed ∞-topoi.

If we only took connective spectra, then these can always be represented by $\mathbb{E}_\infty$ ring objects in $\mathcal{X}$ itself; the possibility of negative homotopy groups prevent this from being the case more generally. On the other hand, we can still view $\mathcal{O}$ as a commutative algebra in Stab$(\mathcal{X})$, as indicated.

Remark 8. As it turns out, there's a classifying ∞-topos *for ringed ∞-topoi*, namely the ∞-topos $\mathrm{Fun}(\mathrm{CAlg}, \mathcal{S})$. (Typically one would take the finitely presented objects of CAlg for set-theoretic reasons, but we'll purposely ignore that here: we actually mean $\mathrm{Fun}(\mathrm{CAlg}, \widehat{\mathcal{S}})$ – so this is actually a presheaf topos in a larger universe.) That is, $\mathcal{O}$ is classified by a geometric morphism

$$\mathcal{X} \xrightarrow{\mathcal{O}} \mathrm{Fun}(\mathrm{CAlg}, \mathcal{S}).$$

In a sense that we'll make precise, this is (close to) the functor represented by $(\mathcal{X}, \mathcal{O})$ itself, i.e. this determines a functor $\mathcal{O} : \mathcal{X}^{op} \to \mathrm{CAlg}$. In fact, they determine each other – the point of this remark is just to observe that CAlg-valued sheaves are classified by an ∞-topos themselves.

We now move towards defining "$\mathrm{Spec}\, R$".

Definition 9. Let $R \in \mathrm{CAlg}$. We say that R is ***local*** if $\pi_0 R$ is local.

In other words, for any $f \in \pi_0 R$, either f or $1 - f$ is a unit (so that there's a unique maximal ideal). The convenience is that this definition generalizes (i.e. globalizes).

Definition 10. If $(\mathcal{X}, \mathcal{O})$ is an ordinary (i.e. non-∞) ringed topos, then $\mathcal{O}$ is ***local*** if it's locally nontrivial (i.e. the element 0 is not a unit) and, writing $i : \mathcal{O}^\times \hookrightarrow \mathcal{O}$ for the inclusion of sheaf of units, the map

$$\mathcal{O}^\times \amalg \mathcal{O}^\times \xrightarrow{(i, 1-i)} \mathcal{O}$$

is an effective epimorphism in $\mathcal{X}$. In other words, this may not be a surjection, but it is after refining any object $X \in \mathcal{X}$ on which we're checking. Then, we say that $\mathcal{X}$ is a ***locally ringed topos*** if $\mathcal{O}$ is local in this sense.

Definition 11. If $\mathcal{O}$ and $\mathcal{O}'$ are *locally ringed* objects in a topos $\mathcal{X}$ (meaning that our map

$$(i, 1 - i) : \mathcal{O}^\times \amalg \mathcal{O}^\times \to \mathcal{O}$$

is an effective epimorphism, and similarly for $\mathcal{O}'$), then a map $\mathcal{O} \to \mathcal{O}'$ is ***local*** if the induced square

$$\begin{array}{ccc} \mathcal{O}^\times & \longrightarrow & (\mathcal{O}')^\times \\ \downarrow & & \downarrow \\ \mathcal{O} & \longrightarrow & \mathcal{O}' \end{array}$$

is a pullback.

The above generalizes readily.

Definition 12. We say that a ringed ∞-topos $(\mathcal{X}, \mathcal{O})$ is a ***locally ringed ∞-topos*** if its "underlying 1-topos"

$$(\mathcal{X}^\heartsuit, \pi_0 \mathcal{O})$$

is a locally ringed 1-topos, where by $\pi_0\mathcal{O}$ we mean the sheafification of the presheaf

$$(\mathcal{X}^\heartsuit)^{op} \hookrightarrow \mathcal{X}^{op} \xrightarrow{\mathcal{O}} \mathrm{CAlg} \xrightarrow{\pi_0} \mathrm{CAlg}(\mathcal{S}\mathrm{et}).$$

A ***morphism*** of locally ringed ∞-topoi $(\mathcal{X}, \mathcal{O}) \to (\mathcal{X}', \mathcal{O}')$ is the data of a geometric morphism

$$f_* : \mathcal{X} \to \mathcal{X}'$$

together with a *local* map

$$\mathcal{O}' \to f_* \mathcal{O}$$

in $\mathrm{CAlg}(\mathrm{Shv}_{\mathcal{S}\mathrm{p}}(\mathcal{X}'))$, where the target object $f_*\mathcal{O}$ is given by the composite

$$(\mathcal{X}')^{op} \xrightarrow{(f^*)^{op}} \mathcal{X}^{op} \xrightarrow{\mathcal{O}} \mathrm{CAlg}.$$

We write

$$\mathcal{T}\mathrm{op}_\infty^{\mathrm{loc}} \subset \mathcal{T}\mathrm{op}_\infty^{\mathrm{CAlg}}$$

for the non-full subcategory of all ringed ∞-topoi consisting of the locally ringed ones and the locally ringed maps between them.

Remark 13. By adjunction, the map on structure sheaves is equivalent to one

$$\mathcal{O}' \to f_* \mathcal{O}$$

in $\mathrm{CAlg}(\mathrm{Shv}_{\mathcal{S}\mathrm{p}}(\mathcal{X}'))$.

Example 14. Let $R \in \mathrm{CAlg}$. We would like to define

$$\operatorname{Spec} R = (\mathrm{Shv}(|\operatorname{Spec} \pi_0 R|), \mathcal{O}).$$

To define $\mathcal{O}$, it suffices to define it on a (sub)basis of the topology on the topological space $|\operatorname{Spec}\pi_0 R|$. We use the standard one: any $f \in \pi_0 R$ determines an open subset

$$U_f = |\operatorname{Spec}(\pi_0 R)[f^{-1}]| \subset |\operatorname{Spec} \pi_0 R|.$$

The magical fact is that we can canonically promote the ordinary ring

$$(\pi_0 R)[f^{-1}] \in \mathrm{CAlg}(\mathcal{S}\mathrm{et})$$

to an $\mathbb{E}_\infty$-ring

$$R[f^{-1}] \in \mathrm{CAlg}.$$

This has to two defining features.

- It corepresents the subspace of $\hom_{\mathrm{CAlg}}(R, A)$ consisting of those maps which take f to an invertible element.
- It can be constructed as a colimit

  $$R[f^{-1}] \simeq \operatorname{colim}\left(R \xrightarrow{f} R \xrightarrow{f} \cdots\right),$$

  which is taken in spectra, but just so happens to carry a canonical $\mathbb{E}_\infty$ structure.

