3 Homotopy (co)limits and n-excisive functors (Aaron Mazel-Gee)

Contents

3.1 Homotopy (co)limits

In this section, we will work in a bicomplete simplicial model category $\mathcal{C}$. That is:

- *(bicomplete)* $\mathcal{C}$ has (small and sequential) limits and colimits.
- *(simplicial)* $\mathcal{C}$ is simplicially enriched, i.e. hom objects are simplicial sets instead of just sets.
- *(model)* $\mathcal{C}$ is a model category.
- *[convention]* By convention, implicit in this terminology is that $\mathcal{C}$ is simplicially bitensored, meaning simplicially tensored and cotensored, meaning that for $X \in \mathcal{C}$ and $S \in \mathtt{sSet} = \mathtt{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$ we have $X \otimes S \in \mathcal{C}$ and $\mathrm{Hom}(S, X) = X^S \in \mathcal{C}$ satisfying the usual exponentiation rules.

Of course, `Top` (or really `sSet`) is the primordial example.

People usually begin with homotopy colimits and then just say "the story of homotopy limits is dual" for the sake of brevity. So that they don't get the short end of the stick yet again, we'll instead begin with homotopy limits and then briefly outline what one needs to change to dualize to homotopy colimits.

3.1.1 Homotopy limits

The motivation for homotopy limits is that ordinary limits of diagrams aren't homotopy invariant. The standard example is that the morphism of corners $(* \to X \leftarrow PX) \longrightarrow (* \to X \leftarrow *)$ consists of equivalences (where PX denotes a based pathspace), but taking limits gives $\Omega X \to *$. Of course, since we're taking a pullback, it's probably a good idea to demand that our maps be fibrations. Indeed, a homotopy limit will represent "homotopy-coherent systems of maps to a diagram", so it'd make sense to want to be able to lift homotopies back through the morphisms in our diagram. So for an arbitrary corner $(Y \xrightarrow{f} X \xleftarrow{g} Z)$, we make the ad hoc definition

$$\mathrm{holim}(Y \xrightarrow{f} X \xleftarrow{g} Z) = \lim(P_f \to X \leftarrow P_g) \cong \{(y, \alpha, z) \in Y \times X^I \times Z : f(y) = \alpha(0), \alpha(1) = z\},$$

where $P_f = \{(y, \omega) \in Y \times X^I : f(y) = \omega(0)\}$ is the usual pathspace construction turning f into a fibration.

To test this, let's consider the diagram $(X \xrightarrow{f} Y \xrightarrow{g} Z)$. This time, to get the holim we might think that we should take

$$\lim(P_f \to P_g \to Z) \cong \{(x, \alpha, y, \beta, z) \in X \times Y^I \times Y \times Z^I \times Z : f(x) = \alpha(0), \alpha(1) = y, g(y) = \beta(0), \beta(1) = z\}.$$

But in fact, there's the hidden morphism $X \xrightarrow{gf} Z$ that we didn't write down, and so instead we might think that we should require a path $\gamma \in Z^I$ from $gf(x)$ to z too; indeed, to give a canonical and robust definition we really have no choice. But this doesn't give somethink equivalent! On the other hand, the fact that $gf = g \circ f$ suggests that in order for our holim to be encoding "homotopy-coherence", we should furthermore require a "higher homotopy" $\delta \in Z^{(\Delta^2)}$ witnessing a homotopy $g(\alpha) \cdot \beta \simeq \gamma$ rel endpoints.

And this is what leads us to the correct general definition. First, we have a few preliminary definitions.

Definition 1. The *cosimplicial indexing category*, denoted by $\mathbf{\Delta}$, is the category whose objects are the nonempty finite ordered sets $[n] = \{0, \ldots, n\}$ and whose morphisms are (weakly) increasing maps. (We can also consider $[n]$ as a poset and hence as a category, and then $\mathbf{\Delta}$ is a full subcategory of $\mathtt{Cat}$ in the obvious way.) The morphisms $d^i : [n] \to [n+1]$ (for $0 \le i \le n+1$; these are the *codegeneracy maps*) and $s^i : [n] \to [n-1]$ (for $0 \le i \le n-1$; these are the *coface maps*) generate all the morphisms in $\mathbf{\Delta}$.

Definition 2. A *cosimplical object* in $\mathcal{C}$ is a functor $X : \mathbf{\Delta} \to \mathcal{C}$. These form the functor category $\mathsf{c}\mathcal{C} = \mathcal{C}^{\mathbf{\Delta}}$.

Definition 3. Given a cosimplicial object $X \in \mathsf{c}\mathcal{C}$, its *corealization* (or *totalization*) is the object of $\mathcal{C}$ given by

$$\mathrm{Tot}(X) = \mathrm{eq}\left(\prod_{[n]} (X_n)^{(\Delta^n)} \rightrightarrows \prod_{[s] \xrightarrow{\varphi} [t]} (X_t)^{(\Delta^s)} \right),$$

where the arrows are induced by the maps $\Delta^s \to X_s \xrightarrow{X(\varphi)} X_t$ and $\Delta^s \xrightarrow{\varphi_*} \Delta^t \to X_t$.

So a point $x \in \mathrm{Tot}(X)$ should be thought of as the following data:

- a point $x_0 : \Delta^0 \to X_0$;
- a path $x_1 : \Delta^1 \to X_1$ connecting the two images of x_0, whose composition to X_0 is $\Delta^1 \xrightarrow{s^0} \Delta^0 \xrightarrow{x_0} X_0$;
- a triangle $x_2 : \Delta^2 \to X_2$ interpolating between the three images of x_1 (and hence between the three images of x_0), whose two compositions to X_1 are $\Delta^2 \xrightarrow{s^i} \Delta^1 \xrightarrow{x_1} X_1$;
- a tetrahedron $x_3 : \Delta^3 \to X_3$ interpolating between the four images of x_2 (and hence between the six images of x_1, and hence between the four images of x_0), whose three compositions to X_2 are $\Delta^3 \xrightarrow{s^i} \Delta^2 \xrightarrow{x_2} X_2$;
- etc.

Thus, x picks out a point in X_0, along with all possible higher homotopies between its $(n+1)$ images in X_n (for all n), such that an n-dimensional homotopy (i.e. a homotopy parametrized by Δ^n) can only be nondegenerate in X_n and above.

Remark 1. In good (but not all) cases, a morphism in $\mathsf{c}\mathcal{C}$ consisting of weak equivalences induces an equivalence of corealizations. (A sufficient condition is for both the source and target to be *Reedy fibrant*.)

