6 Construction of TMF – Aaron Mazel-Gee

Contents

6.0 You could've invented tmf.

We begin by providing some motivation for this entire seminar, in the process of which we'll meet some of the objects that we'll be looking at more closely in this talk – informally at first, but then we'll go back through the relevant parts more carefully. We admit right up front that we'll be ignoring the minor issue of periodification throughout this introduction.

1. $\mathtt{SHC}^{fin}$ is a tensored triangulated category, so we can talk about ideals and thick subcategories. We define $\mathrm{Spec}(\mathtt{SHC}^{fin})$ to be the space of thick triangulated prime ideals.[1] Note that the kernel of any homology theory is a thick ideal: it is thick by the long exact sequence, and it is an ideal because if $X \in \ker(E_*)$ then $E \wedge X \simeq *$ so $E \wedge (X \wedge Y) \simeq * \wedge Y \simeq *$ for any Y. This needn't be prime, however, as e.g. stable homotopy illustrates.

2. By the *nilpotence theorem* of Devinatz-Hopkins-Smith, $\mathrm{Spec}(\mathtt{SHC}^{fin}) \cong \mathrm{Spec}(\mathbb{Z}) \wedge (\mathbb{N}_0 \cup \{\infty\})$.[2]

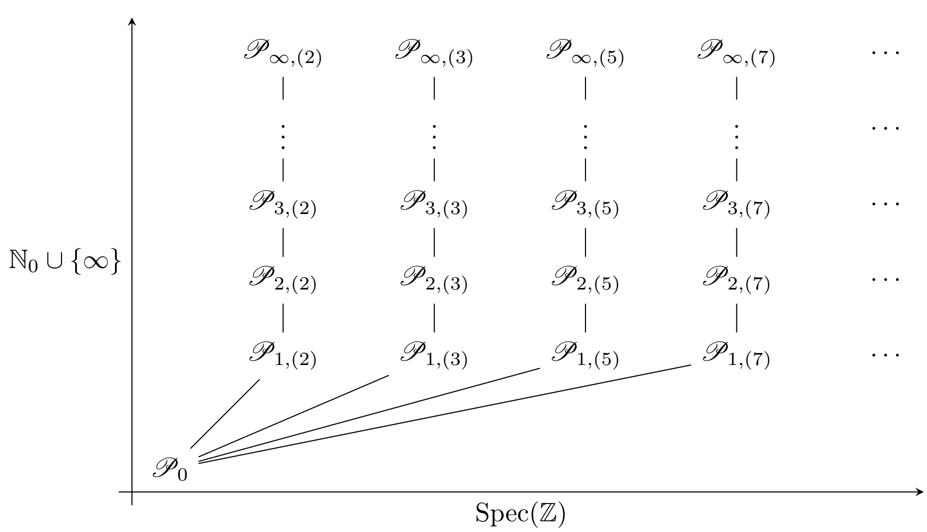

Figure 1: $\mathrm{Spec}(\mathtt{SHC}^{fin})$.

[1] This actually comes with the "opposite Zariski topology": a basis of open sets is $V(X) = \{\mathscr{P} : X \in \mathscr{P}\}$, and the sets $D(X) = \{\mathscr{P} : X \notin \mathscr{P}\}$ are closed. However, when we take our tensored triangulated category to be $K^b(\mathtt{Proj}_R)$, the perfect complexes of R-modules, we recover the space $\mathrm{Spec}(R)$; thus, this is a reasonable definition. And after all, $\mathtt{SHC}^{fin} = K^b(\mathtt{Proj}_{\mathbb{F}_1})$!

[2] Cf. Balmer's *Spectra, spectra, spectra – tensor triangular spectra versus Zariski spectra of endomorphism rings*.

