Model categories for algebraists, or: What's *really* going on with injective and projective resolutions, anyways?

Aaron Mazel-Gee

Abstract

The theory of model categories was originally introduced by Daniel Quillen, that patron saint of the lush pastures between the metropolis of algebra and the jungle of topology. This framework has since become essential on safari, but it's incredibly useful for all you city slickers too – as an organizational tool and more. I'll give a bit of topological background, show how algebraists secretly use model categories all the time without even realizing it, and then indicate how one can use "generalized spaces" to study chain complexes in non-abelian categories. I'll discuss André-Quillen cohomology as a particular example, and if there's time I may even say a few words about motives and $\mathbb{A}^1$-homotopy theory, crystalline cohomology, or algebraic geometry over $\mathbb{F}_1$. Not suitable for children under 13.

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1 Model Categories in Topology

1.1 `Top` and the definition of a model category

In homotopy theory, we study the category `Top` of topological spaces and continuous maps. Our main tools are homotopy groups and (extraordinary) co/homology groups. However, these all take weak equivalences to isomorphisms. (Recall that a map $f : X \to Y$ is a *weak equivalence* if $f_\# : \pi_*(X, x) \xrightarrow{\cong} \pi_*(Y, f(x))$ for all $x \in X$.) So if we want to be honest with ourselves, we should ask: What category are we *actually* studying?

In a sense, the answer is rather obvious and can be phrased in terms of a universal property: we're studying the "localized category" $\texttt{Top}[W^{-1}]$ of topological spaces with weak equivalences inverted; here, a morphism from X to Y is an equivalence class of "zigzag" $X \to Z_1 \xleftarrow{\sim} Z_2 \to \cdots \xleftarrow{\sim} Z_n \to Y$, where the backwards maps are weak equivalences. However, this is awkward for a number of reasons, not the least of which is that there's no reason that the morphisms from X to Y should even form a set! (In general, this construction may yield a proper class.)

However, we are saved by *Whitehead's theorem*, which says that on CW-complexes, a weak equivalence is actually a homotopy equivalence. Moreover, any space X has a CW-replacement QX. This means that $\mathrm{Hom}_{\mathtt{Top}[W^{-1}]}(X, Y) \cong [QX, QY]$ for *any* spaces X and Y. This is great; for one thing it means that $\mathtt{Top}[W^{-1}]$ really is a category, and moreover it gives us a good handle on how to actually compute its hom-sets.

This characterization of certain "well-behaved" objects leads us to make the following definition.

Definition. A *model structure* on a category $\mathcal{C}$ is a choice of three classes of morphisms C (the *cofibrations*, denoted $\rightarrowtail$), W (the *weak equivalences*, denoted $\xrightarrow{\sim}$), and F (the *fibrations*, denoted $\twoheadrightarrow$). These must satisfy a number of properties, but the most important one for us is that any morphism $X \to Y$ admits functorial factorizations $X \overset{\sim}{\rightarrowtail} Z \twoheadrightarrow Y$ and $X \rightarrowtail W \overset{\sim}{\twoheadrightarrow} Y$. A *model category* is a bicomplete category $\mathcal{C}$ equipped with a model structure. In particular, a model category $\mathcal{C}$ has an initial object $\emptyset$ and a terminal object $*$. Then, an object is called *cofibrant* if the unique map from $\emptyset$ is a cofibration, and is called *fibrant* if the unique map to $*$ is a fibration. In particular, the unique maps $\emptyset \to X \to *$ can be functorially factored as $\emptyset \rightarrowtail QX \overset{\sim}{\twoheadrightarrow} X \overset{\sim}{\rightarrowtail} RX \twoheadrightarrow *$; we call Q the *cofibrant replacement* functor and R the *fibrant replacement* functor.

Of course, `Top` is the canonical example of a model category; the cofibrations and fibrations coincide with their usual definitions. The cofibrant objects include CW-complexes, and all objects are fibrant. Maps can be replaced up to weak equivalence by cofibrations by the *mapping cylinder* construction, and maps can be replaced up to weak equivalence by fibrations by the *mapping path space* construction.

However, it turns out that in *any* model category, there's a notion of a *homotopy* between two maps. Then, we define the *homotopy category* $\mathrm{Ho}(\mathcal{C})$ of a model category $\mathcal{C}$ to be the category whose objects are the bifibrant (i.e. cofibrant and fibrant) objects of $\mathcal{C}$ and whose morphisms are homotopy classes of maps. We have the following general theorem.

