Algebraic cobordism, algebraic orientations, and motivic Landweber exactness

§1. Alg. cobordism

Start w/ classical case:

BU_n classifies $\mathbb{C}$-v.b's of rk n; univ. bdl $\gamma_n \downarrow BU_n$.

natural operation: cx-conj, rep^d by $BU_n \xrightarrow{\chi} BU_n$

$\rightsquigarrow$ $C_2 \curvearrowright BU_n$, $C_2 \curvearrowright BU_n^{\gamma_n}$,

i.e. $S^{\mathbb{C}} \wedge -$

$\boxed{\Sigma^{\mathbb{C}}} BU_n^{\gamma_n} \longrightarrow BU_{n+1}^{\gamma_{n+1}}$ is C_2-equiv^t, if $\mathbb{C}$ has conjugation C_2-action

Note: $\mathbb{C} \cong \underline{\mathbb{R}} \oplus \sigma$ $\Rightarrow$ $\Sigma^{\mathbb{C}} = \Sigma^1 \Sigma^{\sigma}$

"sign rep"

$\rightsquigarrow$ Real cobordism, $M\mathbb{R}$, a genuine C_2-spectrum (i.e. indexed over a complete C_2-universe), and for $X \in Top_{C_2}$ get groups

$$M\mathbb{R}^{i+\sigma j} X, \quad i, j \in \mathbb{Z}.$$

Now, algebraic story: $U_n \overset{\simeq}{\hookrightarrow} GL_n(\mathbb{C})$ topologically (as maximal cpct);

U_n isn't ~~cx~~-alg. gp as we'd expect it to be, but GL_n is even defined over $\mathbb{Z}$, so use this.

Same: $\gamma_n \downarrow BGL_n$, and now have map of mspaces

i.e. $S^{\mathbb{A}^1} \wedge -$

$\boxed{\Sigma^{\mathbb{A}^1}} BGL_n^{\gamma_n} \longrightarrow BGL_{n+1}^{\gamma_{n+1}}$,

Now, $S^{\mathbb{A}^1} =: \mathbb{P}^1 \simeq S^1 \wedge G_m =: S^1 \wedge S^{\alpha}$

alt^{la} $S^{1,0} \wedge S^{1,1}$

$\rightsquigarrow$ algebraic cobordism, MGL, a $\mathbb{P}^1$-spectrum, and for $X \in MSpaces_S$ get groups

$$MGL^{i+\alpha j} X, \quad i, j \in \mathbb{Z}.$$

(2)

Rmk.: If $k \subseteq \mathbb{R}$, $\exists$ realiz^n functor $\rho : \mathrm{MSpaces}_k \longrightarrow \mathrm{Top}_{Gal(\mathbb{C}/\mathbb{R})}$,

$$X \longmapsto X(\mathbb{C})$$

and then $\rho(S^1) = S^1 = S^{\mathbb{R}}$,

$\rho(S^\alpha) = S^\sigma$.

Moreover, $M\underline{\mathbb{Z}} \longmapsto H\underline{\mathbb{Z}}$. (constant Mackey functor)

$KGL \longmapsto K\mathbb{R}$.

(Actually, for any base S, get $\rho : \mathrm{MSpaces}_S \longrightarrow \mathrm{Top}_{\pi_1^{et} S}$.)

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§2. Algebraic orientations

Now, k a field of char 0.

