Every love story is a GHOsT story: Goerss–Hopkins obstruction theory for ∞-categories – complementary talk notes

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I should probably begin by saying that I'm really not very well-prepared for this talk.

0 Introduction

0.1 History

Goerss–Hopkins actually announce a relative version of their obstruction theory, but in fact this won't be more general after all. They begin with an $\mathscr{E}_\infty$-ring spectrum R and a map $E_*R \cong A \to B$ of Φ-algebras, and then they look for the space of realizations of $A \to B$ as a map of $\mathscr{E}_\infty$-ring spectra off of R. But this is really just doing the absolute obstruction theory in the category of R-algebras.

If we have access to the Dyer–Lashof algebra for a homotopy commutative and associative ring spectrum E, then (since E is E-local) we can use GHOsT with respect to E_* to put an $\mathscr{E}_\infty$-structure on E itself!

0.2 Overview

0.3 Motivation

1 Blanc–Dwyer–Goerss obstruction theory

1.1 Categorical generalities

In any (∞-)topos, the initial object is empty (in the categorical sense), which is why we topologists generally only think about cobased objects in *pointed* categories. (Motivic spaces don't form a topos, but they're a localization thereof, and in particular their initial object is still empty.)

We'll write S^β for an arbitrary element of $\mathcal{G}$, even though it's really the set of finite coproducts of generators.

The δ in $\mathcal{P}^\delta_\Sigma$ stands for "discrete".

Note that $\mathcal{A}$ is really nothing more or less than the category of models (in $\mathcal{S}\mathrm{et}$) for the multi-sorted algebraic theory $\mathcal{G}^{op}$.

Usually Π-algebras are defined to be product-preserving presheaves of sets on $\mathrm{ho}(\Pi)$ instead of just on Π, but since $\mathcal{S}\mathrm{et}$ is a 1-category then these notions are equivalent.

We'll see that even in the enriched case, when $s\mathcal{A}$ is no a category of certain simplicial presheaves, we can still define the appropriate model structure (or really, we'll still be able to describe the appropriate ∞-category).

Note that the category $\mathcal{A}$ is pointed since the objects of $\mathcal{G}$ are cobased.

1.2 The big picture

To justify the claim about the fibers of the map $B\mathrm{Aut}(A, M) \to \widehat{\mathscr{H}}^{n+2}(A, \Omega^n A)$, note that the group action fixes the path component of $0 \in H^{n+2}$, but can permute the others.

1.3 The E^2-model structure

The trivial model structure on $s\mathcal{S}\mathrm{et}$ can be equivalently described as the Reedy model structure with respect to the trivial model structure on $\mathcal{S}\mathrm{et}$, which is its unique model structure which presents the homotopy category of 0-types.

In the definition of $K \wedge Y$, we need Y to be cobased. This is immediate if $\mathcal{C}$ is pointed.

Note that the functor $S^\beta \mapsto S^n_{\mathbf{\Delta}} \wedge S^\beta$ preserve coproducts up to weak equivalence since a homotopy coproduct is just the coproduct of cofibrant replacements of the individual objects.

1.4 The spiral exact sequence

The first method of constructing the E^1-page – and specifically, the subtlety that this was all taking place in the Reedy model structure on $s\mathcal{C}$, with which the Quillen model structure on $s\mathcal{S}\mathrm{et}$ *isn't* compatible – had me confused for a really, really long time. I'd like to use this podium for just a moment to implore of you all: PLEASE do not tell people that spaces and simplicial sets are the same thing!!

By definition, the natural homotopy groups are corepresentable in $\mathrm{ho}(s\mathcal{C}_{E^2_{\mathcal{G}^E_{\mathcal{C}}}})$. In fact, the classical homotopy groups $\pi_n\pi_*$ are corepresentable for all $n \neq 1$. At $n = 0$ this is true only because $\pi_0\pi_* \cong \pi_{0,*}$, and for $n \geq 2$ we can make a rather weird construction (that involves putting certain cells in 2 simplicial dimensions below, which is where the restriction comes from) that ends up corepresenting them.

1.5 Obstruction theory

The reason for the grading shift – that is, the fact that the "continuous n^{th} k-invariant map" runs $\mathscr{M}_{n-1}(A) \to \widehat{\mathscr{H}}^{n+2}(A, \Omega^n A)$ – is due to the definition of an n-stage, which recall had nontrivial classical homotopy in degrees 0 and $n + 2$.

2 From Blanc–Dwyer–Goerss to Goerss–Hopkins

Goerss informs me that the original Goerss–Hopkins paper wasn't ultimately published because they wanted a diagrams version.