- For every $[(p)] \in \mathrm{Spec}(\mathbb{Z})$ we have a tower of points $\mathscr{P}_{n,(p)}$ for $1 \leq n \leq \infty$, and there is a generic point $\mathscr{P}_0$ corresponding to $[(0)] \in \mathrm{Spec}(\mathbb{Z})$. These are defined by $\mathscr{P}_{n,(p)} = \ker(K(n,p)_*)$, where $K(n,p)$ is the n^{th} *Morava K-theory* at the prime p. (We could also say that $\mathscr{P}_0 = \mathscr{P}_{0,(p)}$ for any prime p, since we always have $K(0,p) = H\mathbb{Q}$.)
- $\overline{\mathscr{P}_{n,(p)}} = \{\mathscr{P}_{m,(p)} : n \leq m \leq \infty\}$ (and $\overline{\mathscr{P}_0} = \mathrm{Spec}(\mathtt{SHC}^{fin})$), and these form a basis of closed sets.
- All closed sets are obtainable as the support of some function $X \in \mathtt{SHC}^{fin}$. In particular, $\mathrm{supp}(X) = \mathrm{Spec}(\mathtt{SHC}^{fin})$ iff $H\mathbb{Q}_*X \neq 0$. Otherwise, $\mathrm{supp}(X) = \bigcup_{p \text{ prime}} \overline{\mathscr{P}_{\mathrm{type}(X,p),(p)}}$, where the union is over finitely many primes.

This is called the *chromatic filtration* of the (finite) stable homotopy category.

3. The *Morava K-theories* are complex-oriented spectra $K(n) = K(n,p)$ for $0 \leq n \leq \infty$. The edge cases are $K(0) = H\mathbb{Q}$, $K(1) = KU/p$, and $K(\infty) = H\mathbb{F}_p$. For $n \geq 1$, $\mathbb{G}_{K(n)} = H_n = H_{n,p}$, the height-$n$ *Honda formal group* over $\mathbb{F}_p$. (If F is a formal group law over a field k of characteristic p, then $[p]_F(x) = ux^{p^n} + \text{h.o.t.}$ (with $u \neq 0$) for some $n \geq 1$, called the *height* of F. This is an isomorphism invariant.[3] We define H_n by saying that $[p]_{H_n}(x) = x^{p^n}$ with x a p-typical coordinate. H_n is defined over $\mathbb{F}_p$, but up to algebraic closure height is a complete isomorphism invariant; thus, up to base change the H_n give all formal groups over $\mathbb{F}_p$.

   The Morava K-theories are of central importance in chromatic homotopy theory. In the sense given above, they are the residue fields of $\mathrm{Spec}(\mathtt{SHC}^{fin})$. Further, they are essentially all of the homology theories admitting Künneth isomorphisms. In fact, for any $X \in \mathtt{SHC}$, $K(n) \wedge X \simeq \bigvee_j \Sigma^{i_j} K(n)$, and this admits no nontrivial retracts – that is, $K(n)$ is a *field* (i.e. all its modules are free) – and moreover, any field takes the form $\bigvee_j \Sigma^{i_j} K(n)$ (as a spectrum). So, these are also the prime fields of $\mathtt{SHC}$ in the same sense that $\mathbb{Q}$ and the $\mathbb{F}_p$ are the prime fields of $\mathtt{Mod}_{\mathbb{Z}}$.

   However, as useful as the Morava K-theories are for detecting information at the various points of $\mathrm{Spec}(\mathtt{SHC}^{fin})$, they do not tell us how to stitch that information back together.