Remark 2. Via the unique maps $\Delta^n \to \Delta^0$, we get a morphism of diagrams

$$\left(\prod_{[n]} X_n \rightrightarrows \prod_{[s]\to[t]} X_t \right) = \left(\prod_{[n]} (X_n)^{(\Delta^0)} \rightrightarrows \prod_{[s]\to[t]} (X_t)^{(\Delta^0)} \right) \longrightarrow \left(\prod_{[n]} (X_n)^{(\Delta^n)} \rightrightarrows \prod_{[s]\to[t]} (X_t)^{(\Delta^s)} \right),$$

which induces a map of equalizers. But the equalizer of the former is just $\lim X = \mathrm{eq}(X_0 \rightrightarrows X_1)$, so we get a canonical map $\lim X \to \mathrm{Tot}(X)$. In the case that $\mathcal{C} = \mathtt{Top}$, this takes a point $x \in \mathrm{eq}(X_0 \rightrightarrows X_1) \subset X_0$ to the point $x \in X_0$, along with the constant path at $d^0(x) \in X_1$, along with the constant triangle at $d^0(d^0(x)) = d^1(d^0(x)) \in X_2$, etc.

We are now ready to return to the original problem of constructing holims.

Definition 4. Let $\mathcal{I}$ be any small indexing category, and let $D : \mathcal{I} \to \mathcal{C}$ be a diagram. We define the *cosimplicial replacement* of D, denoted $\mathrm{crep}(D) \in \mathtt{c}\mathcal{C}$, by setting

$$\mathrm{crep}(D)_n = \prod_{i_0 \to \cdots \to i_n} D(i_n).$$

The coface and codegeneracy maps are "the only thing they could be". (It's clear if you write it out. Maps to a product are determined by what they do on each factor, and everything but d^n ends up being described by identity maps anyways (while changing the indexing tuples of morphisms, of course).)

Definition 5. Let $\mathcal{I}$ be any small indexing category, and let $D : \mathcal{I} \to \mathcal{C}$ be a diagram. We define the *homotopy limit* of D to be

$$\mathrm{holim}_{\mathcal{I}} D = \mathrm{Tot}(\mathrm{crep}(D)).$$

So a point of $\mathrm{holim}_{\mathcal{I}} D$ is a collection of maps $\Delta^n \to D(i_n)$, one for each n-chain of morphisms $i_0 \to \cdots \to i_n$ in $\mathcal{I}$, which provide a compatible system of (higher) homotopies between the various ways of arriving at the object $D(i_n)$.

Remark 3. Unwinding the definitions, we see that we can write this as

$$\mathrm{holim}\, D = \mathrm{eq}\left(\prod_i (D(i))^{B(\mathcal{I}_{\downarrow i})^{op}} \rightrightarrows \prod_{s \to t} (D(t))^{B(\mathcal{I}_{\downarrow s})^{op}} \right).$$

(Recall that for any category $\mathcal{D}$, we define the *classifying space* by $B\mathcal{D} = |N\mathcal{D}|$, where $N\mathcal{D} \in \mathtt{sSet}$ is given by $(N\mathcal{D})_n = \{i_n \leftarrow \cdots \leftarrow i_0\}$. We'll define these symbols in the next section.) In this context, the map $\mathrm{holim}\, D \to \mathrm{hocolim}\, D$ is induced by the unique maps from the classifying spaces to Δ^0. This is a nice point of view, since it illustrates how we need more fat mapping into $D(i)$ depending on the size and shape of the overcategory $\mathcal{I}_{\downarrow i}$.

Exercise 1. Show that this gives our ad hoc constructions when applied to the diagrams $(Y \to X \leftarrow Z)$ and $(X \to Y \to Z)$.

Exercise 2. Make precise the claim that "the hocolim corepresents homotopy-coherent systems of maps to a diagram" (from a single object). Note that the homotopies themselves will be encoded in such a datum.

Remark 4. From Remark 2, we have a map $\lim D = \mathrm{eq}(\mathrm{crep}(D)_0 \rightrightarrows \mathrm{crep}(D)_1) \to \mathrm{Tot}(\mathrm{crep}(D)) = \mathrm{holim}\, D$. This is precisely the map guaranteed by the on-the-nose coherent (and hence homotopy-coherent) system of maps to the diagram from its limit. Of course, the homotopies are all constant here.

Lastly, here are a few comparison results for holims.

Proposition 1. *If $D, D' : \mathcal{I} \to \mathcal{C}$ both consist of fibrant objects and $D \to D'$ is an objectwise equivalence, then* $\mathrm{holim}\, D \xrightarrow{\sim} \mathrm{holim}\, D'$. *(Note that all objects of* `Top` *are fibrant!)*

Proposition 2. *If $D : \mathcal{I} \to \mathcal{C}$, J is another small indexing diagram, and $u : \mathcal{J} \to \mathcal{I}$ is a functor, then the induced map* $\mathrm{holim}_{\mathcal{I}} D \to \mathrm{holim}_{\mathcal{J}} u^* D$ *is an equivalence if u is* homotopy initial *(a/k/a* homotopy cofinal*, a/k/a* homotopy left cofinal*), i.e. for every object $i \in \mathcal{I}$, the category $\mathcal{J} \times_{\mathcal{I}} (\mathcal{I}_{\downarrow i})$ is nonempty and contractible.*

Corollary 1. *If $\mathcal{I}$ has an initial object a, then for every diagram $D : \mathcal{I} \to \mathcal{C}$, the map $D(a) \cong \lim_{\mathcal{I}} D \to$* $\mathrm{holim}_{\mathcal{I}} D$ *is an equivalence.*

3.1.2 Homotopy colimits

The story of hocolims is dual to the story of holims, so we'll simply outline the dual exposition.