Def.: $E \in \mathrm{CAlg}(SH_k^{mot})$ is called algebraically orientable if $\forall n$, $\gamma_n \downarrow BGL_n$ is E-orientable (i.e. $\exists$ Thom class $u \in E^* BGL_n^{\gamma_n}$ s.t. $\forall Y \xrightarrow{f} BGL_n$,

$$E^* Y \xrightarrow[\mathrm{id} \otimes f^* u]{} E^* Y \underset{E^*}{\otimes} E^{*+n(1+\alpha)} Y^{f^*\gamma_n} \xrightarrow[\text{induced by coaction}]{} E^{*+n(1+\alpha)} Y^{f^*\gamma_n}$$

is an iso).

$$\begin{tikzcd}
Y_+ \wedge Y^{f^*\gamma_n} & Y^{f^*\gamma_n} \arrow[l] \\
\mathbb{R}^0 \boxtimes f^*\gamma_n \arrow[u] \arrow[d] & f^*\gamma_n \arrow[l] \arrow[u] \arrow[d] \\
Y \times Y & Y \arrow[l]
\end{tikzcd}$$

Thom spaces

(Can also use spectrum-level definition: need

$$Y^{f^*\gamma_n} \wedge E \xrightarrow{\text{coaction}} Y_+ \wedge Y^{f^*\gamma_n} \wedge E \xrightarrow{\text{id} \wedge u \wedge \mathrm{id}} Y_+ \wedge \Sigma^{n}_{\mathbb{P}^1} E \wedge E \xrightarrow{\mathrm{id} \wedge \mu} \Sigma^{n}_{\mathbb{P}^1} Y_+ \wedge E$$

to be an equivalence of spectra.)

An algebraic orientation of E is a class $u \in E^{1+\alpha}(\mathbb{P}^\infty)$ restricting to $1 \in E^{1+\alpha}(\mathbb{P}^1) \cong E^0(S^0)$.

Computations: mostly formally analogous to classical case.

maps induced by any $\mathbb{P}^{\infty} \to \mathbb{P}^\infty$ (and all here $\mathbb{A}^1$-homotopic)

$\hookrightarrow$ $\underbrace{E^* \mathbb{P}^\infty \cong E^*[\![u]\!]}$, $E^*((\mathbb{P}^\infty)^{\times n}) \cong E^*[\![u_1, \ldots, u_n]\!]$, $E^*\left(\mathrm{colim}_n (\mathbb{P}^\infty)^{\times n}\right) \cong \lim_n E^*(\mathbb{P}^\infty)^{\times n}$

$\cong E^*[\![u_1, u_2, \ldots]\!]$

by cofiber seq. $\mathbb{P}^{n-1} \to \mathbb{P}^n \to S^{n(1+\alpha)} = (\mathbb{P}^1)^{\wedge n}$

$\rightsquigarrow E^*$ lexseq; use induction, and $\lim^1 = 0$ since maps are surjective

↳
$$\begin{tikzcd} \gamma_1^{\boxtimes n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \gamma_n \arrow[d] \\ (\mathbb{P}^\infty)^{\times n} \arrow[r] & BGL_n \end{tikzcd}$$

$$\rightsquigarrow E^*(-)$$

$$\begin{tikzcd} E^*(\mathbb{P}^\infty)^{\times n} \arrow[d, phantom, "\cong" rotate=90] & E^*(BGL_n) \arrow[l] \arrow[d, phantom, "\cong" rotate=90] \\ E^*[[u_1, \dots, u_n]] & E^*[[c_1, \dots, c_n]] \end{tikzcd}$$

i^{th} elem. symm. polynomial in the u_j's ⟷ c_i = i^{th} E-Chern class of γ_n

(e.g. $c_1 \longmapsto u_1 + \dots + u_n$,
$c_2 \longmapsto u_1u_2 + \dots + u_{n-1}u_n$)

Pf.: Gysin sequence $BGL_{n-1} \to BGL_n \to BGL_n^{\gamma_n}$
plus diagram chase.