4. In order to globalize in the *chromatic* direction, there are the complex-oriented *Morava E-theories* $E_n = E_{n,p}$ for $0 \leq n < \infty$.[4] The edge cases are $E_0 = H\mathbb{Q}$ and $E_1 = KU_p^\wedge$. For $n \geq 1$, $\mathbb{G}_{E_n} = \widetilde{H}_n = \widetilde{H}_{n,p}$ is a *universal deformation* of $H_n \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n}$ into complete local rings over $\mathbb{F}_{p^n}$, which lives over the *Lubin-Tate deformation space* $LT_n = LT_{n,p} = \mathrm{Spf}((E_n)_0) \cong \mathrm{Spf}(\mathbb{Z}_p[\zeta_{p^n-1}][\![u_1, \ldots, u_{n-1}]\!])$.[5] That is, if A is any complete local ring with residue field $A/\mathfrak{m}_A$ admitting an inclusion $i : \mathbb{F}_{p^n} \to A/\mathfrak{m}_A$ and $\mathbb{G}/A$ is a formal group which reduces to $i^*H_n/(A/\mathfrak{m}_A)$, then there exists a unique map $f : \mathrm{Spf}(A) \to LT_n$ such that $f^*\widetilde{H}_n \cong \mathbb{G}$ and such that this isomorphism reduces to the identity morphism on special fibers.[6] The formal group $\widetilde{H}_n$ is rather complicated, but it ends up that in its p-series (in a suitable coordinate), the first nonzero term mod $(p, u_1, \ldots, u_{i-1}, u_i^2, \ldots, u_{n-1}^2)$ is $u_i x^{p^i}$. Thus, over the special fiber a deformation of H_n can have any height up to n, and that height is at least i iff the classifying map kills $(p, u_1, \ldots, u_{i-1})$.

   Now, E_n has the same *Bousfield class* as $\bigvee_{i=0}^n K(i)$, meaning that $(E_n)_*X = 0$ iff $K(i)_*X = 0$ for $0 \leq i \leq n$.[7] (This should be vaguely plausible based on the above observation, if you believe that chromatic homotopy theory works out as beautifully as one might hope.) So, E_n detects whether X is supported over $\{\mathscr{P}_0, \mathscr{P}_{1,(p)}, \ldots, \mathscr{P}_{n,(p)}\}$. For any homology theory E, there is an idempotent unital endofunctor $L_E : \mathtt{SHC} \to \mathtt{SHC}$, called *Bousfield localization*, which can be thought of as very roughly analogous to localization of a ring (but instead, $\mathtt{SHC}$) with respect to some multiplicatively closed set (but instead, the thick subcategory $\ker(E_*)$).[8] Then, we can relate the $L_nX = L_{E_n}X$ as n varies via the *chromatic fracture square*, which is the homotopy pullback square

[3] If G is a formal group over an $\mathbb{F}_p$-scheme, then the height of G can also be defined as the number of iterates of the relative Frobenius through which we can factor $[p] : G \to G$.

[4] There isn't an obvious notion of a Morava E_n for $n = \infty$; the universal deformation space of $(\widehat{\mathbb{G}}_a)_{\mathbb{F}_p}$ is a stack, and our usual construction only associates spectra to certain (formal) schemes equipped with formal groups.

[5] The extension of the coefficients of H_n is explained by the fact that we have the sequence of inclusions $\mathrm{Aut}_{\mathbb{F}_p}(H_n) \subseteq \mathrm{Aut}_{\mathbb{F}_{p^2}}(H_n \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}) \subseteq \cdots \subseteq \mathrm{Aut}_{\overline{\mathbb{F}}_p}(H_n \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)$, and this stabilizes precisely at $\mathrm{Aut}_{\mathbb{F}_{p^n}}(H_n \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$.

[6] Note that universal deformations aren't actually universal; there's no canonical choice, although they form a contractible groupoid. For this reason, some people insist on saying *a* universal deformation, as opposed to *the* universal deformation. We may slip up occasionally.

[7] This is a statement about all spectra, not just finite spectra. If $X \in \mathtt{SHC}^{fin}$, then $(E_n)_*X = 0$ iff $K(n)_*X = 0$; this follows from the general fact that if $K(i)_*X = 0$ then $K(i-1)_*X = 0$.

