THE GEOMETRY OF THE CYCLOTOMIC TRACE

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ABSTRACT. This document accompanies the trilogy of articles [1] A naive approach to genuine G-spectra and cyclotomic spectra, [2] Factorization homology of enriched ∞-categories, and [3] The geometry of the cyclotomic trace, which I wrote together with David Ayala and Nick Rozenblyum. In it, I summarize the findings in the articles, demonstrate how they form a coherent body of work, and describe their impact within the field of mathematics. It is intended to be readable by a non-mathematical audience (though it almost surely isn't wholly so).

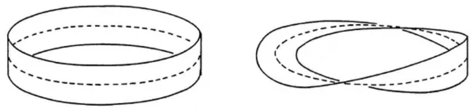
OVERVIEW

Mathematics is the ongoing human endeavor of uncovering the logical structures that govern our universe. This pursuit consists in the two interwoven strands of *theory* and *computation*. For example, the theory of negative numbers was introduced to solve the computational problem of subtracting a larger number from a smaller one. Thereafter, the theory was expanded to include rational, then real, and finally complex numbers to solve the computational problems of division, limits, and polynomial equations.

Algebraic K-theory is an important construction in *algebraic geometry*, which has far-reaching applications throughout mathematics. Our main tool for computing algebraic K-theory is the ***cyclotomic trace***. Though it was introduced nearly 30 years ago, the cyclotomic trace has remained theoretically mysterious. The present work fills this gap by providing a new construction of the cyclotomic trace which admits a precise theoretical interpretation in terms of *derived* algebraic geometry.

BACKGROUND

K-theory. The fundamental goal of geometry is to classify spaces (curves, surfaces, and so on) and find tools for distinguishing them. An important such tool is ***K-theory***, which is the study of *vector bundles* on spaces. A vector bundle is a parametrized family of vector spaces. As a simple example, one may visualize a family of 1-dimensional vector spaces parametrized by the circle as "toothpicks glued to a rubber band". There are two of these: the cylinder and the Möbius band. The former extends from the circle to the disk, while the latter does not: via the Möbius band, K-theory allows us to distinguish the circle from the disk.



The cylinder and the Möbius band.

Algebraic geometry. Given a space X, one can study *functions* on it. A function is just the assignment of a number to each point of X. In fact, the set of functions forms a *ring*: its elements can be added, subtracted, and multiplied, simply by doing so point-by-point along X. Of course, the rings that arise in this way are very special. Grothendieck, one of the most influential mathematicians of the 20th century, worked through the '50s and '60s to develop a robust geometric framework – the theory of ***schemes*** – in which *every* ring arises as the ring of functions on some space. For example, this gives a geometric context for studying questions about prime numbers (which live in the ring of integers). This has yielded great insight into the field of number theory, e.g. it is the basis for Wiles's celebrated proof of Fermat's last theorem.

Algebraic K-theory. The field of ***algebraic K-theory***, introduced by Grothendieck in '56, is a combination of the above two ideas: it is the study of schemes through their vector bundles. This is a powerful tool. For example, a longstanding conjecture in number theory would be resolved by certain (difficult!) algebraic K-theory computations. And in a completely different direction, deep work of Waldhausen has shown that algebraic K-theory also bears on ordinary (as opposed to "algebraic") geometry in surprising ways.

Derived algebraic geometry. A ring is a set in which one can add, subtract, and multiply. Roughly speaking, a "derived ring" is a *space* in which one can add, subtract, and multiply. Whereas rings are the building blocks of algebraic geometry, derived rings are the building blocks of *derived algebraic geometry* – the theory of ***derived schemes***. Crucially, this formalism allows spaces and rings to interact on equal footing. In particular, given a(n ordinary or derived) scheme X, the collection of maps $S^1 \to X$ from the circle assembles into the *loop space* of X, a derived scheme denoted $\mathscr{L}X$.