Theorem. *If $\mathcal{C}$ is a model category then $\mathcal{C}[W^{-1}] \simeq \mathrm{Ho}(\mathcal{C})$, and for any $X, Y \in \mathcal{C}$,*

$$\mathrm{Hom}_{\mathcal{C}[W^{-1}]}(X, Y) \cong \mathrm{Hom}_{\mathcal{C}}(QX, RY)/\simeq .$$

In particular, this means that $\mathcal{C}[W^{-1}]$ is an ordinary category (without passing to a higher set-theoretic universe). Also, this means that $\mathrm{Ho}(\mathcal{C})$ is actually *independent* of the chosen model structure, and only depends on the class W (and the existence of a model structure extending this datum). Perhaps most importantly, though, this tells us that once we've cofibrantly replaced our source and fibrantly replaced our target, then morphisms in the homotopy category are realized by actual morphisms: zigzags are no longer necessary. (Note that in `Top`, all objects are fibrant, which is why the general statement isn't quite a word-for-word generalization of Whitehead's theorem.)

1.2 Simplicial sets

Before seeing how this appears in algebra, we discuss one other closely related model category which will show up in what follows. Recall that a *Δ-complex* is a space which is nicely decomposed as a bunch of simplices (vertices, edges, triangles, tetrahedra, etc.). For instance, the unit interval can be viewed as a Δ-complex with one edge and two vertices.

However, for certain purposes it's actually much cleaner to view a Δ-complex X in terms of all possible *simplicial* maps $\Delta^n \to X$ (where Δ^n is the standard n-simplex), along with the relationships between them. To prevent overcounting, we assume that the vertices of X are totally ordered (or at least consistently locally ordered); then, we write X_n for the set of simplicial maps $\Delta^n \to X$ which preserve the ordering on vertices.

So then for instance the unit interval, thought of in this way, actually has four "degenerate" triangles: once we've ordered our vertices, we can take all three vertices of Δ^2 to one or the other endpoint, and then there are also two order-preserving ways to take one vertex of Δ^2 to one endpoint and the other two vertices of Δ^2 to the other endpoint.

Keeping track of the degenerate simplices allows us to easily describe all maps, even those that collapse simplices; for instance, a map of Δ-complexes might collapse an edge onto a vertex, and it ends up being easier to simply say that we've taken our edge to a *degenerate* edge than to allow for maps that can take simplices to lower-dimensional simplices.

Of course, an order-preserving map $\Delta^m \to \Delta^n$ induces a function $X_n \to X_m$, and these functions behave naturally with respect to maps of Δ-complexes. All of this motivates the following definition.

Definition. Let $\mathbf{\Delta}$ be the category whose objects are the ordered sets $[n] = \{0 \to 1 \to \cdots \to n\}$ and whose morphisms are order-preserving functions (or equivalently, whose objects are the standard simplices Δ^n and whose morphisms are order-preserving simplicial maps). Then, a *simplicial set* is a functor $X : \mathbf{\Delta}^{\mathbf{op}} \to \mathtt{Set}$. The set $X_n = X([n])$ is called the set of *n-simplices* of X. These form the model category $\mathtt{sSet}$. There is an evident *geometric realization* functor $|-| : \mathtt{sSet} \to \mathtt{Top}$, and weak equivalences in $\mathtt{sSet}$ are defined to be those maps which induce weak equivalences in $\mathtt{Top}$. The cofibrations are the levelwise injections, and the fibrations are determined by a certain lifting condition analogous to that in $\mathtt{Top}$.

It turns out that geometric realization admits a right adjoint $\mathtt{sSet} \leftarrow \mathtt{Top} : \mathrm{Sing}_\bullet$, the *singular simplices* functor, whose n^{th} level is the set of continuous maps into our chosen space from the n-simplex Δ^n (so that simplicial chains on $\mathrm{Sing}_\bullet(X)$ are precisely the same thing as singular chains on X). Moreover, the functors $|-|$ and $\mathtt{Sing}_\bullet$ satisfy just the right properties to give us a *canonical* adjunction on homotopy categories: $|-|$ preserves cofibrations and trivial cofibrations, while $\mathrm{Sing}_\bullet$ preserves fibrations and trivial fibrations. Such an adjunction is called a *Quillen adjunction*. The functors on homotopy categories are then called the *derived* functors of the original ones: a Quillen adjunction $F \dashv G$ induces a derived adjunction $LF \dashv RG$. One can compute LF as $F \circ Q$ and RG as $G \circ R$; whereas Quillen adjoints don't generally preserve weak equivalences, their derived functors do (and are universal for this property in an appropriate sense), essentially by construction.