[8] Actually, in this case we're localizing at a relatively open subset in $\mathrm{Spec}(\mathtt{SHC}_{(p)}^{fin})$.

$$\begin{tikzcd}
L_{n+1}X \arrow[r, "\eta_{L_{K(n+1)}}(L_{n+1}X)"] \arrow[d, "\eta_{L_n}(L_{n+1}X)"'] \arrow[dr, phantom, "h", very near start] & L_{K(n+1)}X \arrow[d, "\eta_{L_n}(L_{K(n+1)}X)"] \\
L_nX \arrow[r, "L_n(\eta_{L_{K(n+1)}}(X))"'] & L_nL_{K(n+1)}X
\end{tikzcd}$$

This gives us the *chromatic tower* $\cdots \to L_2X \to L_1X \to L_0X$, and the *chromatic convergence theorem* says that if $X \in \mathtt{SHC}^{fin}$, then $X_{(p)} \simeq \operatorname{holim}_{n<\infty} L_nX$.

5. But this also suggests how to globalize in the *arithemetic* direction. A *global height-($\leq n$) theory* should be a homology theory which allows us to recover the $E_{n,p}$ at all primes p. Obviously, $H\mathbb{Q}$ is a global height-(≤ 0) theory, since $E_0 = H\mathbb{Q}$ at all primes. Next, KU is a global height-(≤ 1) theory: we just p-complete to recover $E_{1,p}$. This should be thought of us the global sections of a quasicoherent spectrum-valued sheaf over $\operatorname{Spec}(\mathbb{Z})$; we obtain $E_{1,p}$ by evaluating on $LT_{1,p} = \operatorname{Spf}(\mathbb{Z}_p) \to \operatorname{Spec}(\mathbb{Z})$, since $E_{1,p} = KU \widehat{\otimes}_{\mathbb{Z}} \mathbb{Z}_p = \lim_n KU \otimes_{\mathbb{Z}} \mathbb{Z}/p^n$. Moreover, the sections of this sheaf are complex-orientable, with formal groups isomorphic to the corresponding sections of the sheaf defined by $\widehat{\mathbb{G}}_m \to \operatorname{Spec}(\mathbb{Z})$. (Note that $\widetilde{H}_{1,p} \cong (\mathbb{G}_m)_{\mathbb{Z}_p}$; we can take the multiplicative formal group law as its own universal deformation.) So, to obtain a global height-(≤ 2) theory, we should look for a scheme or stack $\mathcal{M}$ with a sheaf of formal groups and with maps $LT_{2,p} \to \mathcal{M}$ on which our sheaf evaluates to $\widetilde{H}_{n,p}$ – or equivalently, for a scheme or stack $\mathcal{M}$ with a map $\mathcal{M} \to \mathcal{M}_{FG}$ through which the inclusions of the formal neighborhoods $LT_{2,p}$ all factor.

6. We might think that we could try to define a sheaf of homology theories over $\mathcal{M}_{FG}$, but this is not possible with the current cutting-edge technology, and for various reasons probably not possible at all.[9] We may summarily say simply that $\mathcal{M}_{FG}$ isn't sufficiently rigid.

7. However, it turns out there are appropriate natural maps $LT_{2,p} \to \mathcal{M}^{ss}_{ell,p}$, where the target is the completion of $(\mathcal{M}^{ss}_{ell})_{\mathbb{F}_p} \subset \overline{\mathcal{M}}_{ell}$.[10] We can see this as follows.

   (a) Any abelian variety A/k has a *p-divisible group* $A[p^\infty] = \operatorname{colim} A[p^n]$.

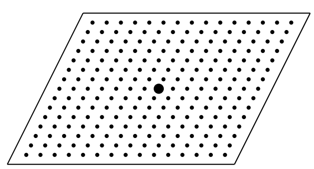

Figure 2: The p-divisible group of a complex elliptic curve.

   (b) If k is a field of characteristic $p > 0$, then the *Serre-Tate theorem* says that $\operatorname{Def}_k(A) \cong \operatorname{Def}_k(A[p^\infty])$.