- The standard counterexample to the homotopy invariance of colimits is that the morphism of corners $(CX \leftarrow X \rightarrow CX) \longrightarrow (* \leftarrow X \rightarrow *)$ consists of equivalences, but taking colimits gives $\Sigma X \rightarrow *$. This time, we'll want to replace our maps by cofibrations; indeed, a hocolim is meant to corepresent "homotopy-coherent systems of maps off a diagram", so it'd make sense to want to be able extend homotopies along the morphisms in our diagram. Once again, we'll run into subtleties with the diagram $(X \rightarrow Y \rightarrow Z)$, which will make us realize that we need to take compositions into account.
- A *simplical object* in $\mathcal{C}$ is a functor $X : \mathbf{\Delta}^{\mathbf{op}} \rightarrow \mathcal{C}$. These form the functor category $\mathtt{s}\mathcal{C} = \mathcal{C}^{\mathbf{\Delta}^{\mathbf{op}}}$.
- A simplicial object $X \in \mathtt{s}\mathcal{C}$ has a (*geometric*) *realization*, which is the object of $\mathcal{C}$ defined as

$$|X| = \operatorname{coeq}\left(\coprod_{[s]\rightarrow[t]} X_t \otimes \Delta^s \rightrightarrows \coprod_{[n]} X_n \otimes \Delta^n\right) = \left(\coprod_{[n]} X_n \otimes \Delta^n\right) \Bigg/ \begin{matrix}(d_i x, t) \sim (x, d^i t)\\ (s_i x, t) \sim (x, s^i t)\end{matrix}$$

  (whenever the last expression makes sense in $\mathcal{C}$).
- A morphism in $\mathtt{s}\mathcal{C}$ consisting of weak equivalences induces an equivalence of realizations when the source and target are *Reedy cofibrant*.
- We get a morphism $|X| \rightarrow \operatorname{colim} X$ by projecting away the simplices in the coequalizer diagram. In the case that $X \in \mathtt{sSet} \subset \mathtt{sTop}$, this is just the π_0 map.
- The *simplicial replacement* of a diagram $D : \mathcal{I} \rightarrow \mathcal{C}$ is the simplicial object $\operatorname{srep}(D) \in \mathtt{s}\mathcal{C}$ defined by

$$\operatorname{srep}(D)_n = \coprod_{i_0 \leftarrow \cdots \leftarrow i_n} D(i_n).$$

  (This is reasonable notation since $\operatorname{srep}(D)$ is a simplicial object so it's a functor off $\mathbf{\Delta}^{\mathbf{op}}$, so the subscripts above correspond with the elements of $[n] \in \mathbf{\Delta}^{\mathbf{op}}$ in a way compatible with the morphisms.) Then, we define the *homotopy colimit* of D to be

$$\operatorname{hocolim}_{\mathcal{I}} D = |\operatorname{srep}(D)|.$$

  So to obtain $\operatorname{hocolim}_{\mathcal{I}} D$, we gluing together a bunch of objects of the form $D(i_n) \otimes \Delta^n$, one for each n-chain of morphisms $i_0 \leftarrow \cdots \leftarrow i_n$. This can be rewritten as

$$\operatorname{hocolim} D = \operatorname{coeq}\left(\coprod_{s\rightarrow t} D(s) \otimes B(\mathcal{I}_{t\downarrow})^{op} \rightrightarrows \coprod_{i} D(i) \otimes B(\mathcal{I}_{i\downarrow})^{op}\right);$$

  now, we're fattening $D(i)$ according to the size and shape of the undercategory $\mathcal{I}_{i\downarrow}$.
- If $D, D' : \mathcal{I} \rightarrow \mathcal{C}$ both consist of cofibrant objects and $D \rightarrow D'$ is an objectwise equivalence, then $\operatorname{hocolim} D \xrightarrow{\sim} \operatorname{hocolim} D'$. If $\mathcal{C} = \mathtt{Top}$, then this is true without the cofibrancy condition (which is not to say that all objects of $\mathtt{Top}$ are cofibrant!).
- If $D : \mathcal{I} \rightarrow \mathcal{C}$, $\mathcal{J}$ is another small indexing diagram, and $u : \mathcal{J} \rightarrow \mathcal{I}$ is a functor, then the induced map $\operatorname{hocolim}_{\mathcal{J}} u^* D \rightarrow \operatorname{hocolim}_{\mathcal{I}} D$ is an equivalence if u is *homotopy terminal* (a/k/a *homotopy final*, a/k/a *homotopy right cofinal*), i.e. for every object $i \in \mathcal{I}$, the category $\mathcal{J} \times_{\mathcal{I}} (\mathcal{I}_{i\downarrow})$ is nonempty and contractible. As a special case, if $\mathcal{I}$ has a terminal object z, then for every diagram $D : \mathcal{I} \rightarrow \mathcal{C}$, the map $\operatorname{hocolim}_{\mathcal{I}} D \rightarrow \operatorname{colim}_{\mathcal{I}} D \cong D(z)$ is an equivalence.

3.1.3 An aside on the derived functor perspective

Suppose we have a diagram $D : \mathcal{I} \to \mathcal{C}$. Write QD for the diagram obtained by replacing each $D(i)$ with $\mathrm{hocolim}_{\mathcal{I}_{\downarrow i}} u_i^* D$, where $u_i : \mathcal{I}_{\downarrow i} \to \mathcal{I}$ is the evident functor; note that there's a natural equivalence $QD \xrightarrow{\sim} D$ since $(i \xrightarrow{\mathrm{id}} i) \in \mathcal{I}_{\downarrow i}$ is a terminal object. Then $\mathrm{colim}\, QD \cong \mathrm{hocolim}\, D$, and moreover $\mathcal{C}^{\mathcal{I}}(QD, D')$ is naturally homeomorphic to the space of homotopy-coherent maps $D \to D'$ for any diagram $D' : \mathcal{I} \to \mathcal{C}$. Thus, we've found a replacement diagram which corepresents homotopy-coherent maps to other diagrams (instead of to a single object, which is the special case obtained by viewing a single object as a constant diagram).

In fact, it turns out that there's a fibrant model structure on $\mathcal{C}^{\mathcal{I}}$ (so weak equivalences and fibrations are checked objectwise) in which Q is a functorial cofibrant replacement functor. There's a Quillen pair $\mathrm{colim} : \mathcal{C}^{\mathcal{I}} \rightleftarrows \mathcal{C} : const.$, and since colim is the left adjoint, we obtain its (left-)derived functor by applying it to cofibrant replacements. In this sense, one can say that "homotopy colimit is the derived functor of colimit". Of course, the dual story is true for homotopy limits. Also, this should alert us to the fact that what we've given is only one model for the homotopy (co)limit; indeed, we could've used any (co)fibrant replacements.

Example 1. Let $\mathbb{N}$ be the natural numbers, considered as poset and hence as a category. Then $\mathcal{C}^{\mathbb{N}}$ is the diagram category of directed sequences. Given $D \in \mathcal{C}^{\mathbb{N}}$, we can form the *mapping telescope replacement diagram* by

$$D_{tel}(n) = \mathrm{coeq}\left(\coprod_{i=1}^{n-1} D(i) \rightrightarrows \coprod_{i=1}^{n} D(i) \otimes I \right).$$

Clearly there's an equivalence $D_{tel} \xrightarrow{\sim} D$, and one can check that D_{tel} is cofibrant (i.e. it satisfies the left lifting property against trivial fibrations). So this is indeed a cofibrant replacement, and hence $\mathrm{colim}_{\mathbb{N}} D_{tel} \simeq \mathrm{hocolim}_{\mathbb{N}} D$. This is a much smaller model than the one above, and indeed it's the one that we should be keeping in mind.

