

Algebraic cobordism, algebraic orientations, and motivic Landweber exactness

§1. Alg. cobordism

Start w/ classical case:

BU_n classifies \mathbb{C} -v.b.s of rk n ; univ. bdd $\gamma_n \downarrow BU_n$.

natural operation: \mathbb{C} -conj, rep^d by $BU_n \xrightarrow{\chi} BU_n$

$\rightsquigarrow C_2 \curvearrowright BU_n, C_2 \curvearrowright BU_n^{\sigma_n}$,

$\xrightarrow{\text{io. } S^{\mathbb{C}} \wedge -} \Sigma^{\mathbb{C}} BU_n^{\sigma_n} \longrightarrow BU_{n+1}^{\sigma_{n+1}}$ is C_2 -equiv^t, if \mathbb{C} has conjugation C_2 -action

Note: $\mathbb{C} \cong \mathbb{R} \oplus \sigma \Rightarrow \Sigma^{\mathbb{C}} = \Sigma^{\mathbb{R}} \Sigma^{\sigma}$

\rightsquigarrow Real cobordism, MR , a genuine C_2 -spectrum (ie. indexed over a complete C_2 -universe), and for $X \in \text{Top}_{C_2}$ get groups

$MR^{i+\sigma j} X, i, j \in \mathbb{Z}$.

Now, algebraic story: $U_n \xrightarrow{\cong} GL_n(\mathbb{C})$ topologically (as maximal cpct); U_n isn't $\alpha\alpha$ -alg. gp as we'd expect it to be, but GL_n is even defined over \mathbb{Z} , so use this.

Same: $\gamma_n \downarrow BGL_n$, and now have map of spaces

$\xrightarrow{\text{io. } S^{\mathbb{A}^1} \wedge -} \Sigma^{\mathbb{A}^1} BGL_n^{\gamma_n} \longrightarrow BGL_{n+1}^{\gamma_{n+1}}$

Now, $S^{\mathbb{A}^1} =: \mathbb{P}^1 \cong S^1 \wedge G_m =: S^1 \wedge S^{\alpha}$
altho S^1, S^{α}

\rightsquigarrow algebraic cobordism, MGL , a \mathbb{P}^1 -spectrum, and for $X \in \text{MSpaces}$ get groups

$MGL^{i+\alpha j} X, i, j \in \mathbb{Z}$.

Remark: If $k \subseteq \mathbb{R}$, \exists realization functor $\rho: \text{MSpaces}_k \rightarrow \text{Top}_{\text{Gal}(k/\mathbb{R})}$,

and then $\rho(S^1) = S^1 = S^{\mathbb{R}}$ $X \mapsto X(\mathbb{C})$
 $\rho(S^\infty) = S^\infty$

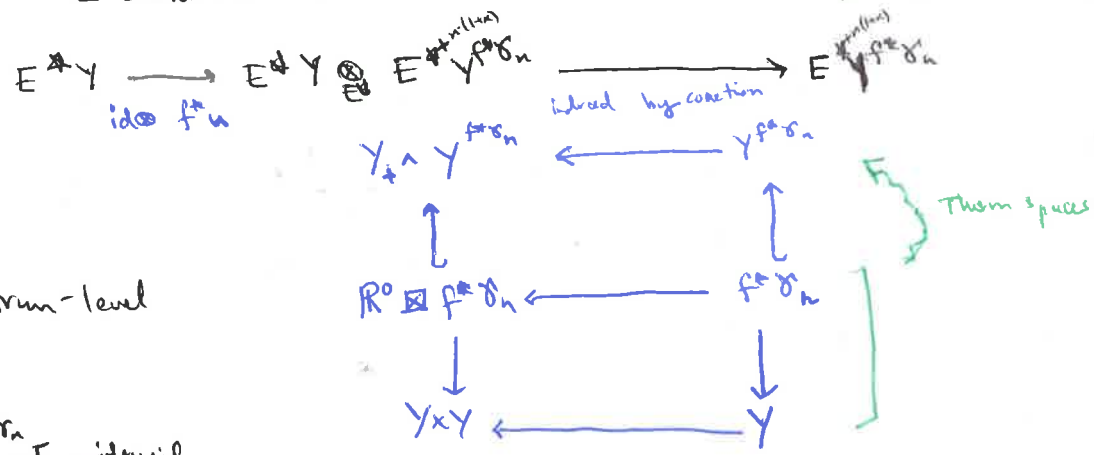
Moreover, $M\mathbb{Z} \hookrightarrow H\mathbb{Z}$ (constant Massey functor)
 $K\mathbb{G}L \hookrightarrow K\mathbb{R}$

(Actually, for any base S , get $\rho: \text{MSpaces}_S \rightarrow \text{Top}_{\pi_1^{\text{ét}} S}$.)