   (c) There is a short exact sequence $0 \to \widehat{A} \to A[p^\infty] \to A_{\text{ét}} \to 0$, under which height is additive; we always have $\operatorname{ht}(A[p^\infty]) = 2\dim A$.

   (d) Elliptic curves are 1-dimensional abelian varieties, so an elliptic curve C/k has $\operatorname{ht}(C[p^\infty]) = 2$. C can be *ordinary*, meaning that $\operatorname{ht}(\widehat{C}) = 1$, or *supersingular*, meaning that $\operatorname{ht}(\widehat{C}) = 2$.

   (e) Thus if C is supersingular then $\widehat{C} = C[p^\infty]$, and hence $\operatorname{Def}_k(\widehat{C}) \cong \operatorname{Def}_k(C[p^\infty]) \cong \operatorname{Def}_k(C)$.

   In particular, the composition $LT_{2,p} \to \mathcal{M}^{ss}_{ell,p} \to \mathcal{M}_{FG}$ is indeed the canonical inclusion.

[9] In the E_∞-version of the Hopkins-Miller theorem, the subgroup structure of the formal groups determines the E_∞-structure, i.e. the associated power operations. But universal deformations are rather special, and in general formal groups don't contain the information necessary to determine these. Lurie's realization theorem applies to maps of stacks to $\mathcal{M}_{FG}$ which factor through some moduli stack $\mathcal{M}_p(n)$ of p-divisible groups of height $\leq n$ with formal component of dimension 1, which sidesteps this problem entirely. Cf. Goerss's *Topological modular forms [after Hopkins, Miller, and Lurie]* and *Realizing families of Landweber exact homology theories.*

[10] Obviously, the map $\mathcal{M}^{ss}_{ell,p} \to \mathcal{M}_{FG}$ factors through $\mathcal{M}_p(2)$.

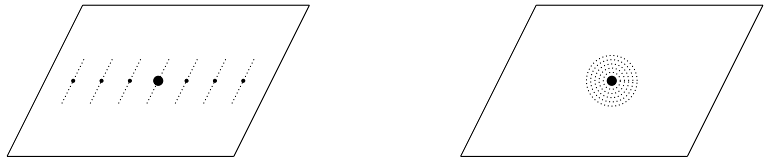

Figure 3: The p-divisible groups of ordinary and supersingular elliptic curves in characteristic p.

8. So, we might hope to get a global height-(≤ 2) theory by taking global sections on the resulting sheaf of homology theories over $\coprod_p \mathcal{M}_{ell,p}^{ss}$. However, since this stack is disconnected, we can't really expect to get any integral behavior from this construction: as things stand, we'd really just be collecting all our constructions together by taking a coproduct. Rather, we would like some single connected object into which all the $\mathcal{M}_{ell,p}^{ss}$ embed. But there is an obvious choice, namely $\mathcal{M}_{ell}$![11] That is:

$$\boxed{\begin{array}{l}\textbf{We pass to Spec}(\mathbb{Z})\textbf{ to put all primes in the game at once, and then we use}\\ \textbf{the ordinary points to interpolate between the supersingular neighborhoods.}\end{array}}$$

Of course, there are a number of other reasons why people care about *tmf*.

- We can use *tmf* to get at the homotopy groups of spheres via the classical mod 2 Adams spectral sequence. There is a sequence of spectra whose mod 2 cohomologies better and better approximate that of the sphere, as follows.

$$\begin{aligned} H\mathbb{F}_2^* H\mathbb{F}_2 &= \mathcal{A} \\ H\mathbb{F}_2^* H\mathbb{Z} &= \mathcal{A} /\!\!/ \langle \mathrm{Sq}^1 \rangle \\ H\mathbb{F}_2^* ko &= \mathcal{A} /\!\!/ \langle \mathrm{Sq}^1, \mathrm{Sq}^2 \rangle \\ H\mathbb{F}_2^* tmf_{(2)} &= \mathcal{A} /\!\!/ \langle \mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4 \rangle \\ &\vdots \\ H\mathbb{F}_2^* S^0 &= \mathcal{A} /\!\!/ \langle \mathrm{Sq}^{2^n} : n \geq 0 \rangle \end{aligned}$$

  In fact, all further quotients of $\mathcal{A}$ in this sequence are obstructed by the Hopf invariant 1 theorem, so $tmf_{(2)}$ is is the best possible approximation along this route.

