p-ADIC MODULAR FORMS AND DIEUDONNÉ CRYSTALS IN STABLE HOMOTOPY THEORY

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Background from stable homotopy theory

The field of *chromatic homotopy theory* is organized around the correspondence between *formal groups* and *complex-oriented cohomology theories*. In particular, completing elliptic curves at the identity induces a map $\overline{\mathcal{M}_{\text{Ell}}} \to \mathcal{M}_{\text{FG}}$, and pulling back yields a sheaf of such cohomology theories over $\overline{\mathcal{M}_{\text{Ell}}}$ (or even better, a sheaf of E_∞*-ring spectra*, the global sections of which are tmf). Its fibers are called *elliptic cohomology theories*.

Let E_2 (the second *Morava E-theory*) denote the value of this sheaf on a formal neighborhood of a supersingular point (a *Lubin-Tate universal deformation space*). Let K denote complex K-theory; $K_p^\wedge = E_1$. The general theory dictates that the cohomology theory $E_1 \otimes E_2$ corresponds to the scheme of isomorphisms between the associated formal groups. Such isomorphisms exist over every point but the origin, and so this scheme is a "covering space" for the *punctured* formal neighborhood. This gives rise to a *monodromy representation* for its fundamental group (that is, a Galois representation), which has been studied extensively using the language of *p-adic modular forms*. My goals are therefore:

- to apply p-adic modular forms to study $E_1 \otimes E_2$;
- to globalize this understanding to $K \otimes tmf$;
- to recast this story in the language of *Dieudonné crystals* in order to generalize to $E_{n-1} \otimes E_n$.

p-adic modular forms

Recall that a *modular form* is a function on the set of isomorphism classes of elliptic curves with *level n structure*, which is natural for pullbacks and which satisfies a certain transformation relation depending on its *weight k*. Equivalently, these may be considered as global sections of a line bundle $\omega^{\otimes k}$ over the moduli $\mathcal{M}_{\text{Ell},n}$ of such elliptic curves.

Number theorists wanted an intelligent p-adic extension of this theory; this was first achieved by Serre, who (via q-expansions) defined a *p-adic modular form* to be the limit of a p-adically convergent sequence classical modular forms. This was generalized by Katz, whose most important insight was to include a *growth condition* on the elliptic curves under consideration. A certain mod-p parameter is the reduction of the modular form E_{p-1}, and vanishes precisely at the points of $\mathcal{M}_{\text{Ell},n} \otimes \mathbb{Z}/p\mathbb{Z}$ classifying supersingular elliptic curves. For any $\varepsilon > 0$, we can consider the *rigid-analytic open subset* of $\mathcal{M}_{\text{Ell},n} \otimes \mathbb{Z}_p$ defined by the condition that $\|E_{p-1}\|_p > \varepsilon$, and then a *generalized p-adic modular form* is simply a section of the restriction of $\omega^{\otimes k}$. (Here, ε controls the growth of the Laurent series coefficients around the supersingular points.)

Katz then studied *p-adic modular functions*, which take values on *trivialized* elliptic curves over p-adic rings. These generalize in the same way, which is the entry point for the topological story above. In fact, there is an ind-étale cover $\mathcal{M}_{\text{Ell,triv}} \to \mathcal{M}_{\text{Ell}}$ (which is just the frame bundle for the universal elliptic curve). The structure group of the cover is $\mathbb{Z}_p^\times$, and so the sheaf corresponds to a character $\pi_1(\mathcal{M}_{\text{Ell}}) \to \mathbb{Z}_p^\times$. The crucial fact for us is that this character is surjective (i.e. "the covering space is connected"). In fact, it is even surjective for a punctured neighborhood of any individual point of $\overline{\mathcal{M}_{\text{Ell}}} \backslash \mathcal{M}_{\text{Ell}}$; in this sense, one might say that "the covering space is as nontrivial as possible".

Dieudonné crystals

Whereas classical Lie theory is governed by the diagram

$$\mathsf{LieGrp} \xrightarrow{H \mapsto \widetilde{H}} \mathsf{LieGrp}_{\text{s.c.}} \underset{\text{BCH}}{\overset{\text{Lie}}{\underset{\sim}{\rightleftarrows}}} \mathsf{LieAlg},$$

the theory of abelian varieties is (p-locally) governed by

$$\mathsf{AbVar}_R \xrightarrow{\Gamma \mapsto \Gamma_e^\wedge} \mathsf{FGrp}_R \underset{\mathcal{G}}{\overset{D}{\underset{\sim}{\rightleftarrows}}} \mathsf{DieuMod}_R.$$

The functor D takes a formal group to its group of *p-typical formal curves* (instead of just tangent vectors), which comes along with certain natural actions F, V, and $[r]$ for $r \in R$. Hence, Dieudonné modules live over the *Cartier algebra*, the quotient of the noncommutative polynomial ring $R\langle V, F\rangle$ by the universal relations. Over a perfect field k of characteristic p, this affords a striking classification: for every height $h \in [1, \infty]$, $\mathsf{FGrp}_k^{\text{ht}=h}/\text{iso.} \cong H^1(\text{Gal}(\overline{k}/k), \text{Aut}(\Gamma_h))$ (so there exist formal groups of every height, and moreover if $k = \overline{k}$ then height is a complete isomorphism invariant).

In order to lift this story to formal neighborhoods (potentially of mixed characteristic), we'd like to consider *families* of Dieudonné modules. This leads us to the notion of a *Dieudonné crystal*, which is a suitably compatible collection of Dieudonné module-like quasicoherent sheaves on the *crystalline site* of a scheme, an object of which consists roughly of an open subscheme along with a nilpotent thickening. Over any scheme on which p is locally nilpotent, these end up being equivalent to the related category of *p-divisible groups*. While the crystalline site is quite large, a Dieudonné crystal is actually completely described by a single *FV-quasicoherent sheaf* equipped with either a suitable *connection* or a *Hodge structure*. A posteriori, one can compute that the Dieudonné crystal of a p-divisible group G is the dual of $(PH^1_{\text{dR}}(\widetilde{G}), \nabla^{\text{GM}})$, the primitives in the first de Rham cohomology of any lift to mixed characteristic along with its Gauss-Manin connection.

This applies to our topological story as follows. Analogously to Lie groups, a p-divisible group is always an extension by its formal component of its étale quotient. And unlike formal groups, the height of a p-divisible group is constant under base change. Now, the height of the p-divisible group associated to E_n is n, which implies (as we've already pointed out) that an isomorphism between the pullbacks of the formal groups associated to E_1 and E_2 must take place over a point where the p-divisible group associated to E_2 is a nontrivial extension, i.e. over the punctured formal neighborhood. Thus, there should be an obstruction theory controlling the impossibility of extending the isomorphism over the origin. As $E_{n-1} \otimes E_n$ lives over an $(n-1)$-dimensional punctured formal disk, Dieudonné modules should give far more traction than these enormous rigid-analytic fundamental groups when we pass to chromatic height n.