Example 15. Let's consider

$$\mathrm{Sym}_{\mathbb{S}}(\mathbb{S}^{\oplus n}) \simeq \bigoplus_{m\geq 0} \left((\mathbb{S}^{\oplus n})^{\otimes m}\right)/\Sigma_m.$$

This is a free object and hence complicated to construct, but it corepresents a canonical functor, namely

$$\hom_{\mathrm{CAlg}}(\mathrm{Sym}_{\mathbb{S}}(\mathbb{S}^{\oplus n}), R) \simeq \hom_{\mathcal{S}\mathrm{p}}(\mathbb{S}^{\oplus n}, R) \simeq (\Omega^\infty R)^{\times n}.$$

In other words, this corepresents "n elements of the underlying space of R". Thus, we might define

$$\mathbb{A}^n = \mathrm{Spec}(\mathrm{Sym}_{\mathbb{S}}(\mathbb{S}^{\oplus n})).$$

Remark 16. Note that

$$\pi_*(\mathrm{Sym}_{\mathbb{S}}(\mathbb{S}^{\oplus n})) \not\cong \pi_*\mathbb{S} \otimes_{\pi_0\mathbb{S}} \pi_0(\mathrm{Sym}_{\mathbb{S}}(\mathbb{S}^{\oplus n})) \cong (\pi_*\mathbb{S})[x_1, \ldots, x_n].$$

Example 17. We define the group scheme

$$GL_1 = \mathrm{Spec}(\mathbb{S}[\Omega^\infty\mathbb{S}]).$$

Here, $\Omega^\infty\mathbb{S}$ is the derived analog of $\mathbb{Z}$. As $\mathbb{S}[\Omega^\infty\mathbb{S}] \simeq \Sigma^\infty_+(\Omega^\infty\mathbb{S})$, then we have

$$\hom_{\mathrm{CAlg}}(\mathbb{S}[\Omega^\infty\mathbb{S}], R) \simeq \hom_{\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{S})}(\Omega^\infty\mathbb{S}, \Omega^\infty_\otimes R) \simeq (\Omega^\infty R)^\times$$

(where $\Omega^\infty_\otimes R$ has underlying space $\Omega^\infty R$, but its structure as an $\mathbb{E}_\infty$ space comes from the multiplicative structure on R).

Example 18. There's a *different* but related object

$$\mathbb{G}_m = \mathrm{Spec}(\mathbb{S}[\mathbb{Z}]).$$

The point here is that $\mathbb{Z}$ is very much not free as an $\mathbb{E}_\infty$ space; this might be thought of as corepresenting the *strict* units. So there's a map $\mathbb{G}_m \to GL_1$. In fact, there's a whole hierarchy of "versions of the multiplicative group", crafted by taking arbitrary truncations of $\Omega^\infty\mathbb{S}$ instead of $\pi_0\Omega^\infty\mathbb{S} \simeq \mathbb{Z}$.

Example 19. We can define

$$\mathbb{P}^n = (\mathbb{A}^{n+1} - \{0\})/GL_1.$$

But there's also a "flat" version,

$$\mathbb{P}^n_{\mathsf{flat}} = (\mathbb{A}^{n+1}_{\mathsf{flat}} - \{0\})/\mathbb{G}_m$$

(corresponding to a flat affine space analogous to $\mathbb{G}_m$). We explain this terminology presently.

3. Spectral algebra

Definition 20. A ***module*** is a pair (R, M) of $R \in \mathrm{CAlg}$ and $M \in \mathrm{Mod}_R$ – so, an object of

$$\mathrm{Gr}\left(\mathrm{CAlg}^{op} \xrightarrow{\mathrm{Mod}_{(-)}} \mathcal{C}\mathrm{at}_\infty\right).$$

Definition 21. We say that a module is ***flat*** if $(\pi_0 R, \pi_0 M)$ is flat in the ordinary sense and the canonical map

$$\pi_* R \otimes_{\pi_0 R} \pi_0 M \to \pi_* M$$

is an isomorphism. A map $R \xrightarrow{f} R'$ in CAlg is called ***flat*** if f exhibits (R, R') as a flat module.

Remark 22. This is a very strong restriction, but it works well in spectral algebra. For instance, given a map $A \to B$ in CAlg, we can form its *Amitsur complex*

$$B \rightrightarrows B \otimes_A B \Rrightarrow B \otimes_A B \otimes_A B \cdots,$$

i.e. the cosimplicial object $B^{\otimes_A(\bullet+1)}$, whose Spec is exactly the Cech complex. We would like to know when the natural map

$$A \to \lim(B^{\otimes_A(\bullet+1)})$$

is an equivalence. In fact, if $A \to B$ is *faithfully* flat (i.e. additionally $\pi_0 A \to \pi_0 B$ is faithfully flat), it turns out that the corresponding Bousfield–Kan spectral sequence collapses, giving the desired equivalence. In other words, the map $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is a *flat cover*.

Remark 23. Under this notion of flatness, free modules are flat, but their suspensions aren't (because the π_0 condition ceases to be satisfied). In this stricter (i.e. unshifted) sense of "free", the Lazard theorem still holds: the flat modules are exactly the filtered colimits of free modules.

Let us recall the following.

Definition 24. A map $A \to B$ in $\mathrm{CAlg}(\mathcal{S}\mathrm{et})$ is called ***étale*** if

- B is finitely presented over A,
- B is flat as an A-module, and
- the multiplication map
$$B \otimes_A B \xrightarrow{\mu} B$$
identifies B as a summand of $B \otimes_A B$, i.e. if there exists a commutative ring $C \in \mathrm{CAlg}(\mathcal{S}\mathrm{et})_{B\otimes_A B/}$ inducing an isomorphism
$$B \otimes_A B \xrightarrow{\cong} B \times C.$$

Remark 25. Geometrically, we can view this as saying that the diagonal map
$$\operatorname{Spec} B \to \operatorname{Spec} B \otimes_A B = \operatorname{Spec} B \underset{\operatorname{Spec} A}{\times} \operatorname{Spec} B$$
induces an isomorphism
$$\operatorname{Spec} B \amalg \operatorname{Spec} B \xrightarrow{\cong} \operatorname{Spec} B \otimes_A B.$$

Definition 26. A map $A \to B$ in CAlg is ***étale*** if it is flat and the map $\pi_0 A \to \pi_0 B$ in $\mathrm{CAlg}(\mathcal{S}\mathrm{et})$ is étale.

Remark 27. Morally, Rognes's definition of "Galois extension" of commutative ring spectra should always be étale, but in fact this is not quite necessarily the case.

4. Spectral stacks

We have the notion of a locally ringed ∞-topos, but this will not quite be enough for spectral (Deligne–Mumford) stacks: we need the following definition as well.

Definition 28. We say that a locally ringed ∞-topos $(\mathcal{X}, \mathcal{O})$ is ***strictly Henselian*** if $(\mathcal{X}^\heartsuit, \pi_0\mathcal{O})$ is strictly Henselian.