§2 Algebraic orientation

Now, k a field of char 0.

Def: $E \in \mathcal{S}\text{Alg}(SH_k^{\text{mot}})$ is called algebraically orientable if $\forall n$, $\gamma_n \downarrow BGL_n$ is E -orientable ($\therefore \exists$ Thom class $u \in E^* BGL_n^{\infty}$ st. $\forall Y \xrightarrow{f} BGL_n$,



is an iso

(Can also use spectrum-level definition: need

$Y^* \delta_n \xrightarrow{\text{coaction}} Y_+ \wedge Y^* \delta_n \xrightarrow{\text{id} \otimes u} E$
 $\rightarrow \sum_{P^1} Y_+ \wedge E \xrightarrow{\text{id} \otimes u} \sum_{P^1} Y_+ \wedge E$

to be an equivalence of spectra.)

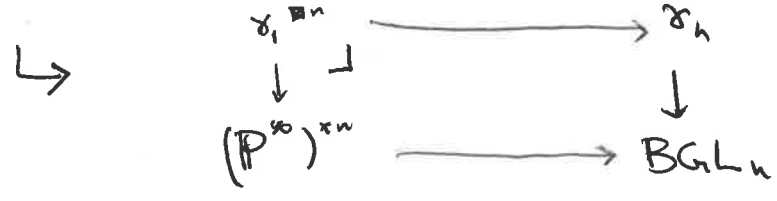
An algebraic orientation of E is a class $u \in E^{1+n}(P^\infty)$ restricting to $1 \in E^{1+n}(P^1) \cong E^0(S^0)$.

Computations: mostly formally analogous to classical case.

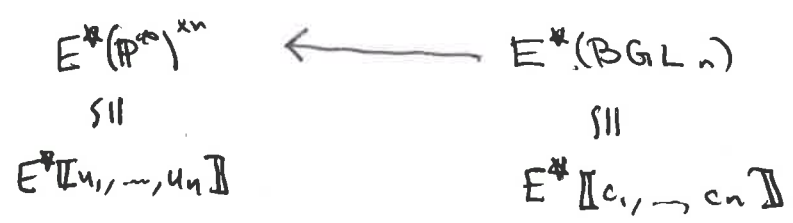
$\hookrightarrow E^* P^\infty \cong E^* [u]$, $E^*((P^\infty)^{\times n}) \cong E^* [u_1, \dots, u_n]$, $E^*(\text{colim}_n (P^\infty)^{\times n}) \cong \varinjlim_n E^*(P^\infty)^{\times n} \cong E^* [u_1, u_2, \dots]$

by cofiber seq. $P^{n-1} \rightarrow P^n \rightarrow S^{n(1+n)} = (P^1)^{\wedge n}$
 $\mapsto E^*$ lex seq; use induction, and $\lim^1 = 0$ since maps are surjective

maps induced by any $f: P^1 \rightarrow P^1$ (and all are \mathbb{A}^1 -homotopic)



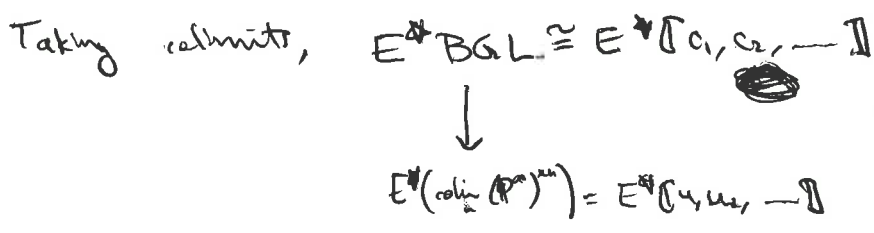
$\{E^*(-)\}$



i^{th} elem. symm. polynomial in the u_j 's $\longleftrightarrow c_i = i^{\text{th}}$ E-Chern class of γ_n