The bar construction (which we probably won't actually discuss, but which is outlined below) provides a particularly nice cofibrant replacement functor. Roughly, the idea is to recognize the colimit as a tensor product and then mimic the algebraic bar construction. (Recall that the realization of the 2-sided bar construction is (usually) the (left-)derived tensor product.)

3.1.4 An aside on the $\mathcal{I}$-module perspective

There is an incredibly slick way to write down (and generalize) everything we've said so far about homotopy (co)limits, due to Hollender and Vogt. Of course, being slick, this has the advantage that certain facts become much cleaner and clearer and it's generally easier to prove things this way, but on the flipside it has the disadvantage that there's so much wrapped up in the notation that it can be somewhat daunting to unwind.

Definition 6. Given diagrams $X : \mathcal{I} \to \mathcal{C}$ and $W : \mathcal{I}^{op} \to \mathcal{C}$, we define their *tensor product* to be

$$W \otimes_{\mathcal{I}} X = \mathrm{coeq}\left(\coprod_{s \to t} W_t \otimes X_s \rightrightarrows \coprod_i W_i \otimes X_i \right).$$

(This is an example of a *coend.*) We think of X as a left $\mathcal{I}$-module and W as a right $\mathcal{I}$-module.

Our primary reason for doing this is the following. If we denote by $* : \mathcal{I}^{op} \to \mathcal{C}$ the constant diagram at the terminal object of $\mathcal{C}$, then $* \otimes_{\mathcal{I}} X \cong \mathrm{colim}\, X$. However, this also subsumes a few other constructions we've seen.

Example 2. If $X \in \mathtt{sTop}$ and $\Delta^* : \mathbf{\Delta} \to \mathtt{Top}$ is the standard inclusion, then $X \otimes_{\mathbf{\Delta}} \Delta^* \cong |X|$.

Example 3. If $D : \mathcal{I} \to \mathcal{C}$ and $B(\mathcal{I}_{-\downarrow})^{op} : \mathcal{I}^{op} \to \mathcal{C}$ is given by $i \mapsto B(\mathcal{I}_{i\downarrow})^{op}$, then $B(\mathcal{I}_{-\downarrow})^{op} \otimes_{\mathcal{I}} D \cong \operatorname{hocolim}_{\mathcal{I}} D$.

Recall that if R is a ring with $M \in \texttt{Mod-}R$ and $N \in R\texttt{-Mod}$, then $M \otimes_R N = \operatorname{coeq}(M \otimes R \otimes N \rightrightarrows M \otimes N)$. Via the unit map $\mathbb{Z} \to R$, this becomes the 1-truncation of the *two-sided bar construction*, a simplicial abelian group $B_\bullet(M, R, N)$ defined by $B_n(M, R, N) = M \otimes R^{\otimes n} \otimes N$. Under mild cofibrancy hypotheses, $|B_\bullet(M, R, N)| = M \otimes^{\mathbb{L}}_R N$. Now that we have identified colimits as tensor products, we can repeat this same story.

Definition 7. Given diagrams $X : \mathcal{I} \to \mathcal{C}$ and $W : \mathcal{I}^{op} \to \mathcal{C}$, we define their *two-sided bar construction* to be $B_\bullet(W, \mathcal{I}, X) \in \texttt{s}\mathcal{C}$, given by

$$B_n(W, \mathcal{I}, X) = \coprod_{i_0 \leftarrow \cdots \leftarrow i_n} W(i_0) \otimes X(i_n)$$

(with completely obvious face and degeneracy maps, except that d_0 and d_n are only mostly obvious). This is covariantly functorial in both X and W.

Now, $B_\bullet(*, \mathcal{I}, X) \cong \operatorname{srep}(X)$. If we write $B(W, \mathcal{I}, X) = |B_\bullet(W, \mathcal{I}, X)|$, then there's a natural map $B(W, \mathcal{I}, X) \to \operatorname{coeq}(B_1(W, \mathcal{I}, X) \rightrightarrows B_0(W, \mathcal{I}, X)) = W \otimes_{\mathcal{I}} X$, which becomes the natural map $\operatorname{hocolim} X \to \operatorname{colim} X$ when we set $W = *$. As in the algebraic setting, the two-sided bar construction is a "derived tensor product".

Example 4. $B_\bullet(*, \mathcal{I}, *) = N(\mathcal{I}^{op})$. Since $B\mathcal{I} = |N\mathcal{I}|$, this should be thought of as analogous to the "freeification" of a group action on a single point.

Remark 5. If we consider $X : \mathcal{I} \to \mathcal{C}$ instead as $X' : \mathcal{I}^{op} \to \mathcal{C}^{op}$, then $B(*, \mathcal{I}, X) \cong B(X', \mathcal{I}, *)$. For this reason, the definition of simplicial replacement given above depended on a few choices, but the resulting hocolims are well-defined up to natural isomorphism.

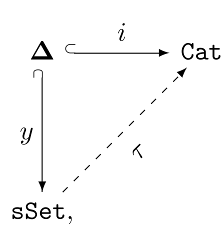
Remark 6. This theory of modules extends quite easily to bimodules. With this language, it takes a single line to prove that our rewriting of the hocolim in terms of classifying space is equivalent to the original definition. This also affords us a simple definition of the cofibrant replacement functor $Q : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}^{\mathcal{I}}$.

There is a dual story of course, where we have a "function space" construction which is adjoint to the tensor product construction; this admits a homotopical replacement as well, which is called the *cobar construction*.

3.1.5 An aside on the ∞-category perspective

Everything one can say about simplicial model categories can be ported over to quasicategories (a/k/a ∞-categories, a/k/a $(\infty, 1)$-categories, a/k/a inner Kan complexes), and some of the future talks will use this latter framework. The most important thing for us to know is that ho(co)lims in a simplicial model category correspond to ordinary (co)lims in quasicategories. Indeed, one might say that "∞-categories don't even know what 'on-the-nose' means".