In fact, the Quillen adjunction $|-| : \mathtt{sSet} \leftrightarrows \mathtt{Top} : \mathrm{Sing}_\bullet$ satisfies just the right properties for its derived adjunction to define an *equivalence* of homotopy categories: for any cofibrant $X \in \mathtt{sSet}$ and any fibrant $Y \in \mathtt{Top}$, $|X| \to Y$ is a weak equivalence in $\mathtt{Top}$ iff $X \to \mathrm{Sing}_\bullet(Y)$ is a weak equivalence in $\mathtt{sSet}$. Such an adjunction is called a *Quillen equivalence*. In this sense, $\mathtt{sSet}$ and $\mathtt{Top}$ both "model the same homotopy theory", and so any homotopy-theoretic statement we can make in one we can make equally well in the other.

2 Model Categories in Algebra

2.1 Chain complexes of R-modules

Now, the first reason we've taken the trouble to explain all this to algebraists is that in algebra there is an obvious and important category with weak equivalences, namely the category $\mathrm{Ch}(R)$ of chain complexes of R-modules, where weak equivalences are quasi-isomorphisms.

In fact, this can be extended to (at least) two distinct model structures.

- In the *standard* model structure, bounded-below complexes of projectives are cofibrant. Fibrations are levelwise surjections, so every object is fibrant.
- In the *injective* model structure, bounded-above complexes of injectives are fibrant. Cofibrations are levelwise injections, so every object is cofibrant.

In fact, the identity functor induces a Quillen equivalence $\mathrm{id} : \mathrm{Ch}(R)_{std} \leftrightarrows \mathrm{Ch}(R)_{inj} : \mathrm{id}$.

2.1.1 Ext

All this means that if we have two objects of $\mathrm{Ch}(R)$, we can try to compute the space of morphisms between them in $\mathrm{Ch}(R)[W^{-1}]$ in two different ways. For instance, suppose we have two R-modules M and N, considered as chain complexes concentrated in degree 0. Let $P_\bullet \to M$ be a projective resolution, and let $N \to I_\bullet$ be an injective resolution. We consider $\mathrm{Ch}(R)$ as enriched over itself, which ends up implying that $\mathrm{Ch}(R)[W^{-1}]$ is also enriched over itself. Then, using our two model structures we obtain that

$$\underline{\mathrm{Hom}}_{\mathrm{Ch}(R)}(P_\bullet, N) \simeq \underline{\mathrm{Hom}}_{\mathrm{Ch}(R)[W^{-1}]}(M, N) \simeq \underline{\mathrm{Hom}}_{\mathrm{Ch}(R)}(M, I_\bullet).$$

Note that we only get weak equivalences here since we're working in a homotopy category. Hence, the only way we can extract any invariants is by taking homology groups; from this, we obtain $H_*(\mathrm{Hom}_{\mathrm{Ch}(R)}(P_\bullet, N)) \cong H_*(\mathrm{Hom}_{\mathrm{Ch}(R)}(M, I_\bullet))$. Of course, these are known as the *Ext groups* of M and N, and we've just proved using model categories that they can be computed either by projectively resolving the source or injectively resolving the target!

2.1.2 Gluing quasicoherent sheaves

Algebraists are prone to working in derived categories of R-modules. This is fine, but doesn't work so well for algebraic geometers, who like to glue algebraic constructions over non-affine schemes. What happens when they're working Zariski-locally in the derived category?

The answer, of course, is that they should be using model categories! If you have a gluing map in the derived category on an overlap, you'd ideally like to realize this as an actual morphism, but morphisms in the derived category are these zig-zag things. So instead, you can choose a model structure that lifts your derived category (which is really just what we've been calling a homotopy category), and then taking bifibrant replacements everywhere will ensure that all our morphisms in the derived category really do lift to actual maps. Of course, the sheaf condition needs to be replaced by what's called the *homotopy* sheaf condition (which we'll discuss later), but this is a relatively small price to pay.

