The geometry of the cyclotomic trace

Aaron Mazel-Gee

with David Ayala and Nick Rozenblyum

X a scheme (derived)

$$\begin{aligned} \mathrm{K}(X) &= \textit{\textbf{algebraic K-theory}} \text{ of } X \\ &:= \mathrm{K}(\mathrm{Perf}_X) \\ &\approx \text{group-completion of } (\mathrm{VBdl}(X)/\text{iso.}, \oplus) \end{aligned}$$

[hard to compute!]

$$\begin{aligned} \mathrm{THH}(X) &= \textit{\textbf{topological Hochschild homology}} \text{ of } X \\ &:= \mathrm{THH}(\mathrm{Perf}_X) := \textstyle\int_{S^1} \mathrm{Perf}_X \\ &\approx \mathscr{O}(\mathscr{L}X), \text{ functions on the free loopspace of } X \end{aligned}$$

[easier to compute]

the ***Dennis trace*** map

$$\begin{array}{ccc} \mathrm{K}(X) & \longrightarrow & \mathrm{THH}(X) \\ (E \downarrow X) & \longmapsto & \left((S^1 \xrightarrow{\gamma} X) \longmapsto \begin{array}{l} \text{trace of} \\ \text{monodromy} \\ \text{of } E|_\gamma \end{array} \right) \end{array}$$

a refinement:

$$\begin{tikzcd} K(X) \arrow[rr, "\text{Dennis trace}"] \arrow[dr, dashed, "\textbf{\textit{cyclotomic trace}}"'] & & THH(X) \\ & TC(X) \arrow[ur] & \end{tikzcd}$$

$\mathrm{TC}(X) =$ ***topological cyclic homology*** of X
$\approx$ .....???!!!?

[computationally accessible, but conceptually mysterious]

why we care about TC

Theorem (Goodwillie/$\mathbb{Q}$ '86, McCarthy/$\mathbb{Z}$ '97, Dundas/$\mathbb{S}$ '97)

The cyclotomic trace is "locally constant": for $\tilde{A} \to A$ a nilpotent extension of associative rings (or of connective ring spectra),

$$\begin{tikzcd} \mathsf{K}(\tilde{A}) \arrow[r] \arrow[d] & \mathsf{K}(A) \arrow[d] \\ \mathsf{TC}(\tilde{A}) \arrow[r] & \mathsf{TC}(A) \end{tikzcd}$$

is a pullback (after p-completion, for any prime p).

"This is how people other than Quillen compute algebraic K-theory."

$\sim$ *A. Blumberg*, algebraic K-theorist

...but what *is* $\mathrm{TC}(X)$, really?

intermediate factorization through *negative cyclic homology*:

$$\mathrm{K}(X) \longrightarrow \mathrm{TC}(X) \longrightarrow \mathrm{HC}^-(X) \longrightarrow \mathrm{THH}(X)$$

	differential algebra	derived algebraic geometry
$\mathrm{THH}(X)$	$\Omega^*_{\mathrm{dR}}(X)$	functions on $\mathscr{L}X$
$\mathrm{HC}^-(X)$ $:= \mathrm{THH}(X)^{\mathrm{h}\mathbb{T}}$	$\mathrm{H}^*_{\mathrm{dR}}(X)$	$\mathbb{T}$-invariant functions on $\mathscr{L}X$
$\mathrm{TC}(X)$ $:= \mathrm{THH}(X)^{\mathrm{hCyc}}$	$???^*_{\mathrm{dR}}(X)$	**TODAY**

constructions of TC

original definition (Bökstedt–Hsiang–Madsen '93):

- uses *genuine-equivariant* stable homotopy theory
  - useful (e.g. equivariant Poincaré duality)...
  - but not conceptual (no DAG interpretation known)
- used opaque point-set manipulations
- based on vague analogy with free loopspaces

firmer categorical footing (Blumberg–Mandell '13):

- define homotopy theory of "cyclotomic spectra"

more recent definition (Nikolaus–Scholze '17?):

- removes genuine-equivariance
- restricts to *connective* ring spectra

this talk, inspired by Nikolaus–Scholze:

- applies to any spectrally-enriched ∞-category
- uses *factorization homology* to keep track of symmetries
- admits direct interpretation in DAG via $\mathscr{L}X$
- suggests higher-dim generalizations ($\rightsquigarrow$ "higher K-theory")

overview

$$\begin{tikzcd} \mathrm{Sp} \arrow[r, shift left, "\mathrm{triv}"] \arrow[r, phantom, "\perp"] & \mathrm{Cyc}(\mathrm{Sp}) \arrow[l, shift left, dashed, "(-)^{\mathrm{hCyc}}"] \\ \mathrm{TC}(X) \arrow[u, phantom, "\Cup"] & \mathrm{THH}(X) \arrow[u, phantom, "\Cup"] \arrow[l, mapsto] \end{tikzcd}$$

$\mathrm{TC}(X) :=$ fixedpoints of ***cyclotomic structure*** on $\mathrm{THH}(X)$
$\rightsquigarrow$ built by "imposing conditions" on functions on $\mathscr{L}X$

main idea: $\mathrm{TC}(X) \approx$ functions on $\mathscr{L}X$ that are...

- invariant under the $\mathbb{T}$-action on $\mathscr{L}X$.
- "sensitive" to the relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$.

$\mathbb{T}$-invariance is easy, but what does "sensitive" mean?