We give only slightly more details. In the case of unenriched categories, we have the left Kan extension diagram



in which τ is uniquely determined by the facts that it makes the diagram commute (i.e. $\tau(\Delta^n) = [n]$) and that it commutes with colimits. (This follows from the general fact that every object of the presheaf category $\mathtt{sSet} = \mathtt{Set}^{\mathbf{\Delta}^{op}} = PreSh_{\mathtt{Set}}(\mathbf{\Delta})$ is a colimit of representable presheaves on $\mathbf{\Delta}$.) This determines an adjunction $\tau : \mathtt{sSet} \rightleftarrows \mathtt{Cat} : N$, where N is the *nerve* functor. In the case of simplicially enriched categories, there is an analogous inclusion $i_{\mathbf{\Delta}} : \mathbf{\Delta} \to \mathtt{Cat}_{\mathbf{\Delta}}$ which lifts i in the sense that $\mathrm{Hom}_{i_{\mathbf{\Delta}}([n])}(j,k)$ is contractible if $j \leq k$ and empty otherwise. Again, we have a left Kan extension diagram

$$\begin{array}{ccc} \mathbf{\Delta} & \xhookrightarrow{i_{\mathbf{\Delta}}} & \mathtt{Cat}_{\mathbf{\Delta}} \\ {\scriptstyle y}\big\downarrow & \overset{\mathfrak{C}}{\nearrow} & \\ \mathtt{sSet}, & & \end{array}$$

which determines an adjunction $\mathfrak{C} : \mathtt{sSet} \rightleftarrows \mathtt{Cat}_{\mathbf{\Delta}} : N_{hc}$, where N_{hc} is the *homotopy-coherent nerve* functor. In fact, this entire construction lifts the previous one in the sense that the diagram

$$\begin{array}{ccc} \mathtt{sSet} & \underset{N_{hc}}{\overset{\mathfrak{C}}{\rightleftarrows}} & \mathtt{Cat}_{\mathbf{\Delta}} \\ \Big\| & & \uparrow \ \big\downarrow {\scriptstyle \pi_0} \\ \mathtt{sSet} & \underset{N}{\overset{\tau}{\rightleftarrows}} & \mathtt{Cat} \end{array}$$

commutes.

Now, the adjoint pair $\mathfrak{C} \dashv N_{hc}$ is in fact a Quillen equivalence (when one endows $\mathtt{Cat}_{\mathbf{\Delta}}$ with the *Bergner model structure*, in which weak equivalences are those functors that are essentially surjective and homotopically fully faithful (with respect to the Quillen model structure), and $\mathtt{sSet}$ with the *Joyal model structure*, in which the weak equivalences are those that become such in $\mathtt{Cat}_{\mathbf{\Delta}}$). Moreover, quasicategories are precisely the fibrant objects of $\mathtt{sSet}$ (namely, their inner horns can be filled).

Thus, given a simplicial model category $\mathcal{C}$, one takes the full simplicial subcategory $\mathcal{C}^0$ of bifibrant objects and then applies N_{hc} to get a fibrant simplicial set, i.e. a quasicategory. Note that $\mathcal{C}^0$ is fibrant by the "corner axioms" for a simplicial model category (which ensures that $N_{hc}(\mathcal{C}^0)$ is fibrant) and indeed the inclusion $\mathcal{C}^0 \to \mathcal{C}$ is a fibrant replacement. (There is a technical condition which ensures that a quasicategory actually corresponds to a simplicial model category.)

For a simplicial model category $\mathcal{C}$ and a diagram $D : \mathcal{I} \to \mathcal{C}$, we therefore obtain the diagram

$$\begin{array}{ccc} N(\mathcal{I}) = N_{hc}(\mathcal{I}) & \xrightarrow{N_{hc}(D)} & N_{hc}(\mathcal{C}) \\ & \searrow & \big\uparrow \wr \\ & & N_{hc}(\mathcal{C}^0). \end{array}$$

Under this correspondence, every homotopy (co)limit diagram in $\mathcal{C}$ yields an ∞-categorical (co)limit diagram in $N_{hc}(\mathcal{C}^0)$, and conversely every ∞-categorical (co)limit diagram in $N_{hc}(\mathcal{C}^0)$ rigidifies to a homotopy (co)limit diagram in $\mathcal{C}$. (Recall that a limit of a functor of ∞-categories is a terminal object in the ∞-category of cones on the induced diagram; a terminal object is an object with a contractible space of maps from any other object, and the ∞-category of these is always empty or contractible. Analogously for colimits.)

3.1.6 Important examples for Goodwillie calculus: cubes and orbits

In Goodwillie calculus, the main ho(co)lims we'll care about are those of cubical and "punctured" cubical diagrams, as well as hocolims of directed diagrams. We'll also care about homotopy orbits of a group action (a homotopy colimit construction). Let's address these each in turn.

Definition 8. Let S be a finite set with $|S| = n$. Then a functor $X : \mathscr{P}(S) \to \mathcal{C}$ is called an *n-cube* in $\mathcal{C}$. Here $\mathscr{P}(S)$ is the power set of S, considered as a poset and hence as a category. We write $\mathscr{P}_0(S) = \mathscr{P}(S) - \{\emptyset\}$ and $\mathscr{P}_1(S) = \mathscr{P}(S) - \{S\}$; diagrams of these shapes are called *punctured* cubes. Often we will take $S = \underline{n} = \{1, \dots, n\}$.

So 0-cubes are just objects, 1-cubes are just morphisms, and 2-cubes are just commutative squares. It's sometimes helpful to think of an $(n+1)$-cube as a morphism of n-cubes, or more generally of an $(m+n)$-cube as an m-cube of n-cubes.

Note that cubes have initial and terminal objects, so their (homotopy) (co)limits are uninteresting. This is why we introduced punctured cubes, though.

Definition 9. An n-cube $X : \mathscr{P}(S) \to \mathcal{C}$ is called *cartesian* if the natural map $X(\emptyset) \to \operatorname{holim}_{\mathscr{P}_0(S)} X$ is an equivalence, and *cocartesian* if the natural map $\operatorname{holim}_{\mathscr{P}_1(S)} X \to X(S)$ is an equivalence. More generally, X is called *k-cartesian* if the former map is k-connected, and *k-cocartesian* if the latter map is k-connected.

Definition 10. An n-cube $X : \mathscr{P}(S) \to \mathcal{C}$ is called *strongly cocartesian* if the restriction $X|_{\mathscr{P}(T)} : \mathscr{P}(T) \to \mathcal{C}$ is cocartesian for all $T \subseteq S$ with $|T| \geq 2$.