Now, let $(\mathcal{X}, \mathcal{O})$ be a ringed 1-topos, say $\mathcal{O} : \mathcal{X}^{op} \to \mathrm{CAlg}^\heartsuit$. This functor admits a left adjoint
$$\operatorname{Spec}_\mathcal{O} : \mathrm{CAlg}^\heartsuit \to \mathcal{X}^{op},$$
which takes $R \in \mathrm{CAlg}^\heartsuit = \mathrm{CAlg}(\mathcal{S}\mathrm{et})$ to "the affine object of $\mathcal{X}$ associated to R": its functor of points is given by
$$\hom_\mathcal{X}(X, \operatorname{Spec}_\mathcal{O}(R)) \simeq \hom_{\mathrm{CAlg}^\heartsuit}(R, \mathcal{O}(X)).$$
We should think of $\mathcal{O}(X)$ as an *affinization* of the object X, i.e. we should think of this set as
$$\hom(\operatorname{Spec} \mathcal{O}(X), \operatorname{Spec} R).$$
For example, if $R = 0$, then the set $\hom_{\mathrm{CAlg}^\heartsuit}(R, \mathcal{O}(X))$ is empty if and only if $0 = 1$ in $\mathcal{O}(X)$: that is, we have a pullback
$$\begin{array}{ccc} \operatorname{Spec}_\mathcal{O}(0) & \longrightarrow & \mathcal{O}^\times \\ \downarrow & & \downarrow \\ 1_\mathcal{X} & \xrightarrow[0]{} & \mathcal{O}. \end{array}$$
If $\mathcal{O}$ is local, this implies that it's locally nontrivial, so that we get $\operatorname{Spec}_\mathcal{O}(0) = \varnothing$.

Now, it's obvious from this adjunction formula that $\operatorname{Spec}_\mathcal{O} : \mathrm{CAlg}^\heartsuit \to \mathcal{X}^{op}$ preserves colimits (i.e. takes colimits in $\mathrm{CAlg}^\heartsuit$ to limits in $\mathcal{X}$). We also observe that if $R = R_0 \times R_1$, then we get $0 = R_0 \otimes_R R_1$, so that the maps $R \to R_i$ are epimorphisms (in $\mathrm{CAlg}^\heartsuit$). If $\mathcal{O}$ is local, this gives us that the map
$$\operatorname{Spec}_\mathcal{O} R_0 \amalg \operatorname{Spec}_\mathcal{O} R_1 \to \operatorname{Spec}_\mathcal{O} R$$
is a monomorphism. If now $\mathcal{O}$ is local, the idempotents corresponding to the decomposition $R = R_0 \times R_1$ can be combined with the axiom that $(i, 1-i) : \mathcal{O}^\times \amalg \mathcal{O}^\times \to \mathcal{O}^\times$ is an epimorphism to show that the map
$$\operatorname{Spec}_\mathcal{O} R_0 \amalg \operatorname{Spec}_\mathcal{O} R_1 \to \operatorname{Spec}_\mathcal{O} R$$
is also an epimorphism. But maps that are both monomorphisms and epimorphisms are isomorphisms. In conclusion, we've shown the following.

Lemma 29. *If $\mathcal{O}$ is local, then $\operatorname{Spec}_\mathcal{O}$ sends finite products to finite coproducts.*

We can now give the following.

Definition 30. Let $(\mathcal{X}, \mathcal{O})$ be a locally ringed topos. Then $\mathcal{O}$ is ***strictly Henselian*** if for any finite étale cover $R \to \prod_i R_i$ the induced map

$$\amalg_i \operatorname{Spec}_{\mathcal{O}} R_i \to \operatorname{Spec}_{\mathcal{O}} R$$

is an (effective) epimorphism.

Remark 31. In other words, we should think of this map as an "étale cover". We're implicitly appealing to the core philosophy of topos theory: any topos comes with a canonical topology, in which the covers are the effective epimorphisms.

This allows us to formulate the following.

Definition 32. Let $(\mathcal{X}, \mathcal{O})$ be a locally ringed ∞-topos. We say that $\mathcal{O}$ is ***strictly Henselian*** if the underlying 1-topos $(\mathcal{X}^{\heartsuit}, \pi_0\mathcal{O})$ is. We write

$$\mathcal{T}\mathrm{op}_{\infty}^{\mathrm{sh}} \subset \mathcal{T}\mathrm{op}_{\infty}^{\mathrm{loc}}$$

for the full subcategory of those locally ringed ∞-topoi for which the sheaf of rings is locally henselian.

Definition 33. A ***spectral scheme*** is a locally ringed ∞-topos $(\mathcal{X}, \mathcal{O})$ which is *locally on* $\mathcal{X}$ of the form

$$\operatorname{Spec} R = (\operatorname{Shv}(|\operatorname{Spec} \pi_0 R|), \mathcal{O}_{\operatorname{Spec} R}).$$

Remark 34. Let us clarify what "locally on $\mathcal{X}$" means. We consider $\mathcal{X} = \mathcal{X}_{/1_{\mathcal{X}}}$ tautologically, and then we mean relative to a covering

$$\amalg U_\alpha \to 1_{\mathcal{X}}$$

within $\mathcal{X}$, i.e. an *effective epimorphism*.

Let us in turn discuss effective epimorphisms. Generally speaking, the advantage of topoi over sites is that the topos is canonical, whereas sites are not really the "true" thing. In particular, they come equipped with a canonical topology. A map $X \to Y$ in an ∞-topos $\mathcal{X}$ is an ***effective epimorphism*** if the canonical augmentation

$$X^{\times_Y(\bullet+1)} \to Y$$

of its Cech nerve induces an equivalence

$$|X^{\times_Y(\bullet+1)}| \xrightarrow{\sim} Y.$$

For instance, in the ∞-topos $\mathcal{S} \simeq \mathcal{P}(\mathrm{pt}) \simeq \operatorname{Shv}(\mathrm{pt})$, a map $X \to Y$ is an effective epimorphism if and only if the map $\pi_0 X \to \pi_0 Y$ is surjective. For example, any connected space is of the form BG, and the map $\mathrm{pt} \to BG$ is an effective epimorphism: indeed, its Cech nerve is precisely the bar complex $G^{\times\bullet}$, which has $|G^{\times\bullet}| \xrightarrow{\sim} BG$ (practically by definition). Note that here, effective epimorphisms in $\mathcal{S}$ are created in $\mathcal{S}\mathrm{et} = \mathcal{S}^{\heartsuit}$. This is true in general.

Remark 35. One can check that a spectral scheme $(\mathcal{X}, \mathcal{O})$ has that $\mathcal{X}$ is a "spatial" ∞-topos: in fact, $\mathcal{X} \simeq \operatorname{Shv}(|\mathcal{X}|)$, where $|\mathcal{X}|$ is the locale (in fact sober space) of equivalence classes of subobjects of $1_{\mathcal{X}} \in \mathcal{X}$. In essence, this is true because it's true locally: $|\operatorname{Spec} R|$ is sober for any $R \in \mathrm{CAlg}^{\heartsuit}$.

It follows that for $(\mathcal{X}, \mathcal{O})$ a spectral scheme, if we write $\mathcal{X} = \operatorname{Shv}(X)$ then $(X, \pi_0\mathcal{O})$ will be an *ordinary scheme*.