(e.g. $c_1 \mapsto u_1 + \dots + u_n$,
 $c_2 \mapsto u_1 u_2 + \dots + u_{n-1} u_n$)

Pf.: Gysin sequence $BGL_{n-1} \rightarrow BGL_n \rightarrow BGL_n^{\mathbb{P}^n}$
 plus diagram chase.



c_i
 \downarrow
 i^{th} elem. symm. power series in the u_j 's

\hookrightarrow detect hopy ring maps:

$$[\Sigma_+^{\infty} BGL, E]_{\text{SHmt}} \cong E^* BGL \cong E^*[c_1, c_2, \dots]$$

$$[\Sigma_+^{\infty} BGL, E]_{\text{CAT}_g(\text{SHmt})} \cong \left\{ g(u_1, u_2, \dots) \in E^*[c_1, c_2, \dots] : g(u_1, u_2, \dots) \circ g(u'_1, u'_2, \dots) = g(u_1, u'_1, u_2, u'_2, \dots) \right\}$$



$$\left\{ \prod_{i=1}^{\infty} f(u_i) : f(x) = 1 + a_1 x + a_2 x^2 + \dots \right\}$$

↳ By the E^* -Thom iso.,

$d_i \xrightarrow{\quad} i^{\text{th}} \text{ elem. symm. power series in the } v_j\text{'s}$

(4)

$$E^* MGL \cong E^* [d_1, d_2, \dots] \xrightarrow{\quad} E^* [v_1, v_2, \dots]$$

$$[MGL, E]_{\text{Catg}(SH^{mot})} \cong \left\{ \prod_{i=1}^{\infty} f(v_i) : f(x) = 1 + a_1 x + a_2 x^2 + \dots \right\}$$

Warning: For $E \in \text{Catg}(SH^{mot})$, E_* need not be (graded-)commutative!

The failure is precisely $[G_m \xrightarrow{z} G_m] \in \pi_{0,0}^{st}$: in switching two coordinates, get

$$[S^{\alpha} \wedge S^{\alpha} \xrightarrow{\tau} S^{\alpha} \wedge S^{\alpha}] \cong \mathbb{1} \neq -1.$$

Prop.: If E is algebraically oriented, then $z = -1$ in E_* . Hence, E_* is a graded-comm. ring (with $|E_{i \times j}| = i+j$).

Pf.: By above, \exists htpy. ring map $MGL \rightarrow E$, so suff. to prove in universal case.

As a " \mathbb{P}^1 -prespectrum", have $(MGL)_{1+\alpha} \cong BGL_1^{\mathbb{R}} = (\mathbb{P}^{\infty})^{\mathbb{R}} \cong \mathbb{P}^{\infty}$. Now,

$$S^{\alpha} \xrightarrow{z} S^{\alpha} \xrightarrow{-\wedge S^1} S^{1+\alpha} \xrightarrow{z \text{ acts}} S^{1+\alpha} = \mathbb{P}^1 \hookrightarrow \mathbb{P}^{\infty} \cong (MGL)_{1+\alpha} \text{ represents } z \in \pi_0 MGL.$$

But $\mathbb{P}^{\infty} \cong K(\mathbb{Z}, 1+\alpha)$, so this must already be -1 . □

Cor.: If E is alg. or^d, get f.g.l. F_E / E_* . Then,

$$\left\{ \text{alg. or}^{\text{ns}} \text{ of } E \right\} \cong \left\{ \text{strict iso's of f.g.l's from } F_E \right\} \cong [MGL, E]_{\text{Catg}(SH^{mot})}.$$

$$u \in E^{1+\alpha} \mathbb{P}^{\infty} \longleftrightarrow u' = g(u) = u r_1 u^2 r_2 \dots \longleftrightarrow f(u) = \frac{g(u)}{u} \in E^* MGL.$$

§3. Motivic Landweber exactness

(5)

Notation: For $E \in \text{SH}^{\text{mot}}$, write $E_n = E_{n(1+\alpha)} = [\Sigma_+^{\infty}(\mathbb{P}^1)^{\wedge n}, E]$ (i/k/a $E_{2n,n}$);

Key computation: $(\text{MGL}_*, \text{MGL}_* \text{MGL})$ is a flat Hopf algebra, and $\text{MGL}_*(-) : \text{SH}^{\text{mot}} \rightarrow \text{Comod}$.