So vacuously, all 0- and 1-cubes are strongly cocartesian. For $n \geq 2$, any strongly cocartesian n-cube $X : \mathscr{P}(S) \to \mathcal{C}$ is determined up to equivalence by $X(T)$ for $T \subset S$ with $|T| \leq 1$: the rest of the cube can be obtained by homotopy pushouts. If we assume our maps $X(\emptyset) \to X(\{i\})$ for $i \in S$ are cofibrations, then we can just take ordinary pushouts.

Next, we've already discussed directed hocolims but there is a fact about them that will be needed in future talks, so we state it now for the record.

Proposition 3 (HTT, 7.3.4.7)**.** *If $\mathcal{C} = \texttt{Top}$ or $\mathcal{C} = \texttt{Spectra}$ (or more generally $\mathcal{C}$ is any ∞-topos), then finite holims commute with directed hocolims. More precisely, if $\mathcal{I}$ is a finite indexing category and $D \in \mathcal{C}^{\mathcal{I} \times \mathbb{N}} \cong (\mathcal{C}^{\mathcal{I}})^{\mathbb{N}} \cong (\mathcal{C}^{\mathbb{N}})^{\mathcal{I}}$, then*

$$\operatorname{hocolim}_{\mathbb{N}} \left(\operatorname{holim}_{\mathcal{I} \times \{n\}} D \right) \simeq \operatorname{holim}_{\mathcal{I}} \left(\operatorname{hocolim}_{\{i\} \times \mathbb{N}} D \right).$$

In the case that $\mathcal{C} = \texttt{Spectra}$ and $\mathcal{I} = \mathscr{P}(\underline{2})$, this follows simply from the fact that a commutative square of spectra is cartesian iff it is cocartesian; this specializes to the fact that fiber sequences and cofiber sequences are the same thing.

Lastly, let us say a word about homotopy orbits. Suppose $X \in \mathcal{C}$ and G is a finite group (or at least a discrete group; for us, G will actually always be a symmetric group). A G-action on X is the same thing as a functor $a : G \to \mathcal{C}$ landing at X, where G is considered as a one-object category. Then the *homotopy orbits* object is defined by $X_{hG} = \operatorname{hocolim}_G a$.

Remark 7. We always have an object $EG \in \mathcal{C}$, which comes equipped with a G-action. Chasing through the definitions, one can verify that $X_{hG} = (X \otimes EG)_G$. This gives the usual definition of homotopy orbits in $\texttt{Top}$, and specializes to the fact that $*_{hG} = (EG)_G = BG$.

Example 5. In $\texttt{Top}$, we get the explicit model $B\mathbb{Z}/2 = *_{h\mathbb{Z}/2} = (S^\infty)_{\mathbb{Z}/2} = \mathbb{R}P^\infty$. (Incidentally, this is a simple counterexample to the reasonable-sounding claim that the homotopy colimit is equivalent to the ordinary colimit if all the maps are cofibrations, since $*_{\mathbb{Z}/2} = *$. The correct statement is wrapped up in our exposition of the derived functor perspective above.)

This concludes our foray into the world of ho(co)lims.

3.2 n-excisive functors

In what follows, $\mathcal{C}$ and $\mathcal{D}$ will be categories of the sort that we studied in the last section; the ones we'll actually care about in the end are `Top`, $\texttt{Top}_*$, and `Spectra`. We will only consider *homotopy* functors (i.e. functors that preserve equivalences).

Definition 11. A functor $F : \mathcal{C} \to \mathcal{D}$ is called *n-excisive* (or *polynomial of degree at most n*) if whenever X is a strongly cocartesian $(n+1)$-cube in $\mathcal{C}$, $F(X)$ is a cartesian cube in $\mathcal{D}$. (A useful mnemonic is that a polynomial of degree at most n is determined by its values on $n+1$ distinct points.)

Proposition 4. *If F is n-excisive then it is also $(n+k)$-excisive for all $k \geq 0$.*

Proof. Clearly it suffices to prove the statement for $k = 1$. Consider an $(n+2)$-cube X as a morphism $Y \to Z$ of $(n+1)$-cubes. If X is strongly cocartesian, then so are Y and Z. By assumption, this means that $F(Y)$ and $F(Z)$ are cartesian. But this implies that $F(X)$ is cartesian too, by an easy lemma [Calc II, 1.6]. □

At the lowest level, F is 0-excisive iff $F(X) \to F(*)$ is an equivalence for all $X \in \mathcal{C}$. In this case we say that F is *homotopy constant.* F is 1-excisive iff it takes homotopy pushout squares to homotopy pullback squares. Following existing terminology, then, we often simply say *excisive* for 1-excisive. The next proposition illustrates why the terminology makes sense; excision and Mayer-Vietoris are both "locality" axioms.

Proposition 5. *If E is any spectrum and $\mathcal{C} = \texttt{Top}$ or $\mathcal{C} = \texttt{Spectra}$, then the functors $F : \mathcal{C} \to \mathcal{D}$ given by $X \mapsto E \wedge X_+$ and $X \mapsto \Omega^\infty(E \wedge X_+)$ are excisive.*

Proof. Suppose that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

is a cocartesian square in $\mathcal{C}$. Then $W \simeq \operatorname{hocolim}(Y \xleftarrow{f} X \xrightarrow{g} Z)$. We can decompose this hocolim in the obvious way, so that $U = M_f \simeq Y$, $V = M_g \simeq Z$, and $U \cap V = X \times I \simeq X$. This decomposition yields a Mayer-Vietoris sequence for the homology theory E_* (where $E_*(U \cup V) \cong E_* W$ via the assumed equivalence). Note that for either choice of F, we can write $E_* A = \pi_* F(A)$ for any $A \in \mathcal{C}$ (where we mean stable homotopy of spectra or unstable homotopy of spaces). Meanwhile, by the following lemma, there is a long exact sequence in homotopy for a homotopy pullback square which has the exact same shape as a Mayer-Vietoris sequence. Then, applying F to the map $X \to \text{holim} = \operatorname{holim}(F(Z) \to F(W) \leftarrow F(Z))$ induces the morphism of long exact sequences

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & \pi_{n+1}F(Y) \oplus \pi_{n+1}F(Z) & \longrightarrow & \pi_{n+1}F(W) & \longrightarrow & \pi_n F(X) & \longrightarrow & \pi_n F(Y) \oplus \pi_n F(Z) & \longrightarrow & \pi_n F(W) & \longrightarrow & \cdots \\ & & \| & & \| & & \downarrow & & \| & & \| & & \\ \cdots & \longrightarrow & \pi_{n+1}F(Y) \oplus \pi_{n+1}F(Z) & \longrightarrow & \pi_{n+1}F(W) & \longrightarrow & \pi_n \,\text{holim} & \longrightarrow & \pi_n F(Y) \oplus \pi_n F(Z) & \longrightarrow & \pi_n F(W) & \longrightarrow & \cdots, \end{array}$$

and so the five lemma implies that the middle map is an isomorphism. Hence $F(X) \to \text{holim}$ is an equivalence. □