- Based on results in physics, Witten was able to define a genus on string manifolds taking values in integral modular forms, i.e. a ring homomorphism $M\mathrm{String}_{2*} \to MF_*$. This refines to the *string orientation*, which sits in the diagram

$$\begin{tikzcd} S^0 = M\mathrm{Framed} \arrow[r, "\mathrm{holim}"] & \cdots \arrow[r] & M\mathrm{String} \arrow[r] \arrow[d] & M\mathrm{Spin} \arrow[r] \arrow[d] & MSO \arrow[r] \arrow[d] & MO \arrow[d] \\ & & tmf & ko & H\mathbb{Z} & H\mathbb{F}_2 \end{tikzcd}$$

  of factorizations of the unit maps of the ring spectra we saw above. Each factorization is easily seen to be sharp with respect to this tower, which gives a refined sense in which these spectra better and better approximate the sphere.

[11] Actually, in keeping with the number theory, where one demands that modular forms evaluate holomorphically at the Tate curve, we work over the Deligne-Mumford compactification $\overline{\mathcal{M}}_{ell}$ in which we allow nodal degenerations but still require our geometric fibers to be irreducible (i.e. no Néron n-gons for $n > 1$). On the one hand, this is reasonable for our purposes, since we really only care about the formal groups, after all, and in fact $tmf_{(p)}$ is an $H\mathbb{F}_p$-spectrum (as we will see shortly for $p = 2$) while $TMF_{(p)}$ is not. But the question remains why number theorists make these same choices in the first place. We're not sure, but e.g. Deligne-Rapoport's *Les schémas de modules de courbes elliptiques* provides some hints. When one allows arbitrary Néron n-gons, the resulting stack is not Artin. Working over $\mathbb{F}_p$, one can allow Néron n-gons for $(n, p) = 1$, and the resulting stack will be Artin (although not separated). Of course, since we're working over $\mathbb{Z}$, then to ensure that our stack is Artin we must require no Néron n-gons for $n > 1$, since any such n is divisible by some prime number.

But this applies to manifold theory as well. It is a classical result that a Spin manifold is nullbordant iff its *ko*- and $H\mathbb{F}_2$-characteristic classes (i.e. its *ko*-Pontrjagin and Stiefel-Whitney classes) vanish.[12,13] One therefore hopes that the *tmf*-characteristic classes might allow us to completely detect bordism classes of String manifolds.

6.1 Overview of the construction.

And so without further ado, we now given an illustration of the construction of the sheaf $\mathcal{O}^{top}$ on $\overline{\mathcal{M}}_{ell}$ whose global sections will define *Tmf* (and whose sections over $\mathcal{M}_{ell}$ will define *TMF*). Recall that by definition, $tmf = \tau_{\geq 0} Tmf$.

The stacks in the diagram all stand for their respective étale sites restricted to affine schemes (which fact we'll address in a moment) and have the following denotations:

- $\overline{\mathcal{M}}_{ell}$ is the moduli of generalized (i.e. irreducible and possible admitting cuspidal singularities) elliptic curves over Spec($\mathbb{Z}$);
- $(\overline{\mathcal{M}}_{ell})_{\mathbb{Q}}$ is the pullback of $\overline{\mathcal{M}}_{ell}$ along $\mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(\mathbb{Z})$;
- $(\overline{\mathcal{M}}_{ell})_p$ is the pullback of $\overline{\mathcal{M}}_{ell}$ along $\mathrm{Spf}(\mathbb{Z}_p) \to \mathrm{Spec}(\mathbb{Z})$;
- $\mathcal{M}_{ell}^{ord} \subset (\overline{\mathcal{M}}_{ell})_p$ is the substack of elliptic curves over p-complete rings with ordinary reduction;
- $\mathcal{M}_{ell}^{ss}$ is the completion of $(\mathcal{M}_{ell}^{ss})_{\mathbb{F}_p} \subset \overline{\mathcal{M}}_{ell}$.