(study $E^*(\text{Gr}_n(\mathbb{A}^d))$ for E alg. or d)

$$\begin{array}{ccc} \rightsquigarrow \text{Spec}(\text{MGL}_*) & \xrightarrow{\quad} & \text{Spec}(\text{MGL}_*) \\ \downarrow & \swarrow \text{2-} & \downarrow \\ \text{Spec}(\text{MGL}_*) // \text{Spec}(\text{MGL}_* \text{MGL}) & \xrightarrow{\quad} & \text{Spec}(\text{MGL}_*) // \text{Spec}(\text{MGL}_* \text{MGL}) \end{array}$$

2-regular p.b.

Thm.: If $A_* \in \text{GrMod}_{\text{MGL}_*}$ is L-exact, then $\text{MGL}_*(-) \otimes_{\text{MGL}_*} A_*$ is a homology theory on SH^{mot} ; if $A_* \in \text{CAlg}(\text{GrMod}_{\text{MGL}_*})$, then ring-valued. Similarly for coh theory, but now only on strongly dualizable objects of SH^{mot} .

Ex.: $\hat{\text{G}}_m / \mathbb{Z}[\beta^{\pm 1}] \rightsquigarrow \text{KGL}_*^{\#}(-)$.

Def.: The cellular spectra are the colocalizations

$$i : \text{SH}_{\text{cell}}^{\text{mot}} \xrightarrow{\quad} \text{SH}^{\text{mot}} \xrightarrow{p} \text{SH}^{\text{mot}} \text{ i.p.}$$

the smallest colocalization containing the sites.

(Note: The counit $pE \rightarrow E$ is a Π_* -iso, "the closest approx to E by cellular inspection".)

Fact: For $E \in \text{SH}_{\text{cell}}^{\text{mot}}$, $E_* F \cong E_* pF$.

Thm.: Continuing above, $\exists E \in \text{SH}_{\text{cell}}^{\text{mot}}$ with $E_*(-) \cong \text{MGL}_*(-) \otimes_{\text{MGL}_*} A_*$; in ring case, E has a quasi-multiplication, i.e. a mult. up to phantom maps.

Pf.: By work of Voevodsky, $(\text{SH}_{\mathbb{Z}}^{\text{mot}})_{\text{cell}}$ is a Brown category (in the sense of Hovey-Palmieri-Strickland). So, get E in absolute case. Then, prove base change results to get all cases.

Note: Commutes w/ base change; in particular, a point $\text{Spec}(\mathbb{C}) \rightarrow S$ induces $\text{SH}^{\text{mot}} \rightarrow \text{SH}$, and this "commutes" with LEFT.

Note: These are all cellular, by construction. (So, all L-exact homology theory factor through $p: SH^{mot} \rightarrow SH^{mot}_{cell}$, so can't hope for global results like nilpotence / ~~pro-... ..~~ thick subcat. then to organize all of SH^{mot} , or even the dualities.)

Weird fact: If A_* is even, also get $E^{Top} \in SH$, in the ring case,

$$E_* E \cong E_* \bigotimes_{E_*^{Top}} E_*^{Top} E_*^{Top}$$

Cor: $KGL^* KGL \cong KGL^* \widehat{\bigotimes}_{KU^*} KU^* KU$

Cor: Motivic Chern characters (after identifying $M\mathbb{Q}$ as L-exact).
 (agrees with "K-theory to Chow groups" Chern character)

Cor: There may be some hope of making e.g. motivic Morava E-theory into an E_∞ -ring; classical version uses vanishing of $H^*_{AQ, E_*}(E_* E, \Omega^n E_* E)$, which uses crucially that $\Omega^n E_* E$ is an extended comodule: $\Omega^n E_* E \cong \Omega^n E_* \bigotimes_{E_*} E_* E$.