Lemma 1. *If* $A = \operatorname{holim}(C \to D \leftarrow B)$*, then there is a long exact sequence*

$$\cdots \to \pi_n(A) \to \pi_n(B) \oplus \pi_n(C) \to \pi_n(D) \to \pi_{n-1}(A) \to \cdots .$$

Proof. Recall that we consider $A \subset B \times D^I \times C$. Then we have an (honest) pullback diagram

$$\begin{array}{ccc} A & \longrightarrow & D^I \\ \downarrow & & \downarrow \\ B \times C & \longrightarrow & D \times D \end{array}$$

The right vertical map is a fibration with fiber ΩD, so the left vertical map is as well. The long exact sequence in homotopy for this fibration is the desired long exact sequence. □

Corollary 2. *We can take* $E = S$*, and then the functors* $\Sigma^\infty_+ : \mathtt{Top} \to \mathtt{Spectra}$ *and* $\Omega^\infty\Sigma^\infty_+ : \mathtt{Top} \to \mathtt{Top}_*$ *are excisive.*

In fact, we will see in another talk that if $F : \mathcal{C} \to \mathcal{D}$ is 1-excisive and *finitary* (i.e. commutes with filtered holims, a/k/a *continuous*), then there necessarily exist *coefficient spectra* C_0 and C_1 so that either $F(X) \simeq C_0 \vee (C_1 \wedge X)$ or $F(X) \simeq \Omega^\infty(C_0 \vee (C_1 \wedge X))$.

Exercise 3. If E is any spectrum, show that the Bousfield localization functor $L_E : \mathtt{Spectra} \to \mathtt{Spectra}$ is excisive. (Use that L_E preserves (co)fiber sequences, and that a square in spectra is cartesian iff it is cocartesian.) Incidentally, finitary Bousfield localizations are usually called *smashing*; in this case, $C_0 \simeq *$ and $C_1 \simeq L_E S$.

We now work towards a slightly less trivial result.

Lemma 2. *Suppose* $\mathcal{A} = \bigcup_{s \in S} \mathcal{A}_s$ *is a covering of a poset by subposets which are all either concave or convex. Then for any functor* $F : \mathcal{A} \to \mathcal{D}$*, the cube defined by*

$$\begin{array}{rcl} T & \mapsto & \operatorname{holim}\left(F|_{\bigcap_{s\in T} \mathcal{A}_s}\right) \\ \emptyset & \mapsto & \operatorname{holim}(F) \end{array}$$

is cartesian.

This is proved by an easy induction.

Lemma 3. *If* $F : \mathcal{C} \to \mathcal{D}$ *is* n*-excisive, then for any strongly cocartesian* m*-cube* $X : \mathscr{P}(S) \to \mathcal{C}$*, the natural map*

$$F(X(\emptyset)) \to \operatorname{holim}_{\{T \in \mathscr{P}(S) : |S-T| \leq n\}} F(X(T))$$

is an equivalence.

Proof. For $m \leq n$, this is true since the holim of the target is indexed over all of $\mathscr{P}(S)$, which has the initial object $\emptyset$. For $m = n+1$, this is true by definition of n-excisiveness. For $m > n+1$, we prove the statement by induction on m.

Define a cube $Y : \mathscr{P}(S) \to \mathcal{D}$ by setting

$$Y(T) = \operatorname{holim}_{\{U \in \mathscr{P}(S) : U \supset T, |S-U| \leq n\}} F(X(U))$$

for each $T \subset S$. Then there is a morphism of cubes $F(X) \to Y$ given at $T \in \mathscr{P}(S)$ by

$$F(X(T)) \xrightarrow{\sim} \operatorname{holim}_{\{U \in \mathscr{P}(S) : U \supset T\}} F(X(U)) \to Y(T),$$

where the second map is the restriction.

Our goal is to show that this is an equivalence at $T = \emptyset$. By the inductive hypothesis, it is an equivalence at all $T \in \mathscr{P}_0(S)$, so this will follow if both $F(X)$ and Y are cartesian. But $F(X)$ is cartesian since n-excisiveness implies m-excisiveness for all $m \geq n$, and it is routine to verify that Y is cartesian from the previous lemma by taking $\mathcal{A} = \mathscr{P}(S)$ and $\mathcal{A}_s = \{U \in \mathscr{P}(S) : s \in U, |S - U| \leq n\}$. □

Proposition 6. *Suppose $F : \mathcal{C}^r \to \mathcal{D}$ is n_i-excisive in the i^{th} variable. Then $F \circ \Delta : \mathcal{C} \to \mathcal{D}$ is n-excisive, where $n = \sum n_i$.*

Proof. Let $X : \mathscr{P}(S) \to \mathcal{C}$ be a strongly cocartesian cube in $\mathcal{C}$ with $|S| = r > n$. Consider the morphism of cubes given at $U \in \mathscr{P}(S)$ by

$$F \circ \Delta(X(U)) = F(X(U), \ldots, X(U)) \to \operatorname{holim}_{\{(T_1,\ldots,T_r) \in \mathscr{P}(S)^r : T_i \supset U, |S - T_i| \leq n_i\}} F(X(T_1), \ldots, X(T_r)).$$

We will first show that this is an equivalence, and then we will show that the target cube is cartesian.

At any fixed $U \in \mathscr{P}(S)$, we prove that the map is an equivalence by applying the latter lemma r times. To explain, we first define the cube $X' : \mathscr{P}(S - U) \to \mathcal{C}$ by $X'(T') = X(T' \cup U)$, so in particular $X'(\emptyset) = X(U)$. Then the target can be rewritten as

$$\operatorname{holim}_{\{T_1' \in \mathscr{P}(S-U) : |(S-U) - T_1'| \leq n_1\}} \left(\cdots \left(\operatorname{holim}_{\{T_r' \in \mathscr{P}(S-U) : |(S-U) - T_r'| \leq n_r\}} F(X'(T_1'), \ldots, X'(T_r')) \right) \cdots \right),$$

and the canonical map from $F(X(U), \ldots, X(U)) = F(X'(\emptyset), \ldots, X'(\emptyset))$ is an equivalence.