Localizations are applied sectionwise. A number of comments are in order.

- Lest this construction seem unmotivated or ad hoc, we note that the geometry of sheaves on stacks implies that this necessarily recovers the stack we started with. That is, given a derived module sheaf $\mathcal{F}$ over $\overline{\mathcal{M}}_{ell}$, it must be that $\mathcal{F} \simeq \mathrm{holim}(\mathcal{F}_{\mathbb{Q}} \to (\prod_p \mathcal{F}_p)_{\mathbb{Q}} \leftarrow \prod_p \mathcal{F}_p)$ and that $\mathcal{F}_p \simeq \mathrm{holim}(\mathcal{F}_{ord} \to (\mathcal{F}_{ss})_{ord} \leftarrow \mathcal{F}_{ss})$, where subscripts denote derived completions along the appropriate substack.[14] In fact, it is the same setup applied to $\mathcal{M}_{FG}$ that yields the chromatic fracture squares.
- One of the most important points – indeed, what makes the construction of *tmf* so technically difficult – is that there is no immediate notion of global sections, since (by Yoneda) there is no terminal affine scheme with an étale map to $\overline{\mathcal{M}}_{ell}$.[15] We might instead try to extract a homotopy limit over all affine covers, but unfortunately the category of homology theories isn't complete. However, Brown representability tells us that all homology theories are associated to spectra, and so if we lift our presheaf to this category then we might have some renewed hope of a universal elliptic homology theory.[16]

  In fact, it turned out that it was easier[17] to prove a seemingly stronger result: our presheaf valued in homology theories actually lifts to a *sheaf* valued in E_∞*-ring spectra*. It seems that in this seminar we will mostly take these as a black box[18], but for the moment we will simply say that these are ring spectra which are "commutative up to all possible coherent homotopies". The main point here is that E_∞-rings (and their morphisms) are much more rigid than ordinary spectra (and their morphisms), and so in the immortal words of Lurie: "Although it is much harder to write down an E_∞-ring than a spectrum, it is also much harder to write down a map between E_∞-rings than a map between spectra. The practical effect of this, in our situation, is that it is much harder to write down the *wrong* maps between E_∞-rings and much easier to find the right ones." Indeed, the Goerss-Hopkins obstruction theory for E_∞-rings will dictate that all of our choices will be made from contractible spaces thereof.

[12] Cf. Anderson-Brown-Peterson's *The structure of the Spin cobordism ring.*

[13] Note that really we should be writing wedges of suspensions of these ring spectra, or actually even wedges of suspensions of various quotient spectra, if we want to capture all the characteristic classes.

[14] Given an ideal sheaf $\mathcal{I}$ on $\mathcal{M}$ we have the inclusions $j_n : \mathcal{M}/\mathcal{I}^n \to \mathcal{M}$, and then the derived completion of $\mathcal{F}$ along $\mathcal{I}$ is given by $\mathcal{F}_{\mathcal{I}} = \mathrm{holim}(j_n)_*(j_n)^*\mathcal{F}$.

[15] In the usual setup, a presheaf on a topological space X is just a contravariant functor on its associated category $\mathtt{Open}(X)$ of open subsets and inclusions. Then, its global sections are by definition given by its evaluation on the initial object $X \in \mathtt{Open}(X)^{op}$.