2.2 "Chain complexes" in non-abelian categories

Let $\mathcal{A}$ be an abelian category. Then the *Dold-Kan correspondence* defines an equivalence of categories $\mathrm{Ch}_{\geq 0}(\mathcal{A}) \simeq \mathtt{s}\mathcal{A}$ between nonnegatively-graded chain complexes in $\mathcal{A}$ and simplicial objects in $\mathcal{A}$. This isn't especially useful for abelian categories since chain complexes are rather easier to work with, but this tells us how to study "chain complexes" in non-abelian categories – we just work with simplicial objects!

2.2.1 Simplicial rings, the cotangent complex, and André-Quillen cohomology

One immediate example of an interesting and non-abelian category is `Ring` of (let's say commutative, though this can be vastly generalized) rings. Here's a cool example of what we can do with simplicial rings.

First, recall that if S is an *augmented* R-algebra (i.e. there's a retraction $\rho : S \to R$ of the structure map $\iota : R \to S$), we can define the module of *relative R-linear derivations* $\mathrm{Der}_R(S, M)$: these are the R-module homomorphisms $d : S \to M$ such that $d(\iota(r)) = 0$ which satisfy the *Leibniz rule* $d(s_1 s_2) = \rho(s_1) \cdot d(s_2) + d(s_1) \cdot \rho(s_2)$. In fact, there is a *universal derivation* $S \to \Omega^1_{S/R}$, in the sense that this induces an isomorphism $\mathrm{Der}_R(S, M) \cong \mathrm{Hom}_R(\Omega^1_{S/R}, M)$ which is natural in M. This R-module $\Omega^1_{S/R}$ is called the *module of relative Kähler differentials*. This is easy to construct based on its universal property: it's just the free S-module on the symbols ds for all $s \in S$, subject to the relations $d(s_1 + s_2) = ds_1 + ds_2$, $d(s_1 s_2) = d(s_1) \cdot s_2 + s_1 \cdot d(s_2)$, and $d(\iota(r)) = 0$. (If we flip to algebraic geometry, then we're studying a morphism of schemes $\iota : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ equipped with a section $\rho : \mathrm{Spec}(R) \to \mathrm{Spec}(S)$, and $\Omega^1_{S/R}$ is the *conormal sheaf* of $\mathrm{Spec}(R)$ in $\mathrm{Spec}(S)$; this should be thought of as the sheaf of vertical 1-forms.)

Now, André-Quillen cohomology is the *derived functor* of derivations. How can we compute it? Well, there is an embedding $\mathrm{const} : \mathtt{Ring} \hookrightarrow \mathtt{sRing}$ given by taking *constant* simplicial rings (i.e. the functor $\mathbf{\Delta^{op}} \to \mathtt{Ring}$ which takes all objects to our chosen ring and all morphisms to its identity morphism – this is like considering a set as a constant simplicial set, which is really just considering the set as the obvious discrete topological space). As you might guess, `sRing` has a model structure. Hence we can take a cofibrant replacement $\mathrm{const}(R) \rightarrowtail P \overset{\sim}{\twoheadrightarrow} \mathrm{const}(S)$ (of $\mathrm{const}(S)$ as a $\mathrm{const}(R)$-algebra), and we can form the module of Kähler differentials $\Omega^1_{P/\mathrm{const}(R)}$ simply by working levelwise. This is called the *cotangent complex* (or rather, the associated complex of R-modules is called this – technically we're still looking at a simplicial R-module, but Dold-Kan now applies so it doesn't really matter), and is denoted $\mathbb{L}_{S/R}$. Then, *derived* derivations of S into M are given by $\mathrm{Hom}_{\mathtt{sMod}_R}(\mathbb{L}_{S/R}, M)$. Of course, since we had to choose a cofibrant replacement, $\mathbb{L}_{S/R}$ is only well-defined up to weak equivalence, and so this isn't quite canonical either. But as before, we simply take its homology groups $H_*(\mathrm{Hom}_{\mathtt{sMod}_R}(\mathbb{L}_{S/R}, M))$, and these are by definition the *André-Quillen cohomology groups* $H^*_{AQ}(S/R, M)$.