Before defining spectral stacks, we must define the étale spectrum of a commutative ring spectrum. Recall that for the Zariski topos of $R \in \mathrm{CAlg}$ we simply took $\operatorname{Shv}(|\operatorname{Spec} \pi_0 R|)$ and refined the structure sheaf from $\pi_0 R$ to R. Recall that we defined a map $A \to B$ in CAlg to be

- *flat* if it's "strongly flat" (i.e. it's flat on π_0 and $\pi_* B \cong \pi_* A \otimes_{\pi_0 A} \pi_0 B$);
- *étale* if it's flat and $\pi_0 A \to \pi_0 B$ is étale; and
- an *étale cover* if it's étale and $\pi_0 A \to \pi_0 B$ is faithfully flat.

Definition 36. Given $R \in \mathrm{CAlg}$, its ***étale spectrum*** is defined to be

$$\operatorname{Sp\acute{e}t} R = (\operatorname{Shv}_R^{\mathrm{et}}, \mathcal{O}),$$

where

$$\operatorname{Shv}_R^{\mathrm{et}} = \operatorname{coShv}(\mathrm{CAlg}_R^{\mathrm{et}}) \simeq \operatorname{Shv}((\mathrm{CAlg}_R^{\mathrm{et}})^{op})$$

is the (small) étale site of R (i.e. sheaves on affines), and

$$(F : \mathrm{CAlg}_R^{\mathrm{et}} \to \mathcal{S}) \in \operatorname{coShv}_R^{\mathrm{et}}$$

is an étale sheaf if for all étale covers $A \to B$ we have

$$F(A) \xrightarrow{\sim} \lim F(B^{\otimes_A(\bullet+1)}).$$

Remark 37. Here, representables are sheaves: the topology is subcanonical.

Remark 38. If R is discrete and $R \to R'$ is étale, then R' is discrete.

We won't talk about this next result much, but it's the first thing that really makes the machinery of the Artin–Lurie representability theorem run.

Theorem 39 (Goerss–Hopkins). *If R is discrete, then the ∞-category $\mathrm{CAlg}_R^{\mathrm{et}}$ is actually a 1-category, canonically equivalent to that of étale $\pi_0 R$-algebras.*

We have the following (expected) universal property of $\mathrm{Spét}\, R$.

Proposition 40. *Let $(\mathcal{X}, \mathcal{O})$ be a strictly Henselian ringed ∞-topos. Then the global sections functor*

$$\Gamma : (\mathcal{X}, \mathcal{O}) \to \mathrm{CAlg}$$

(given by $X \mapsto \Gamma(X; \mathcal{O}) = \mathcal{O}(X)$) determines an equivalence

$$\hom_{\mathcal{T}\mathrm{op}_\infty^{\mathrm{sh}}}((\mathcal{X}, \mathcal{O}), \mathrm{Spét}\, R) \xrightarrow{\sim} \hom_{\mathrm{CAlg}}(R, \Gamma(\mathcal{X}, \mathcal{O})),$$

where we write $\Gamma(\mathcal{X}) = \Gamma(1_{\mathcal{X}})$.

In other words, we have an adjunction

$$\Gamma : \mathcal{T}\mathrm{op}_\infty^{\mathrm{sh}} \rightleftarrows \mathrm{CAlg}^{op} : \mathrm{Spét},$$

and what we've given above is a *construction* of this right adjoint.

Definition 41. A strictly Henselian ringed ∞-topos $(\mathcal{X}, \mathcal{O})$ is a ***spectral (Deligne–Mumford) stack*** if $(\mathcal{X}, \mathcal{O})$ is locally (on $\mathcal{X}$) equivalent (in $\mathcal{T}\mathrm{op}_\infty^{\mathrm{sh}}$) to $\mathrm{Spét}\, R$ for $R \in \mathrm{CAlg}$. We say that $(\mathcal{X}, \mathcal{O})$ is ***connective*** if all these R are connective. We write

$$\mathsf{Stk} \subset \mathcal{T}\mathrm{op}_\infty^{\mathrm{sh}}$$

for the full subcategory on the spectral stacks.

Remark 42. An ordinary Deligne–Mumford stack can be defined in exactly the same way, only replacing ∞ with 1 everywhere.

Remark 43. There's a difference in this story between covers and hypercovers. We haven't discussed the latter because we haven't needed it yet; in general, objects *won't* satisfy hyperdescent.

Remark 44. Now, any spectral stack (or scheme) $(\mathcal{X}, \mathcal{O})$ represents a ***moduli functor***

$$\mathscr{M}_{(\mathcal{X},\mathcal{O})} : \mathrm{CAlg} \to \mathcal{S},$$

given by

$$\mathscr{M}_{(\mathcal{X},\mathcal{O})}(R) = \hom_{\mathsf{Stk}}(\mathrm{Spét}\, R, (\mathcal{X}, \mathcal{O})).$$

As we have an equivalence

$$\hom_{\mathsf{Stk}}(\mathrm{Spét}\, A, \mathrm{Spét}\, B) \simeq \hom_{\mathrm{CAlg}}(B, A) \simeq \hom_{\mathsf{Sch}}(\mathrm{Spec}\, A, \mathrm{Spec}\, B),$$

it follows that we get a fully faithful functor

$$\mathscr{M} : \mathsf{Stk} \to \mathrm{Fun}(\mathrm{CAlg}, \mathcal{S}).$$

Now, a ***representability theorem*** is a characterization of an object of $\mathrm{Fun}(\mathrm{CAlg}, \mathcal{S})$ as being in the image of the functor $\mathscr{M}$. However, it's better is to restrict as

$$\mathscr{M} : \mathsf{Stk}^{\mathrm{cn}} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S}).$$

As we'll see, this is because the cotangent complex is only well-behaved on connective objects: its negative homotopy measures the stackiness of an object, so if we allow ourselves all commutative algebras then it'll be hard to tell "where" the nonconnectivity is coming from – i.e. whether it's coming from the nonconnective rings or from actual stackiness.

Remark 45. Note that the sections of tmf are *non*connective. Thus, its construction using this representability theorem goes through two stages: one (i.e. Lurie) constructs a connective object that localizes to give the desired spectral stack (i.e. by inverting an element of its structure sheaf).

5. DEFORMATION THEORY

Let $(\mathcal{X}, \mathcal{O})$ be a ringed ∞-topos. Then $\mathcal{O} \in \mathrm{CAlg}(\mathrm{Shv}_{\mathcal{S}\mathrm{p}}(\mathcal{X}))$, so we can consider

$$\mathrm{Mod}_{\mathcal{O}} = \mathrm{Mod}_{\mathcal{O}}(\mathrm{Shv}_{\mathcal{S}\mathrm{p}}(\mathcal{X})).$$

This is equipped with a map to $\mathcal{X}$, whose restriction to $\mathcal{X}_{/X}$ will be written as

$$\mathrm{Mod}_{\mathcal{O}_X} = \mathrm{Mod}_{\mathcal{O}}|_{\mathcal{X}_{/X}}.$$

We'll also write

$$\mathcal{T}\mathrm{op}_\infty^{\mathrm{Mod}} \to \mathcal{T}\mathrm{op}_\infty^{\mathrm{CAlg}}$$

for "moduled" ∞-topoi, equipped with its forgetful map to ringed ∞-topoi. So the fiber over $(\mathcal{X}, \mathcal{O})$ is the ∞-category of triples of the form $(\mathcal{X}, \mathcal{O}, \mathcal{M})$.