The Dennis trace. The "twistedness" of a vector bundle around a loop is encoded by a matrix. Taking its trace (the sum of its diagonal entries) determines a *number*. (For example, the trace of the cylinder is 1, while the trace of the Möbius band is -1.) Thus, a vector bundle on a scheme X determines a *function* on $\mathscr{L}X$. This association defines the ***Dennis trace***: the functions on $\mathscr{L}X$ assemble into an object denoted $\mathsf{THH}(X)$, and the Dennis trace from the algebraic K-theory of X runs $\mathsf{K}(X) \to \mathsf{THH}(X)$.

The cyclic trace. The Dennis trace is invariant under rotation of loops: if γ' denotes the loop $S^1 \to S^1 \xrightarrow{\gamma} X$ where the first map is a rotation of the circle, then the traces of E around γ and γ' agree. The collection of rotation-invariant functions on $\mathscr{L}X$ is denoted $\mathsf{THC}^-(X)$, and this observation gives us a refinement

$$\begin{array}{ccc} \mathsf{K}(X) & \xrightarrow{\text{Dennis trace}} & \mathsf{THH}(X) \\ & {\scriptstyle\textit{cyclic trace}}\searrow \qquad \nearrow & \\ & \mathsf{THC}^-(X) & \end{array} .$$

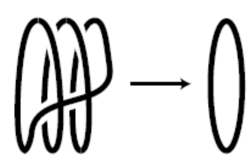
In '86, the cyclic trace was shown by Goodwillie to be an "infinitesimal equivalence" after a process called *rationalization*, which removes much (but not all) of the interesting information from $\mathsf{K}(X)$. Since $\mathsf{THC}^-(X)$ is relatively computable, the cyclic trace allows us to compute the rationalization of $\mathsf{K}(X)$.

The cyclotomic trace. Computations showed that Goodwillie's theorem could not hold before rationalization. A further refinement of the cyclic trace was therefore sought. In '89, Bökstedt–Hsiang–Madsen came up with such a refinement

$$\begin{array}{cccc} \mathsf{K}(X) & \xrightarrow{\text{Dennis trace}} & & \mathsf{THH}(X) \\ & {\scriptstyle\text{cyclic trace}}\searrow & & \nearrow \\ {\scriptstyle\textit{cyclotomic trace}}\searrow & & \mathsf{THC}^-(X) & \\ & \mathsf{TC}(X) & \nearrow & \end{array} ,$$

which was shown by Dundas–McCarthy in '97 to be an "infinitesimal equivalence" even without rationalization. However, for reasons described below, this lacked a corresponding conceptual description. *In [3], we show that* $\mathsf{TC}(X)$ *consists of those functions on* $\mathscr{L}X$ *which are not merely rotation-invariant, but satisfy an intricate compatibility condition regarding covering maps of circles.* An r-fold covering map $S^1 \xrightarrow{r} S^1$ (for a positive integer r) allows us to obtain from each loop $S^1 \xrightarrow{\gamma} X$ a new loop $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$, denoted $r^*\gamma$. We identify $\mathsf{TC}(X)$ as consisting of those functions on $\mathscr{L}X$ for which the value on each loop γ determines the value on the loop $r^*\gamma$ "to the greatest extent possible".

A naive approach to genuine G-spectra and cyclotomic spectra. The construction of TC from THH relies on the "cyclotomic spectrum" structure of the latter. This is defined in terms of "genuine G-spectra", which are widely useful but conceptually mysterious. *As a key ingredient for our conceptual description of* TC, *in [1] we recast these notions in terms of "naive G-spectra", which are conceptually transparent.*



A 3-fold covering map of circles.

Factorization homology of enriched ∞-categories. The original definition of THH was given in terms of combinatorics (the mathematics of counting), but it can also be given in terms of geometry as *spectrum-enriched factorization homology* over the circle. The advantage of the latter is that it is manifestly related to covering maps of circles. *In [2], we develop the general theory of enriched factorization homology. We also use space-enriched factorization homology to obtain the (relatively simple) "unstable cyclotomic trace", from which we show in [3] that the (relatively complicated) cyclotomic trace naturally arises.*