Taking colimits, $E^*BGL \cong E^*[[c_1, c_2, \dots]]$ $\qquad c_i$
$$\downarrow \qquad\qquad\qquad \updownarrow$$
$E^*(\mathrm{colim}\,(\mathbb{P}^\infty)^{\times n}) = E^*[[u_1, u_2, \dots]]$ $\qquad$ i^{th} elem. symm. power series in the u_j's

↳ detect htpy ring maps:

$$\begin{tikzcd} {[\Sigma^\infty_+ BGL, E]_{SHmot}} \arrow[r, phantom, "\cong"] & E^*BGL \cong E^*[[c_1, c_2, \dots]] \\ {[\Sigma^\infty_+ BGL, E]_{CAlg(SHmot)}} \arrow[u, hook] \arrow[r, phantom, "\cong"] & \{ g(u_1, u_2, \dots) \in E^*[[c_1, c_2, \dots]] : g(u_1, u_2, \dots) \cdot g(u_1', u_2', \dots) = g(u_1, u_1', u_2, u_2', \dots) \} \arrow[u, hook] \end{tikzcd}$$

$$\cong$$

$$\left\{ \prod_{i=1}^{\infty} f(u_i) \; : \; f(x) = 1 + a_1x + a_2x^2 + \dots \right\}.$$

$$\begin{tikzcd} \Sigma^\infty_+ BGL \wedge \Sigma^\infty_+ BGL \arrow[r] \arrow[d, "\oplus"'] & E \wedge E \arrow[d, "\mu_E"] \\ \Sigma^\infty_+ BGL \arrow[r] & E \end{tikzcd}$$

↳ By the E^*-Thom iso.,

$$\begin{array}{ccc} E^*MGL \cong E^*[[d_1, d_2, \ldots]] & \hookrightarrow & E^*[[v_1, v_2, \ldots]] \\ \nearrow \quad\quad \updownarrow & & \\ [MGL, E]_{CAlg(SH^{mot})} \cong \left\{ \prod_{i=1}^{\infty} f(v_i) : f(x) = 1 + a_1 x + a_2 x^2 + \ldots \right\} & & \end{array}$$

$d_i \longmapsto$ ith elem. symm. power series in the v_j's

Warning: For $E \in CAlg(SH^{mot})$, E_* need not be (graded-) commutative!

The failure is precisely $[\mathbb{G}_m \xrightarrow{\tau} \mathbb{G}_m] \in \pi_{0,0}^{st}$: in switching two coordinates, get

$$[S^\alpha \wedge S^\alpha \xrightarrow{\tau} S^\alpha \wedge S^\alpha] \simeq \tau \neq -1.$$

Prop.: If E is algebraically oriented, then $\tau = -1$ in E_*. Hence, E_* is a graded-comm. ring (with $|E_{i+\alpha j}| = i + j$).

Pf.: By above, $\exists$ htpy-ring map $MGL \to E$, so suff. to prove in universal case. As a "$\mathbb{P}^1$-prespectrum", have $(MGL)_{1+\alpha} \simeq BGL_1^{\gamma_1} = (\mathbb{P}^\infty)^{\gamma_1} \simeq \mathbb{P}^\infty$. Now, $S^\alpha \xrightarrow{\tau} S^\alpha \overset{-\wedge S^1}{\rightsquigarrow} S^{1+\alpha} \xrightarrow{\tau \wedge id_{S^1}} S^{1+\alpha} = \mathbb{P}^1 \hookrightarrow \mathbb{P}^\infty \simeq (MGL)_{1+\alpha}$ represents $\tau \in \pi_0 MGL$. But $\mathbb{P}^\infty \simeq K(\mathbb{Z}, 1+\alpha)$, so this must already be -1. $\square$

Cor.: If E is alg. or^d, get f.g.l. F_E / E_*. Then,

$$\left\{ \begin{array}{c} \text{alg. or}^{ns} \\ \text{of } E \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{strict iso's} \\ \text{of fgl's from } F_E \end{array} \right\} \simeq [MGL, E]_{CAlg(SH^{mot})}.$$

$$u' \in E^{1+\alpha}\mathbb{P}^\infty \longleftrightarrow u' = g(u) = u + a_1 u^2 + \ldots \longleftrightarrow f(u) = \frac{g(u)}{u} \in E^* MGL.$$

§3. Motivic Landweber exactness

Notation: For $E \in SH^{mot}$, write $E_n = E_{n(1+\alpha)} = [\Sigma^\infty_+ (\mathbb{P}^1)^{\wedge n}, E]$ (a/k/a $E_{2n,n}$).