Now, let $(R, M) \in \mathrm{Mod}$ be a module, and define its étale spectrum

$$\mathrm{Spét}(R, M) \in \mathcal{T}\mathrm{op}_\infty^{\mathrm{Mod}}$$

via the right adjoint to the functor

$$\Gamma : \mathcal{T}\mathrm{op}_\infty^{\mathrm{Mod,sh}} = \mathcal{T}\mathrm{op}_\infty^{\mathrm{Mod}}|_{\mathcal{T}\mathrm{op}_\infty^{\mathrm{sh}}} \to \mathrm{Mod}^{op}$$

(where the source is the ∞-category of ***moduled, strictly Henselian*** ringed ∞-topoi) given by

$$(\mathcal{X}, \mathcal{O}, \mathcal{M}) \mapsto (\Gamma(\mathcal{X}, \mathcal{O}), \Gamma(\mathcal{X}, \mathcal{M})).$$

In other words, for any $(\mathcal{X}, \mathcal{O}, \mathcal{M})$ we have an equivalence

$$\hom_{\mathcal{T}\mathrm{op}_\infty^{\mathrm{Mod,sh}}}((\mathcal{X}, \mathcal{O}, \mathcal{M}), \mathrm{Spét}(R, M)) \simeq \hom_{\mathrm{Mod}}((R, M), \Gamma(\mathcal{X}, \mathcal{O}, \mathcal{M})).$$

Definition 46. A ***quasicoherent sheaf*** over a spectral stack $(\mathcal{X}, \mathcal{O}) \in \mathsf{Stk}$ is a triple $(\mathcal{X}, \mathcal{O}, \mathcal{F})$ which is locally of the form $\mathrm{Spét}(R, M)$. These assemble into an ∞-category $\mathsf{QC}_{(\mathcal{X}, \mathcal{O})}$.

Remark 47. If in fact $(\mathcal{X}, \mathcal{O}) = \mathrm{Spét}\, R$ is affine, then we have

$$\mathsf{QC}_{(\mathcal{X}, \mathcal{O})} \simeq \mathrm{Mod}_R \hookrightarrow \mathrm{Mod}_{\mathcal{O}}.$$

In other words, an arbitrary $\mathcal{O}$-module has no reason to be quasicoherent – as usual.

Indeed, if $(\mathcal{X}, \mathcal{O}, \mathcal{F}) \simeq \mathrm{Spét}(R, M)$, then for any affine object

$$U \in \mathcal{X} = \mathrm{Shv}_R^{\mathrm{et}} \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{et}}, \mathcal{S})$$

the canonical map

$$M \otimes_R \mathcal{O}(U) \to \mathcal{F}(U)$$

is an equivalence. This is, of course, precisely the restriction on $\mathcal{O}$-modules which makes them quasicoherent.

Remark 48. The subcategory $\mathsf{QC}_{\mathcal{O}} \subset \mathrm{Mod}_{\mathcal{O}}$ is the smallest coreflective subcategory (i.e. its inclusion admits a right adjoint) containing the symmetric monoidal unit $\mathcal{O}$. In particular, colimits of quasicoherents can be computed in $\mathrm{Mod}_{\mathcal{O}}$. On the other hand, only finite limits in $\mathsf{QC}_{\mathcal{O}}$ can be computed in $\mathrm{Mod}_{\mathcal{O}}$. (These are both stable, and all functors in sight are exact.)

Theorem 49 (Quillen, André, Basterra–Mandell)**.** *Let* $R \in \mathrm{CAlg}$. *Then*

$$\mathsf{Stab}(\mathrm{CAlg}_{/R}) = \mathcal{S}\mathrm{p}(\mathrm{CAlg}_{R//R}) \simeq \mathrm{Mod}_R,$$

via the functors

$$\{(A_n \xrightarrow{f_n} R)\}_{n \geq 0} \mapsto \mathrm{fib}(f_0)$$

and

$$R \oplus M \mapsfrom M$$

(the trivial square-zero extension of R *by* M*).*

The right adjoint is actually hard to write down in general – it's hard to write down an $\mathbb{E}_\infty$ structure on $E \oplus M$.

Remark 50. The "square-zero" condition really is just the assertion that the composite

$$M^{\otimes 2} \to (R \oplus M)^{\otimes 2} \xrightarrow{\mu} R \oplus M$$

is the zero map.

Corollary 51. *If* $X : \mathrm{CAlg}^{(\mathrm{cn})} \to \mathcal{S}$, *then*

$$\mathcal{S}\mathrm{p}(\mathrm{Fun}(\mathrm{CAlg}^{(\mathrm{cn})}, \mathcal{S})_{/X}) \simeq \mathrm{QC}_X = \lim_{x : \operatorname{Spec} R \to X} \mathrm{Mod}_R.$$

Moreover, this is a limit of symmetric monoidal functors among symmetric monoidal ∞*-categories, so gives a symmetric monoidal structure on* QC_X.

We now use these square-zero extensions to define the cotangent complex.

Recall that in ordinary algebra, if $R \in \mathrm{CAlg}^{\heartsuit}$ and $A \in \mathrm{CAlg}^{\heartsuit}_R$, we can form the module of *Kähler differentials* $\Omega_{A/R} \in \mathrm{Mod}_R$, which receives a universal R-linear derivation from A: that is, $d(xy) = xd(y) + yd(x)$ and $d(r) = 0$. Recall too that such a derivation is equivalent to a section

$$\begin{array}{ccc} & & R \oplus M \\ & \nearrow & \downarrow \\ A & \longrightarrow & R \end{array}$$

of the projection map.

Things work identically in spectral algebraic geometry. Given a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, a module (R, M) with $R \in \mathrm{CAlg}^{\mathrm{cn}}$, and an element $x \in X(R) = \hom(\operatorname{Spec} R, X)$, let us consider the fiber

$$\begin{array}{ccc} \mathrm{fib} & \longrightarrow & X(R \oplus M) \\ \downarrow & & \downarrow \\ \{x\} & \hookrightarrow & X(R). \end{array}$$

For instance, if $X = \operatorname{Spec} R$ and $x = \mathrm{id}$, this is giving the space of sections of the projections $R \oplus M \to R$: in other words, we have a fiber sequence

$$\mathrm{Der}(R, M) \to \hom(R, R \oplus M) \to \hom(R, R).$$

Considering M as a variable, we obtain a functor

$$\mathrm{Der}(R, -) : \mathrm{Mod}_R \to \mathcal{S},$$

and we're interested in the corepresentability of this functor. Here, we might like to restrict to *connective* modules – in fact, it suffices to restrict to the *almost connective* modules, i.e. those which are n-connective for some $n \in \mathbb{Z}$. If this is corepresentable, we define the corepresenting object to be the (***absolute***) ***cotangent complex*** of R, denoted

$$\mathbb{L}_R = \mathbb{L}_{R,\mathrm{id}} \in \mathrm{Mod}_R^{a\mathrm{cn}}.$$

In fact, we'll *globalize* this, to ask the question for all connective modules (R, M) and all points $x : \operatorname{Spec} R \to X$. We might hope for the corepresentability

$$\hom_{\mathrm{Mod}_R}(\mathbb{L}_{X,x}, M) \to X(R \oplus M) \to X(M)$$

for some $\mathbb{L}_{X,x} \in \mathrm{Mod}_R^{a\mathrm{cn}}$. By definition of QC_X (as a limit), this will then piece together to an object $\mathbb{L}_X \in \mathrm{QC}_X$, which will be *the* cotangent complex of X.

