Introduction to Supermanifolds

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This talk is an introduction to supermanifolds — these are supposed to encode "odd fuzz" on regular manifolds. What you do is you enlarge the notion of "functions": functions are usually sections of a trivial bundle, and we will instead enlarge that bundle.

1 Super Algebra

Definition: *A* super vector space *is a $\mathbb{Z}/2$-graded vector space (over $\mathbb{R}$) $V = V_0 \oplus V_0$, the* even *and* odd *parts. Morphisms are grading-preserving linear maps. There is a "parity reversing functor" Π, for which $(\Pi V)_0 = V_1$ and $(\Pi V)_1 = V_0$.*

Renato: These are just the even maps? **Aaron:** You can reverse parity, which is how you can talk about the other maps.

This category has a $\otimes$ structure, which makes it equivalent as a monoidal category to the usual category of representations of $\mathbb{Z}/2$. But we choose different *commutativity isomorphisms*:

$$\begin{aligned} c_{V,W} : V \otimes W &\xrightarrow{\sim} W \otimes V \\ v \otimes w &\mapsto (-1)^{|w|\cdot|v|} w \otimes v \qquad \text{(assuming } v, w \text{ homogenous)} \end{aligned}$$

We can now talk about super algebras, commutative super algebras — use the commutativity isomorphism, so "commutative" means "skew commutative on odd things" —, super modules, super Lie algebras, etc. We can then talk about $\mathrm{Ssym}^\bullet(V)$ and $\mathrm{S}\Lambda^n(V)$.

We now introduce the Berezinian. It will connect with integration — it's a generalization of determinant, and when you change variables in an integral, you need to introduce a determinant.

Definition: *A* free module *over a superalgebra $\mathcal{A}$ is a module which is free as an ungraded module, but we do ask there to exist a homogenous basis.*

Example: $A^{p|q}$ is freely generated by $x_1, \dots x_p$ even and $\theta_1, \dots, \theta_q$ odd. ◊

A morphism $T : \mathcal{A}^{p|q} \to \mathcal{A}^{r|s}$ will be given by a matrix

$$\begin{pmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{pmatrix}$$

Given such a linear transformation, its *supertrace* is

$$\mathrm{STr}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{Tr}(A) - \mathrm{Tr}(D)$$

This is an even element of the underlying algebra $\mathcal{A}$.

Definition: *Let L be a free module (of finite type) over a commutative superalgebra $\mathcal{A}$. Then we get a notion as above of* $\mathrm{GL}(L)$*. The* Berezinian *is a homomorphism* $\mathrm{Ber} : \mathrm{GL}(L) \to \mathcal{A}_0^\times$*. We ask it to satisfy:*

1. *If ϵ is even and $\epsilon^2 = 0$, then* $\mathrm{Ber}(1 + \epsilon T) = 1 + \epsilon\,\mathrm{STr}(T)$.
2. *If T is diagonalizable $T = \left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$ then* $\mathrm{Ber}(T) = (\det A)(\det D)^{-1}$.
3. *If $0 \to L' \to L \to L' \to 0$ is a short exact sequence, and (T', T, T'') is an automorphism, then* $\mathrm{Ber}(T) = \mathrm{Ber}(T')\,\mathrm{Ber}(T'')$.

Harold: Which is the definition? **Aaron:** You can do this for arbitrary A, B, C, D, and it satisfies all of those, and we think those are enough to nail down the function.

Oh, I should have said: the canonical example of a supervector space is $\Lambda^*(\mathbb{R}^q)$ — this is a commutative superalgebra, where the parity of the number of tensors is the grading.

2 Super Manifolds

The right way to do this is with locally ringed spaces. We will write $(|M|, \mathcal{O}_M)$, where $|M|$ is the topological space, and $\mathcal{O}_M$ is the sheaf of functions. We say that such a locally ringed space is a *supermanifold* of dimension $p|q$ if it is locally modeled on $\mathbb{R}^{p|q}$. Here $\mathbb{R}^{p|q}$ is the sheaf on $|\mathbb{R}^{p|q}| = \mathbb{R}^p$ which assigns to an open set U the ring $\mathscr{C}^\infty(U) \otimes \Lambda^\bullet(\mathbb{R}^q)$. Sometimes you ask for Hausdorff, second countable, etc.