Contrary to what one might expect, stability does *not* play any crucial role in Goerss–Hopkins obstruction theory. Well, it wouldn't play any role if we were still using homotopy instead of homology. But as we'll see, it will be crucial for us that homology is given by smashing with a spectrum and then taking homotopy. But if this were our chosen algebraic invariant (which would then be very non-linear!) in an unstable category, then everything else would go through identically.

2.1 The key ideas

We don't just want to work in the "E_*-E^2-model structure" on $s\mathcal{C}$ directly; we really do need to have our hands on explicit generators.

Relatedly, we could just resolve everything using $\mathcal{G}$ (whose E-homology is of course free and hence projective), but using $\mathcal{G}^E_{\mathcal{C}}$ will be more natural because it builds in a way of recovering E-homology (by taking colimits).

We have to be careful in defining the functor $\mathrm{Alg}_T(s\mathcal{C}) \to \mathrm{Alg}_O(\mathcal{C})$: note that if $Y \in \mathrm{Alg}_T(s\mathcal{C})$, then we have $Y_i \in \mathrm{Alg}_{T_i}(\mathcal{C})$: these constituents are algebras over *different* operads.

The Σ-freedom of the operads T_n first of all implies that the one doesn't need to worry about the difference between the quotient and the homotopy quotient. But also, when a group action isn't free, then there's a homotopy orbits spectral sequence for the homology or homotopy of the homotopy quotient, and we certainly don't want to have to mess around with that.

More loosely, we could replace our assumption on T by the more abstract condition that we have a monad T_E on $s\mathcal{A}$ such that we have a functor $E_* : \mathrm{Alg}_T(s\mathcal{C}) \to \mathrm{Alg}_{T_E}(s\mathcal{A})$ that preserves cofibrancy.

We have to be careful with what we mean by cofibrancy in $\mathrm{Alg}_{T_E}(s\mathcal{A})$: now that T is involved, we can no longer give explicit projective generators on the algebra side. So, we have to work in the usual model structure on $s\mathcal{A}$, which is of course fine but technically speaking slightly less clean.

We need for $\otimes$ to commute with colimits in the left variable so that T_n being Σ-free implies that Σ_k acts freely on $T_n(k) \otimes X^{\otimes k}$.

2.2 Complications

3 GHOsT for ∞-categories

Okay, I lied: actually I'll talk just a little bit about how complicated the model-categorical version gets.

3.1 ∞-categorical generalities

Presentable ∞-categories are equivalent to cofibrantly generated model categories, in the appropriate sense.

Of course, usually our monoidal structure on $\mathcal{C}$ will be symmetric, and will commute with colimits in both variables.

To have our left localizations, we'll probably actually need our monoidal product on $\mathcal{C}$ to commute with colimits in both variables, since equivalences will actually be created by $E \otimes - : \mathcal{C} \to \mathcal{C}$.

We need to identify $\mathcal{A} \subset \mathcal{P}^{\delta}_{\Sigma}(\mathcal{G}_{\mathcal{A}})$ as a full subcategory. Identifying it as the subcategory of sheaves for some topology is actually pretty natural; I have no idea how one would expect to do this otherwise, anyways.

These $\mathcal{P}_{\Sigma}$ categories are also called *homotopy varieties* (in the sense of universal algebra), and have been studied by Rosicky under that name.

Writing $\mathcal{D}^{\wedge}$ for the Ind-completion of $\mathcal{D}$, the equivalence $\mathscr{D}^{-}_{\geq 0}(\mathcal{D}^{\wedge}) \xrightarrow{\sim} \mathcal{P}_{\Sigma}(\mathcal{D}_{\mathrm{proj}})$ is induced from the Yoneda embedding $\mathcal{D}^{\wedge}_{\mathrm{proj}} \hookrightarrow \mathcal{P}_{\Sigma}(\mathcal{D}_{\mathrm{proj}})$ by the universal property of $\mathscr{D}^{-}_{\geq 0}$.

I have no idea how to do all this unstably for homology, unless we can somehow get a handle on the functor $X \mapsto \Omega^{\infty} E \otimes X$. Note that Ω^{∞} is a right adjoint, so won't preserve the decomposition $E \simeq \operatorname{colim} E_{\alpha}$.