Example 52. If $R = \mathrm{Sym}_{\mathbb{S}} M$ for some connective $\mathbb{S}$-module M, then $\operatorname{Spec} R : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ admits a cotangent complex $\mathbb{L}_{\operatorname{Spec} R} \simeq M \otimes R$. This is a basic exercise in manipulating adjunctions, using the freeness of R.

More generally, any connective affine scheme admits a resolution by free ones, and it is possible to construct the cotangent complex for arbitrary connective affines in this way. But we'll take a different route, using nontrivial square-zero extensions.

Let (R, M) be a connective module, and consider a map

$$d : R \to \Sigma M$$

which happens to be classified by a map $\mathbb{L}_R \to M$. Then,

$$R' = \mathrm{fib}(d) \simeq \lim \left(\begin{array}{ccc} & & R \\ & & \downarrow{\scriptstyle (\mathrm{id}, d)} \\ R & \xrightarrow[(\mathrm{id}, 0)]{} & R \oplus \Sigma M \end{array} \right).$$

This gives R' the structure of a connective $\mathbb{E}_\infty$-ring (with the guarantee of connectivity coming from the fact that we suspended M). Let's write this as R^d: the canonical map

$$R^d \to R$$

is a *not necessarily trivial* square-zero extension of R by M.

We'll take this as a definition of a not-necessarily-trivial square-zero extension. On the other hand, it's not a priori so clear how unique the data of M and $d : R \to \Sigma M$ are.

This leads us to the following result, which is also a major piece of input into our story. Let's consider the functor

$$\mathrm{Der} \to \mathrm{Fun}([1], \mathrm{CAlg})$$

which sends a derivation $(d : R \to \Sigma M)$ (into the suspension of an R-module) to the map $R^d \to R$. Write $\mathrm{Der}^+ \subset \mathrm{Der}$ for the subcategory on those objects for which M is connective, whose morphisms are *cocartesian*, i.e. they must be *pushout* squares

$$\begin{array}{ccc} R & \xrightarrow{d} & \Sigma M \\ \downarrow & & \downarrow \\ R' & \longrightarrow & \Sigma M \otimes_R R'. \end{array}$$

Let's also write $\mathrm{Fun}^+([1], \mathrm{CAlg}) \subset \mathrm{Fun}([1], \mathrm{CAlg})$ for the subcategory whose maps are likewise pushouts.

Theorem 53. *The functor*

$$\mathrm{Der}^+ \to \mathrm{Fun}^+([1], \mathrm{CAlg})$$

is a left fibration.

In other words, there might be several derivations that give rise to the same square-zero extension, but all of their endomorphisms are automorphisms.

This implies that we can classify commutative algebras *over a square-zero extension* in the following way.

Corollary 54. *Let $d : R \to \Sigma M$ be a derivation. Then*

$$(\mathrm{Der}^+)_{d/} \xrightarrow{\Phi} \mathrm{CAlg}^{\mathrm{cn}}_{R^d}$$

is an equivalence.

We now make a few comments about the cotangent complex.

Definition 55. Given $f : A \to B$, if we had considered $\operatorname{Spec} B \in \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_A, \mathcal{S})$, we would obtain the ***relative cotangent complex***, denoted $\mathbb{L}_{B/A}$.

Remark 56. This sits in a cofiber sequence

$$\mathbb{L}_A \otimes_A B \to \mathbb{L}_B \to \mathbb{L}_{B/A},$$

and so one could also define $\mathbb{L}_{B/A} = \mathbb{L}_B/(\mathbb{L}_A \otimes_A B)$.

Proposition 57. *If $f : A \to B$ in $\mathrm{CAlg}^{\mathrm{cn}}$ has that* cofib(f) *(taken in Mod_A) is n-connective, then*

$$\mathrm{cofib}(f) \otimes_A B \to \mathbb{L}_{B/A}$$

is $2n$-connective.

Corollary 58. *If $f : A \to B$ in $\mathrm{CAlg}^{\mathrm{cn}}$ has that* cofib(f) *is n-connective, then $\mathbb{L}_{B/A}$ is n-connective. Moreover, the converse holds if $\pi_0 A \to \pi_0 B$ is an isomorphism.*

Corollary 59. *If $f : A \to B$ in $\mathrm{CAlg}^{\mathrm{cn}}$ has that* cofib(f) *is n-connective, then f is an equivalence if and only if $\mathbb{L}_{B/A} \simeq 0$ and $\pi_0 A \to \pi_0 B$ is an isomorphism.*

This follows from the converse statement in the previous result, by taking $n \to \infty$.

Proposition 60. *Let $f : A \to B$ in $\mathrm{CAlg}^{\mathrm{cn}}$. Then, if B is locally (resp. almost) of finite presentation over A, then $\mathbb{L}_{B/A}$ is (resp. almost) perfect. The converse holds provided that $\pi_0 B$ is of finite presentation over $\pi_0 A$.*

6. The representability theorem

Question 61. Given a functor
$$X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S},$$
when is there a (connective) spectral Deligne–Mumford stack $(\mathcal{X}, \mathcal{O}) \in \mathsf{Stk}^{\mathrm{cn}}$ such that $X \simeq \hom(-, (\mathcal{X}, \mathcal{O}))$, i.e. such that
$$X(R) \simeq \hom(\mathrm{Sp\acute{e}t}\, R, (\mathcal{X}, \mathcal{O}))?$$

Remark 62. If X is represented by $(\mathcal{X}, \mathcal{O})$, then its restriction $\mathcal{X}|_{\mathrm{CAlg}^{\heartsuit}}$ is represented by $(\mathcal{X}, \tau_{\leq 0}\mathcal{O}) \simeq (\mathcal{X}, \pi_0\mathcal{O})$. So this can be used to prove representability theorems in the classical setting.