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$

Q.: M an $n \times n$ matrix, compare $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

Ex. 1: $r = 2$, $M = \mathrm{diag}(m_1, \ldots, m_n) \in \mathrm{M}_{n\times n}(R)$

$$\mathrm{tr}(M)^2 = \sum_{i,j} m_i m_j \quad , \quad \mathrm{tr}(M^2) = \sum_k (m_k)^2$$

- both *cyclically invariant*, i.e. lie in the fixedpoints $(R \otimes R)^{C_2}$
- difference is *norms*: image of $\sum_{i<j}[m_i \otimes m_j]$ under

$$\begin{aligned} (R \otimes R)_{C_2} &\xrightarrow{\mathrm{Nm}} (R \otimes R)^{C_2} \\ [x \otimes y] &\longmapsto \sum_{\sigma \in C_2} \sigma(x \otimes y) \end{aligned}$$

$\rightsquigarrow$ become equal in the ***Tate construction***, the cofiber

$$(R \otimes R)_{C_2} \xrightarrow{\mathrm{Nm}} (R \otimes R)^{C_2} \longrightarrow (R \otimes R)^{\mathrm{t}C_2}$$

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$

Q.: M an $n \times n$ matrix, compare $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

Ex. 2: $M = \mathrm{diag}(m_1, m_2)$, $r \in \mathbb{N}^\times$ arbitrary

now, difference between

$$\mathrm{tr}(M)^r = (m_1 + m_2)^r \quad , \quad \mathrm{tr}(M^r) = ((m_1)^r + (m_2)^r)$$

governed by binomial coefficients $\binom{r}{i}$ for $0 < i < r$

fact: these are never coprime to r

$\rightsquigarrow$ quotient $(R^{\otimes r})^{C_r}$ by norms from *all* proper subgroups of C_r

$\rightsquigarrow$ $tr(M^r) \equiv \mathrm{tr}(M)^r$ in the *generalized* Tate construction $(R^{\otimes r})^{\tau C_r}$

$\star$ for $\mathcal{C}$ a spectrally enriched ∞-category, a covering map

$$S^1_b \xleftarrow{r} S^1_a$$

of oriented circles induces a ***cyclotomic structure map***

$$\mathrm{THH}(\mathcal{C}) := \int_{S^1_b} \mathcal{C} \longrightarrow \left(\int_{S^1_a} \mathcal{C} \right)^{\tau C_r} =: \mathrm{THH}(\mathcal{C})^{\tau C_r}$$

Theorem 1 (A & M-G & R)

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \lim^{\text{r.lax}} \left(\mathrm{Fun}(\mathrm{B}\mathbb{T}, \mathrm{Sp}) \overset{\text{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times} \right).$$

⋆ an object of $\lim^{\text{r.lax}}$ is given by $T \in \mathrm{Fun}(\mathrm{B}\mathbb{T}, \mathrm{Sp})$ equipped with:

- for each $r \in \mathbb{N}^{\times}$, a cyclotomic structure map $T \xrightarrow{\sigma_r} T^{\tau C_r}$;
- for each $r, s \in \mathbb{N}^{\times}$, the *data* of a commutative square

Thm. [Nikolaus–Scholze] for T connective and $r=s=p$ prime
Cor.: suff to specify just σ_p
(since $\sigma_{p^n} = (\sigma_p)^{\circ n}$, and n-cubes *canonically* commute $\forall\ n \geq 2$)

$$\begin{tikzcd} T \arrow[r, "\sigma_r"] \arrow[d, "\sigma_{rs}"'] & T^{\tau C_r} \arrow[d, "(\sigma_s)^{\tau C_r}"] \\ T^{\tau C_{rs}} \arrow[r, "\sim", "\text{can.}"'] & (T^{\tau C_s})^{\tau C_r} \end{tikzcd}$$

- for each $r_1, \ldots, r_n \in \mathbb{N}^{\times}$, the *data* of a commutative n-cube...

⋆ **slogan:** $\mathrm{TC}(X)$ is built from $\mathrm{THH}(X) \approx \mathscr{O}(\mathscr{L}X)$ by selecting just those functions:

- that are $\mathbb{T}$-invariant;
- whose values on $S^1 \xrightarrow{\gamma} X$ determine their values on $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$ "to the greatest extent possible", subject to all possible coherences between these determinations.

Theorem 2 (A & M-G & R)

1.

$$\textbf{\textit{diagonal package}} \text{ for spaces} \rightsquigarrow \begin{tikzcd} \mathrm{Cat}(\mathcal{S}) \arrow[r, dashed] \arrow[dr, "\mathrm{THH}_{\mathcal{S}}"'] & \mathrm{Cyc}^{\mathrm{h}}(\mathcal{S}) \arrow[d, "\mathrm{fgt}"] \\ & \mathcal{S} \end{tikzcd}$$

$\mathrm{Cyc}^{\mathrm{h}}(\mathcal{S})$:= "unstable cyclotomic spaces"

2.

$$\begin{tikzcd} \textbf{\textit{diagonal package}} \text{ for spaces} \arrow[d, squiggly, "\substack{\text{linearization} \\ \text{(à la Goodwillie calculus)}}"'] & & \\ \textbf{\textit{Tate package}} \text{ for spectra} \arrow[r, squiggly] & \mathrm{Cat}(\mathcal{S}p) \arrow[r, dashed] \arrow[dr, "\mathrm{THH}"'] & \mathrm{Cyc}(\mathcal{S}p) \arrow[d, "\mathrm{fgt}"] \\ & & \mathcal{S}p \end{tikzcd}$$

$\mathrm{Cyc}(\mathcal{S}p)$:= cyclotomic spectra