The beautiful thing about this is that André-Quillen cohomology actually tells you things about algebraic geometry! For instance, an open inclusion has contractible cotangent cotangent complex, i.e. $\mathbb{L}_{T^{-1}R/R} \simeq 0$ for any multiplicatively closed subset $T \subset R$; thus, André-Quillen cohomology for open inclusions vanishes identically. More generally, if R is noetherian and S has finite type as an R-algebra, then $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is étale iff $\mathbb{L}_{S/R} \simeq 0$ and is smooth iff $\mathbb{L}_{S/R} \simeq \mathrm{const}(\Omega^1_{S/R})$ and $\Omega^1_{S/R}$ is a projective S-module. And this construction glues (in the homotopical way alluded to above), since the cotangent complex is local: if $(R_1, T_1) \to (R_2, T_2)$ is a map of rings and multiplicative systems, then $\mathbb{L}_{T_2^{-1}R_2/T_1^{-1}R_1} \simeq \mathbb{L}_{R_2/R_1} \otimes_{R_2} T_2^{-1}R_2$. Lastly, a pair of composable morphisms of rings induces a cofiber sequence of cotangent complexes, which induces a long exact sequence in André-Quillen cohomology.

3 Topology + Algebra = ♡

3.1 A cute example between algebra and topology

Suppose X is a topological space. Then there is a model structure on the category $\mathtt{Psh}(X)$ of presheaves on X. For simplicity we'll stick to the case that these presheaves take values in some algebraic category, but this can be generalized to the homotopical situation too. The cofibrations are just the sectionwise inclusions, so all objects are cofibrant. Equivalence is defined stalkwise, and the sheaves are precisely the fibrant objects. So, sheafification (which, remember, preserves stalks) is nothing more or less than fibrant replacement!

3.2 Algebraic geometry over $\mathbb{F}_1$

If $\mathcal{C}$ is any tensored triangulated category, then it admits the notions of a *thick* subcategory (i.e. a subcategory which has the 2-out-of-3 property for distinguished triangles and which is closed under retracts), of an *ideal* subcategory (via the tensor structure) and of a *prime ideal* subcategory (with the obvious definition). Putting this all together, we can define a topological space TTSpec($\mathcal{C}$): its points are the thick prime ideal subcategories of $\mathcal{C}$, and it has the "opposite-Zariski topology" (so a basis of open sets is given by $V(X) = \{\mathscr{P} : X \in \mathscr{P}\}$, and the sets $D(X) = \{\mathscr{P} : X \notin \mathscr{P}\}$ are closed).

This is a sensible definition for the following reason. Given a ring R, we can consider the tensored triangulated category $K^b(\mathtt{Proj}_R)$ of perfect complexes of R-modules (i.e. bounded chain complexes of finite-rank projective R-modules). Then, with the above definition, $\mathrm{TTSpec}(K^b(\mathtt{Proj}_R)) \cong \mathrm{Spec}(R)$; that the opposite-Zariski topology on the former coincides with the Zariski topology on the latter can be attributed to *Hochster duality*.

Now, of course $\mathbb{F}_1$ isn't an actual mathematical object, but the cognoscenti tend to agree that "finite-rank projective $\mathbb{F}_1$-modules" should just be pointed finite sets. As we now know, even though these don't form an abelian category, we can nevertheless study "chain complexes" in this category by studying simplicial objects – and in fact, a simplicial pointed-set is precisely the same thing as a pointed simplicial-set! When we take the triangulated structure into account, we end up with the *finite stable homotopy category*, denoted $\mathtt{SHC}^{fin}$: this can be recovered as the stabilization of the category of finite CW-complexes, and is one of the primary objects of study of stable homotopy theory. The *nilpotence theorem* of Devinatz, Hopkins, and Smith, one of the foundational theorems of the subfield of *chromatic homotopy theory*, implies that TTSpec($\mathtt{SHC}^{fin}$) (which, morally, is Spec($\mathbb{F}_1$)) is the smash product of Spec($\mathbb{Z}$) with ($\mathbb{N}_0 \cup \{\infty\}$). More specifically:

- there's a generic point $\mathscr{P}_0$;
- for every prime p and for every $n \in [1, \infty]$, there's a point $\mathscr{P}_{n,(p)}$;
- the closure of $\mathscr{P}_{n,(p)}$ is $\{\mathscr{P}_{m,(p)} : m \in [n, \infty]\}$.

Thus, Spec($\mathbb{F}_1$) is something like a fattened-up (and incredibly non-artinian) version of Spec($\mathbb{Z}$).