To see this, consider the adjunction
$$\pi_0 : \mathrm{CAlg}^{\mathrm{cn}} \rightleftarrows \mathrm{CAlg}^{\heartsuit} : H.$$
This induces an adjunction
$$(-)^{\mathrm{der}} : \mathsf{Sch}^{\heartsuit} \rightleftarrows \mathsf{Sch}^{\heartsuit} : (-)^{\heartsuit}.$$
(This is clear affinely, but it extends non-affinely too.) In fact, étale-locally, the same is true for stacks:
$$(-)^{\mathrm{der}} : \mathsf{Stk}^{\heartsuit} \rightleftarrows \mathsf{Stk}^{\mathrm{cn}} : (-)^{\heartsuit}.$$
So if X is represented by $(\mathcal{X}, \mathcal{O})$ and $R \in \mathrm{CAlg}^{\heartsuit}$, then we have
$$\begin{aligned} X(R) &\simeq \hom(\mathrm{Sp\acute{e}t}^{\delta}(R)^{\mathrm{der}}, (\mathcal{X}, \mathcal{O})) \\ &\simeq \hom(\mathrm{Sp\acute{e}t}\, HR, (\mathcal{X}, \tau_{\leq 0}\mathcal{O})). \end{aligned}$$
This uses connectiveness is an essential way, in the adjunction $\pi_0 \dashv H$.

Remark 63. If X is represented, then it admits a cotangent complex
$$\mathsf{QC}_X \ni \mathbb{L}_X \simeq \mathbb{L}_{(\mathcal{X}, \mathcal{O})} \in \mathsf{QC}_{\mathcal{O}}.$$
Of course, $\mathbb{L}_X$ controls the deformation theory of X. Recall that square-zero extensions of $\mathbb{E}_\infty$ ring spectra are certain pullback diagrams, in which one arrow corresponds to a derivation. So, the "deformation theory" of X can be expressed in terms of the following limit-preservation conditions on X.

(1) If X is represented, then it is ***cohesive***. To explain this, suppose that
$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$
is a pullback square in $\mathrm{CAlg}^{\mathrm{cn}}$ with both maps surjective on π_0 (i.e. "nil-thickenings", so giving closed immersions on Spét). Then, the opposite diagram
$$\begin{array}{ccc} \mathrm{Sp\acute{e}t}\, A' & \longleftarrow & \mathrm{Sp\acute{e}t}\, A \\ \uparrow & & \uparrow \\ \mathrm{Sp\acute{e}t}\, B' & \longleftarrow & \mathrm{Sp\acute{e}t}\, B \end{array}$$
is a pushout square of stacks. So applying $X(-)$ must give a pullback
$$\begin{array}{ccc} X(A') & \longrightarrow & X(A) \\ \downarrow & & \downarrow \\ X(B') & \longrightarrow & X(B) \end{array}$$
must be a pullback in spaces. In fact, there's a strictly weaker condition, called ***infinitesimally cohesive***: this only refers to surjections on π_0 with nilpotent ideal, and these present "nil-immersions".

(2) If X is represented, then it is ***nil-complete***. Roughly, this means that it preserves limits of Postnikov towers. More precisely, given the Postnikov tower $R \simeq \lim_n \tau_{\leq n} R$, representability implies that we have an equivalence
$$X(R) \xrightarrow{\sim} \lim_n X(\tau_{\leq n} R).$$
(This is because $\mathrm{Sp\acute{e}t}\, R \simeq \mathrm{colim}_n\, \mathrm{Sp\acute{e}t}\, \tau_{\leq n} R$.)

Theorem 64 (Lurie)**.** *A functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is represented by a connective spectral stack if and only if the following conditions are satisfied.*

(1) There exists a connective spectral stack representing a functor $Y_0 : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ and an equivalence $X|_{\mathrm{CAlg}^\heartsuit} \simeq Y_0|_{\mathrm{CAlg}^\heartsuit}$.
(2) X admits a cotangent complex $\mathbb{L}_X$.
(3) X is nil-complete.
(4) X is infinitesimally cohesive.

Remark 65. There are stronger versions of this theorem. In particular, the first hypothesis will imply something about $X|_{\mathrm{CAlg}^\heartsuit}$, and ask that that be representable. In those stronger versions, one can do away with that hypothesis. On the other hand, in examples of interest, one generally already knows this to be satisfied.

Further, the ordinary Artin representability theorem can be applied to check this hypothesis, in an analogous way. In fact, the stronger versions of this theorem basically just incorporate this.

Remark 66. There's some precedent to this result, in Toën–Vezzozi's HAG-II.

The key ingredient in the proof is the fact that if we have a map $Y \to X$ which is "close to being étale", it can be approximated by a composite $Y \to Y' \to X$ such that $Y' \to X$ is étale and $Y \to Y'$ is "close (enough) to being an equivalence". This is formally encoded by the following.

Lemma 67. *Let Y_0 be represented by $(\mathcal{Y}, \mathcal{O}_0)$, and suppose we are given $f_0 : Y_0 \to X$ such that $\pi_n \mathbb{L}_{Y_0/X} = 0$ for $n \leq 1$. Suppose moreover that X is an étale sheaf. Then f_0 factors as a composite $Y_0 \xrightarrow{g} Y \xrightarrow{f} X$ where*

- *the map g is induced by a 1-connective map $\mathcal{O} \to \mathcal{O}_0$ (i.e. Y is represented by some $(\mathcal{Y}, \mathcal{O})$), and*
- *$\mathbb{L}_{Y/X} = 0$ (so that f is formally étale).*

Remark 68. Recall that $\pi_0 \mathbb{L}_{Y_0/X}$ will be the ordinary Kähler differentials of the stack, so the assertion that it vanishes is the assertion that it's unramified.

We also need the following result.

Lemma 69. *If X admits a cotangent complex $\mathbb{L}_X$, is infinitesimally cohesive, and is nil-complete, then X is an étale sheaf if and only if $X|_{\mathrm{CAlg}^\heartsuit}$ is an étale sheaf.*

Proof of Theorem 64. We have that Y_0 is represented by $(\mathcal{Y}, \mathcal{O}_0)$ and $X|_{\mathrm{CAlg}^\heartsuit} \simeq Y_0|_{\mathrm{CAlg}^\heartsuit}$. By an adjunction argument, we can replace $\mathcal{O}_0$ with $\tau_{\leq 0}\mathcal{O}_0$, so that we may suppose that $\mathcal{O}_0$ is discrete.