Key computation: $(MGL_*, MGL_* MGL)$ is a flat Hopf algebroid, and $MGL_*(-) : SH^{mot} \longrightarrow \mathrm{Comod}$.

(study $E^*(Gr_n(\mathbb{A}^d))$ for E alg. or^d)

$$\rightsquigarrow \begin{tikzcd} \mathrm{Spec}(MGL_*) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{2-categorical p.b.}"] & \mathrm{Spec}(MU_*) \arrow[d] \\ \mathrm{Spec}(MGL_*) // \mathrm{Spec}(MGL_* MGL) \arrow[r] & \mathrm{Spec}(MU_*) // \mathrm{Spec}(MU_* MU) \end{tikzcd}$$

Thm.: If $A_* \in \mathrm{GrMod}_{MU_*}$ is L-exact, then $MGL_*(-) \otimes_{MU_*} A_*$ is a homology thy on SH^{mot}; if $A_* \in \mathrm{CAlg}(\mathrm{GrMod}_{MU_*})$, then ring-valued. Similarly for coh thy, but now only on strongly dualizable obj's of SH^{mot}.

Ex.: $\hat{\mathbb{G}}_m / \mathbb{Z}[\beta^{\pm}] \rightsquigarrow KGL^*(-)$.

Def.: The cellular mspectra are the colocaliz^n

$$i : SH^{mot}_{cell} \underset{\longleftarrow}{\overset{\hookrightarrow}{\perp}} SH^{mot} : p,$$

the smallest colocaliz^n containing the $S^{i+j\alpha}$.

(Note: The counit $pE \to E$ is a π_*-iso, "the closest approx to E by cellular mspectra".)

Fact: For $E \in SH^{mot}_{cell}$, $E_* F \cong E_* pF$.

Thm.: Continuing above, $\exists$ $E \in SH^{mot}_{cell}$ with $E_*(-) \cong MGL_*(-) \otimes_{MU_*} A_*$; in ring case, E has a quasi-multiplication, i.e. a mult. up to phantom maps.

Pf.: By work of Voevodsky, $(SH^{mot}_{\mathbb{Z}})_{cell}$ is a Brown category (in the sense of Hovey–Palmieri–Strickland). So, get E in absolute case. Then, prove base-change results to get all cases.

Note: Commutes w/ base change; in particular, a point $\mathrm{Spec}(\mathbb{C}) \longrightarrow S$ induces $SH^{mot} \longrightarrow SH$, and this "commutes" with LEFT.

Note: These are all cellular, by construction. (So, all L-exact homology thys factor through $p: SH^{mot} \to SH^{mot}_{cell}$, so can't hope for global results like nilpotence / ~~periodicity~~ thick subcat. thm to organize all of SH^{mot}, or even the dualizables.)

Weird fact: If A_* is even, also get $E^{Top} \in SH$; in the ring case,

$$E_*E \cong E_* \underset{E_*^{Top}}{\otimes} E_*^{Top} E^{Top}.$$

Cor.: $KGL^* KGL \cong KGL^* \underset{KU^*}{\hat{\otimes}} KU^* KU$.

Cor.: Motivic Chern character (after identifying $M\mathbb{Q}$ as L-exact).
(agrees with "K-thry to Chow gps" Chern character)

Cor.: There may be some hope of making e.g. motivic Morava E-theory into an $\mathcal{E}_\infty$-ring; classical version uses vanishing of $H^*_{AQ, \mathcal{E}_\infty}(E_*E, \Omega^n E_*E)$, which uses crucially that $\Omega^n E_*E$ is an extended comodule: $\Omega^n E_*E \cong \Omega^n E_0 \underset{E_0}{\otimes} E_*E$.