More generally, $\mathbb{F}_1$-algebras are supposed to be the same thing as monoids, and so we can repeat this whole story for pointed finite sets with a suitable monoid action: this would yield an equivariant version of the finite stable homotopy category.

It's possible to endow this space TTSpec($\mathcal{C}$) with a structure sheaf of rings, but the truth is that this is all a bit outdated: really one should be working with tensored stable ∞-categories (instead of their associated tensored triangulated categories) and sheaves of E_∞-rings (instead of sheaves of ordinary rings). Nevertheless, this still seemed cool enough to be worth sharing anyways.

3.3 Crystalline cohomology and the homotopy sheaf condition

Let X be a scheme. We'd like to define the crystalline cohomology of X as the value of a crystal on the terminal object of the crystalline topos $\mathtt{Crys}_X$ of X, but unfortunately this doesn't exist. On the other hand, suppose that $X \hookrightarrow Y$ is any embedding into a smooth scheme. If $U \subset X$ is a Zariski open subset and $U \hookrightarrow \mathfrak{U}$ is a nilpotent PD-thickening (i.e. an object of $\mathtt{Crys}_X$), then there is an non-unique map $\mathfrak{U} \to D_X(Y)$ (where $D_X(Y)$ is the PD-completion of $X \hookrightarrow Y$), and any two such maps agree upon composition with the map $D_X(Y) \to D_X(Y \times Y)$ induced by the diagonal on Y. So, the "final object" is really $D_X(Y)//D_X(Y \times Y)$; more precisely, we may say that global sections are given by the equalizer of $\Gamma(D_X(Y)) \rightrightarrows \Gamma(D_X(Y \times Y))$. The standard way to proceed from here is to take alternating sums to form the cochain complex $0 \to \Gamma(D_X(Y)) \to \Gamma(D_X(Y \times Y)) \to \Gamma(D_X(Y \times Y \times Y)) \to \cdots$. On the other hand, by Dold-Kan this is associated to the simplicial object $\Gamma(D_X(Y^{\bullet+1}))$, which (recall) is a "generalized space". This is obtained by applying global sections levelwise to the evident *cosimplicial* object $D_X(Y^{\bullet+1})$ (adding the "co" since now we're talking about a functor $\mathbf{\Delta} \to \texttt{Crys}_X$), which we can therefore think of as the *derived* terminal object of $\texttt{Crys}_X$.

In fact, this illustrates beautifully how the homotopy sheaf condition is defined in full generality. If $\mathcal{F}$ is a presheaf of sets, then recall that $\mathcal{F}$ is a sheaf if for all covers $\{U_\alpha \to U\}$ we have that $\mathcal{F}(U) \xrightarrow{\cong} \mathrm{eq}(\prod \mathcal{F}(U_\alpha) \rightrightarrows \prod \mathcal{F}(U_{\alpha\beta}))$. But this diagram of which we're taking the equalizer is really just $\mathcal{F}$ applied to the 1-truncation of a cosimplicial object $U_{\bullet+1}$ (whose n^{th} level is the coproduct of the $U_{\alpha_0\ldots\alpha_n}$). So if $\mathcal{F}$ is instead a presheaf valued in some model category, then we say $\mathcal{F}$ is a *homotopy sheaf* if for all covers $\{U_\alpha \to U\}$ we have that $\mathcal{F}(U) \xrightarrow{\sim} \mathrm{holim}(\mathcal{F}(U_{\bullet+1}))$.

3.4 Motives and $\mathbb{A}^1$-homotopy theory

The idea of $\mathbb{A}^1$-homotopy theory is to apply the techniques of homotopy theory to schemes. The appropriate setting for this is the *Nisnevich site* of a base scheme S, denoted $\texttt{Sm}_{\mathrm{Nis},S}$, whose objects are the schemes which are separated, smooth, and of finite type over S. Then, a *motivic space* is just a homotopy sheaf $\texttt{Sm}_{\mathrm{Nis},S} \to \texttt{sSet}$. These form a model category $\texttt{MSpaces}_S$, meaning that we can apply all the usual constructions of homotopy theory.

Note first that any $K \in \texttt{sSet}$ defines a motivic space, simply by sheafifying the constant presheaf at K. On the other hand, so does any $T \in \texttt{Sm}_{\mathrm{Nis},S}$ define a motivic space, simply by considering its functor of points as a presheaf of discrete simplicial sets and then sheafifying. Thus, motivic spaces provide a setting in which ordinary spaces and schemes can interact on even footing.