To complete the proof, we apply the former lemma to show that the target cube is cartesian, taking $\mathcal{A} = \{(T_i) \in \mathscr{P}(S)^r : |S - T_i| \leq n_i\}$ and $\mathcal{A}_s = \{(T_1, \ldots, T_r) \in \mathcal{A} : s \in T_i\}$. □

Corollary 3. *For any spectrum E, the functors $X \mapsto E \wedge (X_+^n) = E \wedge (X_+)^{\wedge n}$ and $X \mapsto \Omega^\infty(E \wedge X_+^n)$ off* `Top` *and the functors $X \mapsto E \wedge X^{\wedge n}$ and $X \mapsto \Omega^\infty(E \wedge X^{\wedge n})$ off* `Top`$_*$ *are all n-excisive. (In particular, we can take $E = S$.)*

3.3 The generalized Blakers-Massey theorem

The idea of the Blakers-Massey theorem is as follows. Although homotopy doesn't satisfy excision, it does in the so-called *stable range*. (Indeed, stable homotopy is a homology theory!) Precisely, we have the following statement.

Theorem 1 (Blakers-Massey). *Let $(X; A, B, x)$ be a triad such that $(A, A \cap B)$ is an n-connected relative CW complex (for $n \geq 1$) and $(B, A \cap B)$ is an m-connected relative CW complex. Then $\pi_r(A, A \cap B, x) \to \pi_r(X, B, x)$ is an isomorphism for $1 \leq r < m + n$ and an epimorphism for $r = m + n$.*

(Recall that the n^{th} *relative homotopy* of a pointed pair is defined as homotopy classes of maps from the pointed pair $(D^n, S^{n-1}, *)$.) In the context of Goodwillie calculus, we can view this as giving us a partial answer to the question of comparing k-cartesianness with k-cocartesianness: the Blakers-Massey theorem tells us that these notions coincide in the stable range.

Theorem 2 (Blakers-Massey, take 2). *Let $X : \underline{2} \to$* `Top` *be a square of spaces. If X is cocartesian and $X(\emptyset) \to X(\{i\})$ is k_i-connected, then X is $(k_1 + k_2 - 1)$-cartesian. Dually, if X is cartesian and $X(\{i\}) \to X(\underline{2})$ is k_i-connected, then X is $(k_1 + k_2 + 1)$-cocartesian.*

The first assertion is the one which is equivalent to the previous statement; this is also called the *homotopy excision theorem*, since it's telling us how highly connected the map $X(\emptyset) \to \operatorname{holim}(X(\{2\}) \to X(\underline{2}) \leftarrow X(\{1\}))$ is. Recall that the homotopy of this holim sits in a sort of Mayer-Vietoris sequence, so the homotopy of the source does too up through a certain dimension.

The Blakers-Massey theorem implies the following result, which is really the entire reason for the existence of stable homotopy theory.

Corollary 4 (Freudenthal suspension theorem)**.** *For every n-connected CW complex X, the suspension homomorphism $\pi_r(X) \to \pi_{r+1}(\Sigma X)$ is an isomorphism for $1 \le r \le 2n$ and an epimorphism for $r = 2n+1$.*

The Blakers-Massey theorem is also called the *triad connectivity theorem*, which suggests the appropriate generalization. An $(n+1)$-*ad* is a tuple $(X, \{X_s\}_{s\in S})$ of a space X and n subspaces X_s, where $|S| = n$. This determines an n-cube $\mathcal{X} : \mathscr{P}(S) \to \mathtt{Top}$ by setting $\mathcal{X}(S) = X$ and $\mathcal{X}(S - T) = \bigcap_{s\in T} X_s$ for $T \subsetneq S$. In fact, to give an $(n+1)$-ad is precisely the same thing as to give an n-cube such that:

- all the maps $\mathcal{X}(T) \to \mathcal{X}(S)$ are inclusions of subspaces, and
- $\mathcal{X}(T \cap U) = \mathcal{X}(T) \cap \mathcal{X}(U)$.

Goodwillie provides the following generalization of the Blakers-Massey theorem.

Theorem 3 (generalized Blakers-Massey)**.** *Let $X : \mathscr{P}(S) \to \mathtt{Top}$ be an n-cube of spaces, with $n \ge 1$. If X is strongly cocartesian and $X(\emptyset) \to X(\{s\})$ is k_s-connected for each $s \in S$, then X is k-cartesian with $k = 1 - n + \sum k_s$. Dually, if X is strongly cartesian and $X(S - \{s\}) \to X(S)$ is k_s-connected for each $s \in S$, then X is k-cocartesian with $k = n - 1 + \sum k_s$.*

Remark 8. Of course, the formula for the holim of a punctured n-cube reduced to a particularly nice form when $n = 2$, but it becomes already fairly intractable when $n = 3$. Nevertheless, this gives us a stable range in which our n-cube behaves as if it were both cartesian and cocartesian. Let's see what this buys us.

Recall that, dual to the skeletal filtration of a realization, the totalization of a cosimplicial space $K : \mathbf{\Delta} \to \mathtt{Top}$ admits a map to a tower of fibrations, called the *coskeletal cofiltration*, the q^{th} space of which is

$$\operatorname{cosk}^q(K) = \operatorname{eq}\left(\prod_{\{[n]:n\le q\}} (K_n)^{(\Delta^n)} \rightrightarrows \prod_{\{[s]\to[t]:s,t\le q\}} (K_t)^{(\Delta^s)} \right).$$

If we write $F_q = \operatorname{fib}(\operatorname{cosk}^q(K) \to \operatorname{cosk}^{q-1}(K))$, then taking homotopy yields a first-quadrant spectral sequence of the form $E_1^{p,q} = \pi_p(F_q) \Rightarrow \pi_p(\operatorname{Tot}(K))$ with differentials $d_r^{p,q} : E_r^{p,q} \to E_r^{p-1,q+r}$.

Now, if $K = \operatorname{crep}(D)$ for some diagram $D : \mathcal{I} \to \mathtt{Top}$, then of course $\operatorname{Tot}(K) = \operatorname{holim}_{\mathcal{I}} D$. So in our stable range, $\pi_* \operatorname{holim}_{\mathscr{P}_0(S)} X \cong \pi_*(X(\emptyset))$, so the vertical stripe roughly given by $0 \le p \le k$ (where our cube is k-cartesian) should be thought of as analogous to the partial Mayer-Vietoris sequence for homotopy when $n = 2$. (Note that the homotopy of the spaces in the diagram show up in the F_q. In fact,

$$F_q = \Omega^q \left(\prod_{i_0\to\cdots\to i_q} D(i_q) \right),$$

but unfortunately the signature of our spectral sequence precludes this fact from implying strong convergence.)