Given $M = (|M|, \mathcal{O}_M)$, the *reduced supermanifold* is $M^{\mathrm{red}} = (|M|, \mathcal{O}_M/\mathrm{Nil})$, where Nil is the (sheaf of) ideal of nilpotent elements. **Renato:** So this is a manifold that we're used to. **Aaron:** Yes, it's an ordinary manifold. Or rather it's a supermanifold of dimension $p|0$.

Zach: Is this really an ordinary manifold, or can it have twisting? **Aaron:** It is a locally ringed space modeled on $\mathbb{R}^p$.

We always have $M^{\mathrm{red}} \hookrightarrow M$. This is the map that is the identity on topological spaces, and the quotient of rings.

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Let $\mathbb{R}^q \hookrightarrow E \twoheadrightarrow X$ be a vector bundle over an ordinary manifold. Then ΠE is the "oddification" of the vector bundle. It is a supermanifold with $|\Pi E| = X$ and $\mathcal{O}_{\Pi E} = \Gamma(\Lambda^\bullet E^*)$, where E^* is the dual vector bundle, and so on. This has dimension $p|q$ if $\dim X = p$ as an ordinary manifold.

Theorem (Batchelor '79): *We already described* Π : *vector bundles* $\to$ *supermanifolds. We define a map* J *in the other way, which sends* M *to the vector bundle over* M^{red}*, with sections* $\Gamma(U, J(M)) = \text{Nil}(U)/(\text{Nil}(U))^2$.

The theorem is that $J \circ \Pi \cong \text{id}$ *is a natural isomorphism of functors. For each object* M*,* $\Pi \circ J(M) \cong M$*, but unnaturally so — so we get a bijection on objects between* {*vector bundles*} *and* {*supermanifolds*}.

In fact, if $\dim M = p|q$ *for* $p \geq 1$ *and* $q \geq 2$*, then there does not exist a contraction* $M \to M^{\text{red}}$ *that is compatible with all automorphisms of* M*. There are more supermanifold-morphisms than vector bundle morphisms.*

Theo: Is it right to say that $M^{\text{red}} \hookrightarrow M$ is a closed submanifold, and $J(M)$ is the first-order neighborhood of M^{red}, like the pull back of the tangent bundle? **Aaron:** That sounds right.

Theo: Is Batchelor's theorem true in the analytic category? I see how to prove it in the smooth case, but my proof would require partitions of unity. **Aaron:** Nobody I've read said it wasn't true, but I'm not sure.

3 Functors of points

I gave a local characterization, but local things can be hard to work with.

Given S, M supermanifolds, we have a bijection between $\text{SuperMan}(S, M) \cong \text{SuperAlg}(\mathcal{O}_M(M), \mathcal{O}_S(S))$. This clearly only holds in the smooth category. We call the elements of $\text{SuperMan}(S, M)$ the *S-points of M*.

We think of arbitrary functors $\text{SuperMan}^{\text{op}} \to \text{Set}$ as "generalized" supermanifolds. It's Yoneda's lemma that the generalized supermanifold corresponding to a supermanifold is no loss of data.

Example: $\underline{\text{SM}}(B, C)$ is the generalized supermanifold that sends A to $\text{SM}(A \times B, C)$. ◊

I.e. $(C^B)^A = C^{A\times B}$. 3^2 is the number of maps from the 2-element set to the 3-element set.

Theorem: $\underline{\text{SM}}(\mathbb{R}^{0|1}, M) = \Pi \text{T} M$.

We will now define the right hand side.

First, and aside on vector bundles:

Definition: *A* super vector bundle *over a supermanifold is a locally free sheaf* $\mathcal{E}$ *of* $\mathcal{O}_M$ *modules. This is the same as the usual non-super case.*

Example: $\mathcal{T}M$ the tangent bundle is the sheaf that on $U \in |M|$ gives $\mathrm{Der}(\mathcal{O}_M(U))$. An *even derivation* of a superalgebra is an even linear map D satisfying $D(fg) = D(f)g + fD(g)$. An *odd derivation* is an odd map satisfying $D(fg) = D(f)g + (-1)^{|f|}fD(g)$. This gives the supermodule of derivations. ♢

Definition: *The* total space *of* $\mathcal{E}$ *has* S*-points* $E(S) = \{(f,g)$ *s.t.* $f \in \mathrm{SM}(S,M)$ *and* $g \in \Gamma(S, f^*\mathcal{E}^{\mathrm{ev}})\}$.