This isn't mentioned explicitly in the slides, but we'll of course need functorial Postnikov truncations. These localizations exist (and are essentially unique) for abstract reasons, but in fact one can also construct them by taking the "fundamental" map to the appropriate Eilenberg–MacLane object and then taking its fiber. (This is actually what makes the Postnikov tower convenient, is that its maps are all *fiber inclusions*: that is, up to homotopy it is the inclusion of the fiber of a fibration.) On the other hand, note that one *cannot* take the usual functorial construction where one cones off all possible maps in from high-dimensional spheres: to be totally invariant, we can only say that ∞-categories are only enriched in the ∞-category of spaces. (On the other hand, any ∞-category can be presented as a topological category, and once we've made this (noncanonical) choice then we *do* get such a functorial construction of Postnikov towers.)

3.2 Model ∞-categories

We need model ∞-categories in a pretty serious way to identify the various module structures as the expected ones (and their various maps as maps of modules); in turn, we need these module structures to know what André–Quillen cohomology groups we need to compute!

But also, note that without knowing that $\mathcal{P}_{\Sigma}(T(\mathcal{G}^{E}_{\mathcal{C}}))$ underlies $\mathrm{Alg}_T(s\mathcal{C})_{E^2_{\mathcal{G}^E_{\mathcal{C}}}}$, it's pretty mysterious how one might hope to talk about T-algebras in $\mathcal{P}_{\Sigma}(\mathcal{G}^{E}_{\mathcal{C}})$: T itself is an honest simplicial object, but the objects of $\mathcal{P}_{\Sigma}(\mathcal{G}^{E}_{\mathcal{C}})$ themselves are only noncanonically presentable as simplicial objects. This is another sense in which model ∞-categories are necessary to give a complete picture of what's going on here.

Model ∞-categories are a model-independent notion! That is, they don't depend on what you mean by "∞-category".

In re co/simplicial resolutions, originally from Dwyer–Kan's *Function complexes in homotopical algebra*: really, we only need a relative category $(\mathcal{C}, \mathbf{W})$, and we only need that all cosimplicial resolutions $\Gamma^\bullet(d_1)$ and simplicial resolutions $\Lambda_\bullet(d_2)$ determine functors

$$\hom^{\mathrm{lw}}(\Gamma^\bullet(d_1), -) : (\mathcal{C}, \mathbf{W}) \to (s\mathcal{S}\mathrm{et}, \mathbf{W}_{\mathrm{Quillen}}) \quad \text{and} \quad \hom^{\mathrm{lw}}(-, \Lambda_\bullet(d_2)) : (\mathcal{C}, \mathbf{W})^{op} \to (s\mathcal{S}\mathrm{et}, \mathbf{W}_{\mathrm{Quillen}})$$

of relative categories. But then of course, by far the easiest way to ensure these conditions is to just have an extension of the relative category to a model structure in the first place.

To prove that given $d \overset{\approx}{\rightarrowtail} d'$ we obtain $\hom^{\overset{r}{\sim}}_{\mathcal{D}}(d', d_2) \xrightarrow{\sim} \hom^{\overset{r}{\sim}}_{\mathcal{D}}(d, d_2)$, we actually need to be careful with basepoints, since the E^2-model structure is for based simplicial spaces. We can actually use Dwyer–Kan's original argument to show that this map is an isomorphism on π_0, and then at any given basepoint in the source we can use the argument alluded to in the slides to show that we have an E^2-equivalence with respect to it.

Of course, the whole point is that the complete Segal space functor (taking $\mathbf{W}$ to be trivial) defines an *equivalence* between ∞-categories and simplicial spaces satisfying the Segal condition. The backwards functor is given by taking a simplicial space $X_\bullet$ to the coend $\int_{\mathbf{\Delta}} X_\bullet \times [\bullet]$ (taken in $\mathcal{C}\mathrm{at}_\infty$).

3.3 Applications and generalizations

The inclusion $\mathscr{X}_{\text{ét,aff}} \hookrightarrow \mathscr{X}_{\text{ét}}$ is the inclusion of a subsite assuming the stack $\mathscr{X}$ is *separated* – that is, its diagonal map is closed. (This implies that the intersection of two affines is again affine.)

Well, enriching the obstruction theory should at least be purely formal when the enriching ∞-category has the cartesian monoidal structure. Otherwise, things get a lot crazier: this source of complication is what earned Haugseng his PhD.

Note that discrete motivic spaces probably don't form a topos. So, we need our various theories (e.g. of André–Quillen cohomology group objects, and then later of operator categories) to be somewhat more general.

Via Elmendorf's theorem, the equivariant analog of taking global sections is taking G-fixedpoints.

We might also hope to produce a version for algebraic theories (which are more general than operads). The main thing to explore is the relationship between E_*X and E_*VX for V (the free functor of the monadic adjunction associated to) an algebraic theory.