Now, we need to get an actual map $Y_0 \to X$ (i.e. beyond $\mathrm{CAlg}^\heartsuit \subset \mathrm{CAlg}$). To do this, define the left Kan extension

$$\begin{array}{ccc} \mathrm{CAlg}^\heartsuit & \xrightarrow{Y_0|_{\mathrm{CAlg}^\heartsuit}} & \mathcal{S}. \\ \downarrow & \nearrow_{Y_0'} & \\ \mathrm{CAlg}^{\mathrm{cn}} & & \end{array}$$

Hence, our equivalence $Y_0|_{\mathrm{CAlg}^\heartsuit} \xrightarrow{\sim} X|_{\mathrm{CAlg}^\heartsuit}$ induces a map $Y_0' \to X$. Moreover, as X is an étale sheaf, this factor through the étale sheafification to give a map

$$\begin{array}{ccc} Y_0' & \longrightarrow & X. \\ & \searrow \quad \nearrow_{f_0} & \\ & Y_0 & \end{array}$$

Now, we want to show that $f_0^* \mathbb{L}_X$ is connective and $\pi_0 \mathbb{L}_{Y_0} \to \pi_0 f_0^* \mathbb{L}_X$ is an isomorphism. To check that this latter map of quasicoherent sheaves over Y_0 is an equivalence, it suffices to check on a point of Y_0. So, choose a map $\eta : \mathrm{Spec}\, R \to Y_0$; it turns out that without loss of generality we can actually assume that $R \in \mathrm{CAlg}^\heartsuit$. Now by definition, the composite

$$\mathrm{Spec}\, R \xrightarrow{\eta} Y_0 \xrightarrow{f_0} X$$

gives

$$\eta^* f_0^* \mathbb{L}_X \in \mathrm{Mod}_R^{a\mathrm{cn}},$$

i.e. an *almost connective* R-module. We would like to show that this is actually connective. So, if it is not actually connective, then there exists a discrete R-module M such that

$$\hom_R(\eta^* f_0^* \mathbb{L}_X, M)$$

is *not* discrete (i.e. has some nonvanishing positive homotopy group). Now, note that $\eta^* \mathbb{L}_{Y_0}$ is connective, since Y_0 is represented by a spectral *Deligne–Mumford* stack (i.e. its stabilizer groups are all étale – this is in contrast with spectral *Artin* stacks). By definition, we have the map of fiber sequences

$$\begin{array}{ccc}
\hom(\eta^* \mathbb{L}_{Y_0}, M) & \dashrightarrow & \hom(\eta^* f_0^* \mathbb{L}_X, M) \\
\downarrow & & \downarrow \\
Y_0(R \oplus M) & \longrightarrow & X(R \oplus M) \\
\downarrow & & \downarrow \\
Y_0(R) & \longrightarrow & X(R)
\end{array}$$

over the point $\eta \in Y_0(R)$. The middle and lower horizontal maps are equivalences because we have assumed that $X|_{\mathrm{CAlg}^\heartsuit} \simeq Y_0|_{\mathrm{CAlg}^\heartsuit}$, and moreover the lower square is a pullback. Thus, the dashed map on fibers is also an equivalence. So $f_0^* \mathbb{L}_X$ is connective, and

$$\pi_0 \eta^* \mathbb{L}_{Y_0} \cong \pi_0 \eta^* f_0^* \mathbb{L}_X,$$

so that

$$\pi_0 \mathbb{L}_{Y_0} \cong \pi_0 f_0^* \mathbb{L}_X.$$

Now, since X and Y_0 both admit cotangent compmlexes, then the map $f_0 : Y_0 \to X$ admits a relative cotangent complex, fitting into the cofiber sequence

$$f_0^* \mathbb{L}_X \to \mathbb{L}_{Y_0} \xrightarrow{\mathrm{cof}} \mathbb{L}_{Y_0/X}.$$

The long exact sequence in homotopy now implies that $\mathbb{L}_{Y_0/X}$ is 1-connective. So, in order to appeal to Lemma 67, it suffices to show that $\pi_1 \mathbb{L}_{Y_0/X} = 0$. This lives in $\mathrm{QC}_{Y_0}^\heartsuit$. To simplify notation, let us write

$$\mathcal{F} = \pi_1 \mathbb{L}_{Y_0/X} \in \mathrm{QC}_{Y_0}^\heartsuit.$$

Suppose that $\mathcal{F} \neq 0$. Then the Postnikov section

$$\mathbb{L}_{Y_0/X} \xrightarrow{\pi_1} \Sigma \mathcal{F}$$

is nonzero by assumption, and we can precomposite it to obtain a composite

$$f^* \mathbb{L}_X \xrightarrow{i} \mathbb{L}_{Y_0} \to \mathbb{L}_{Y_0/X} \xrightarrow{\pi_1} \mathcal{F}$$

which must be zero since it's the postcomposition of the composite defining a cofiber sequence. On the other hand, the map

$$\gamma : \mathbb{L}_{Y_0} \to \mathbb{L}_{Y_0/X} \xrightarrow{\pi_1} \mathcal{F}$$

must be nonzero since the map $\mathbb{L}_{Y_0} \to \mathbb{L}_{Y_0/X}$ is surjective.

Recall that maps out of the cotangent complex classify derivations. So again choose a discrete ring $R \in \mathrm{CAlg}^\heartsuit$ and a map $\eta : \operatorname{Spec} R \to Y_0$, which detects the nontriviality of $\mathcal{F}$ in the sense that

$$M = \eta^* \mathcal{F} \in \mathrm{Mod}_R^\heartsuit$$

is nonzero. This gives a derivation

$$\mathbb{L}_R \to \Sigma M$$

classifying a square-zero extension $R^\gamma \to R$ such that $R^\gamma \in \mathrm{CAlg}^\heartsuit$. Recall that this sits in a fiber square

$$\begin{array}{ccc} R^\gamma & \longrightarrow & R \\ \downarrow & & \downarrow {\scriptstyle (1,\gamma)} \\ R & \xrightarrow[(1,0)]{} & R \oplus \Sigma M. \end{array}$$

Now, the obstruction theory associated to the cotangent complex implies that since $\gamma \neq 0$, then the point $\eta \in Y_0(R)$ does not lift to a point of $Y_0(R^\gamma)$. However, its further restriction $f_0(\eta) \in X(R)$ must lift to a point of $X(R^\gamma)$ since $\gamma \circ i = 0$. Thus, we have arrived at a contradiction to the assertion that f is nonzero, since the diagram

$$\begin{array}{ccc} X(R^\gamma) & \xleftarrow{\sim} & Y_0(R^\gamma) \\ \uparrow & & \uparrow \\ X(R) & \xleftarrow[\sim]{} & Y_0(R) \end{array}$$

commutes. So in fact, $\mathbb{L}_{Y_0/X}$ is 2-connective.

Now, we have the map

$$Y_0 \xrightarrow{f_0} X$$

with $\mathbb{L}_{Y_0/X}$ is 2-connective, and X is an étale sheaf, so Lemma 67 provides a factorization

$$\begin{array}{ccc} Y_0 & \xrightarrow{\quad f_0 \quad} & X \\ & {\scriptstyle g}\searrow \quad \nearrow{\scriptstyle f} & \\ & Y & \end{array}$$

in which $\mathbb{L}_{Y/X} = 0$ and g is induced by a 1-connective map $\mathcal{O} \to \mathcal{O}_0$. In particular, $\pi_0\mathcal{O} \xrightarrow{\cong} \pi_0\mathcal{O}_0$, so these determine the same underlying scheme. Hence, $(\mathcal{Y}, \mathcal{O})$ is a connective spectral stack representing Y, and we have a étale map $Y \to X$ (which is surjective by checking just on discrete rings, or even just on fields). So X is representable. $\square$