Then $\Pi : \mathrm{SV{\scriptstyle ECT}}_M \to \mathrm{SV{\scriptstyle ECT}}_M$ is the functor that reverses parities. It is $\Pi\mathcal{E} = \underline{\mathbb{R}^{0|1}} \otimes \mathcal{E}$, where $\underline{\mathbb{R}^{0|1}}$ is the trivial bundle with fiber $\mathbb{R}^{0|1}$.

Renato: The reduced manifold of $\mathrm{T}M$ is $\mathrm{T}M^{\mathrm{red}}$. But you told me just a sheaf on M^{red}. **Theo:** He says that in general, a vector bundle over M is a sheaf over M^{red}. Then he said: any vector bundle has a total space, which might have a much larger reduced space than M has.

Example: The total space of $\mathcal{T}M$ is $\mathrm{T}M$, and is $2p|2q$-dimensional. ♢

Proof (of theorem about TM):

$$\begin{aligned}
\mathrm{SM}(S, \underline{\mathrm{SM}}(\mathbb{R}^{0|1})) &= \mathrm{SM}(S \times \mathbb{R}^{0|1}, M) && \text{defn of } \underline{\mathrm{SM}} \\
&\cong \mathrm{S{\scriptstyle UPER}A{\scriptstyle LG}}(\mathcal{O}_M(M), \mathcal{O}_{S\times\mathbb{R}^{0|1}}(S \times \mathbb{R}^{0|1})) && \text{proposition} \\
&\cong \mathrm{S{\scriptstyle UPER}A{\scriptstyle LG}}(\mathcal{O}_M(M), \mathcal{O}_S(S) \otimes \mathcal{O}_{\mathbb{R}^{0|1}}(\mathbb{R}^{0|1})) && \text{defn of } M \times M' \\
&\cong \mathrm{S{\scriptstyle UPER}A{\scriptstyle LG}}(\mathcal{O}_M(M), \mathcal{O}_S(S) \otimes \mathrm{E}(\theta)) && \mathrm{E}(\theta) = \text{exterior algebra on } \theta
\end{aligned}$$

Suppose $\phi : \mathcal{O}_M(M) \to \mathcal{O}_S(S) \otimes \mathrm{E}(\theta)$. Write $\phi = f + \theta g$, where $f, g : \mathcal{O}_M(M) \to \mathcal{O}_S(S)$ are even,odd respectively linear maps. What is it to be an algebra map?

$$\begin{aligned}
f(ab) + \theta g(ab) = \phi(ab) = \phi(a)\phi(b) = (f(a) + \theta g(a))(f(b) + \theta g(b)) = \\
= f(a)f(b) + \theta(g(a)f(b) + (-1)^{|f(a)||\theta|}f(a)g(b) = f(a)f(b) + \theta(g(a)f(b) + (-1)^{|a|}f(a)g(b))
\end{aligned}$$

So f is an algebra map, and g is an odd derivation with respect to f.

But $\mathrm{Der}_f^{\mathrm{odd}}(U) = \Pi(\mathcal{T}M)(U)$, so this is $\mathrm{SM}(S, \Pi(\mathcal{T}M))$. □

The reason I like $\mathbb{R}^{0|1}$ is that it (conjecturally) connects to homotopy theory. One goal in this class is to gain homotopical information from functors on the bordism category.

Theorem: $0|1\text{-EFT}^n(M) = \begin{cases} \Omega^{\mathrm{even}}_{\mathrm{closed}}(M), n \text{ even} \\ \Omega^{\mathrm{odd}}_{\mathrm{closed}}(M), n \text{ odd} \end{cases}$. *So we can define* $0|1\text{-EFT}^n[M]$ *by imposing that* $0|1\text{-EFT}^n$ *is supposed to be a homotopy functor. You get even/odd de Rham cohomology of* M*, which is the same as that of* M^{red}*. So Euclidean Field Theories are a form of "cocycles" for de Rham cohomology.*

Theorem (Stolz-Teichner): $1|1\text{-EFT}^n[M] \cong \mathrm{K}^*(M)$.

This is very tantelizing. De Rham cohomology and K-theory are the 0th and 1st rungs on a ladder, called "chromatic homology".

Conjecture (Stolz-Teichner): $2|1\text{-EFT}^n[M] \cong \mathrm{TMF}^*(M)$.