In fact, recall that we consider simplicial objects as generalized spaces. A simplicial object $X : \mathbf{\Delta}^{op} \to \texttt{Sm}_{\mathrm{Nis},S}$ in $\texttt{Sm}_{\mathrm{Nis},S}$ defines a motivic space, again simply by sheafifying. These don't quite give all of $\texttt{MSpaces}_S$, but they almost do: the only issue is that $\texttt{Sm}_{\mathrm{Nis},S}$ has finiteness conditions on its objects, whereas to represent an object of $\texttt{MSpaces}_S$ we may need to take arbitrary coproducts.

In ordinary homotopy theory, we can define homotopies using the unit interval $[0,1]$. This isn't a scheme, but in the motivic setting we can still define a homotopy between two maps $f_0, f_1 : X \to Y$ to be a map $H : \mathbb{A}^1 \times X \to Y$ which restricts to $f_n : \{n\} \times X \to Y$ for $n \in \{0,1\}$. This gives us the *motivic homotopy category*, denoted $\texttt{MHC}_S$, in which we have the defining relation $\mathbb{A}^1 \simeq \mathrm{pt}$.

One way in which this is richer than the model category of ordinary spaces is that there are now more "spheres" than before. We denote them by $S^{n,t}$: the first coordinate is the *simplicial dimension*, and the second coordinate is the *Tate dimension*. The ordinary sphere coming from the simplicial set S^n is denoted $S^{n,0}$. On the other hand, we can also construct $S^{1,0}$ (up to weak equivalence) by intersecting three copies of $\mathbb{A}^1$ at their respective points 0 and 1 or by taking a single copy of $\mathbb{A}^1$ and identifying 0 and 1. (Remember that $\mathbb{A}^1$ is contractible, so this should make sense from a Čech point of view.) Then, the scheme $\mathbb{G}_m$ (over S) defines the sphere $S^{1,1}$, and the scheme $\mathbb{P}^1$ (also over S) defines the sphere $S^{2,1}$. In other words, $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$! Thus, there is a motivic sphere in every bidegree (n,t) with $n \geq t \geq 0$. Of course, these give rise to (bigraded) motivic homotopy groups. However, note that these do not determine the weak equivalences in $\texttt{MSpaces}_S$: these are determined stalkwise. (The "points" of the Nisnevich site of S are associated to the henselizations of its local rings in the Zariski topology; these are not actually objects of the Nisnevich site in general, but rather are pro-objects thereof.)

Now that we have a notion of spaces, we can define spectra simply by following the usual definitions in stable homotopy theory. (To be precise, one usually uses symmetric spectra, since these tend to be the most manageable.) Of course, we're now stabilizing our (pointed) motivic spaces in both the simplicial and Tate directions. By definition, these represent the "motivic extraordinary cohomology theories". Of these, there are certain "ordinary" theories, which are characterized by the *Bloch-Ogus axioms*; in particular, any Weil cohomology theory is ordinary.

Taking as a given that the ordinary cohomology theories are an interesting object of study, it is natural to ask for the universal category through which they all factor and in which they all become representable. By definition, this is the category of *motives*... except that we don't actually know that it exists. What we do have is something that might be called the "derived category of motives", although for all we know it's not the derived category of anything at all.

The urge to find an abelian category lifting the derived category of motives seems to stem from the fact that when Grothendieck originally came up with the idea of motives, abelian categories were the state of the art of homological algebra, and so he envisioned motivic cohomology as being given by Ext in this abelian category. But there are now generalizations of the relevant constructions to triangulated categories, and so this is not altogether necessary. On the other hand, having such an abelian category would still have nice implications, such as the vanishing of certain motivic cohomology groups which are otherwise difficult to compute.

3.5 A few words on algebraic K-theory

Since we're already all the way there, we can't resist mentioning that algebraic K-theory is far and away most cleanly cast in the language of motivic homotopy theory: it's just a Nisnevich sheaf. In fact, it's represented by a motivic E_∞-ring spectrum which is constructed in much the same way as ordinary complex K-theory. Moreover, there's a motivic version of (so far just the beginning of) the whole chromatic homotopy theory story, and algebraic K-theory is just the Landweber exact motivic cohomology theory corresponding to the formal multiplicative group!

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