[16] Actually, one can define an étale morphism of stacks (on $\mathtt{Sch}_{\text{ét}}$), so that one can extend the étale site of $\overline{\mathcal{M}}_{ell}$ to include stacks; this makes it possible to literally take global sections by evaluating on the identity map. Cf. Douglas's *Sheaves in homotopy theory.*

[17] "Easier" is a relative term.

[18] Although we generally work in categories of spectra where the categories of commutative ring spectra and E_∞-rings are Quillen-equivalent, the obstruction theory is built using the precise structure of the operads in question.

$$
\begin{tikzcd}
\text{height 2} & & \mathcal{O}^{top} \arrow[r] \arrow[ddd] \arrow[dr, phantom, "\ulcorner h", very near start] & \prod_p (\iota_p)_* \mathcal{O}_p^{top} \arrow[ddd, "L_{\mathbb{Q}}"] & & \mathcal{O}_p^{top} \arrow[ll, squiggly] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner h", very near start] & (\iota_{ss})_* \mathcal{O}_{K(2)}^{top} \arrow[d, "L_{K(1)}"] & \mathcal{O}_{K(2)}^{top} \arrow[l, squiggly] \\
\text{height 1} & & & & \mathcal{O}_{K(1)}^{top} \arrow[r, squiggly] & (\iota_{ord})_* \mathcal{O}_{K(1)}^{top} \arrow[r, "\alpha_{\mathrm{chrom}}"'] & \left((\iota_{ss})_* \mathcal{O}_{K(2)}^{top} \right)_{K(1)} & \\
& & & & & & & \\
\text{height 0} & \mathcal{O}_{\mathbb{Q}}^{top} \arrow[r, squiggly] & (\iota_{\mathbb{Q}})_* \mathcal{O}_{\mathbb{Q}}^{top} \arrow[r, "\alpha_{\mathrm{arith}}"'] & \left(\prod_p (\iota_p)_* \mathcal{O}_p^{top} \right)_{\mathbb{Q}} & & & & \\
& & \overline{\mathcal{M}}_{ell} & & & & & \\
& (\overline{\mathcal{M}}_{ell})_{\mathbb{Q}} \arrow[ur, "\iota_{\mathbb{Q}}"] & & & & & (\overline{\mathcal{M}}_{ell})_p \arrow[ullll, "\iota_p"'] & \\
& & & & \mathcal{M}_{ell}^{ord} \arrow[urr, "\iota_{ord}"] & & & \mathcal{M}_{ell}^{ss} \arrow[ul, "\iota_{ss}"']
\end{tikzcd}
$$

Figure 4: The Diagram.

- What is this sheaf, anyways?
  - If $f : \mathrm{Spec}(R) \to \overline{\mathcal{M}}_{ell}$ classifies a generalized elliptic curve C/R, then $E = \mathcal{O}^{top}(\mathrm{Spec}(R))$ is a structured version of the homology theory obtained from the Landweber exact functor theorem. In particular:
    - $\pi_0(E) \cong R$ (so we may call $\mathcal{O}^{top}$ a *derived enhancement* of the ordinary structure sheaf on $\overline{\mathcal{M}}_{ell}$).
    - E is *weakly even-periodic*, i.e. $\pi_2 E \otimes_{\pi_0 E} \pi_n E \xrightarrow{\cong} \pi_{n+2} E$ and $\pi_{2n+1} E = 0$ for all $n \in \mathbb{Z}$. In particular, all even homotopy groups are rank-1 projective $\pi_0 E$-modules. (They aren't necessarily free since a formal group over a ring is only guaranteed to have a coordinate Zariski-locally.)
    - $\mathbb{G}_E \cong \widehat{C}$.
  - In fact, the algebraic geometry tells us that the inclusion $\mathbb{C}P^1 \to \mathbb{C}P^\infty$ corresponds to the projection

    $$\widetilde{E}^*(\mathbb{C}P^\infty) \cong x \cdot E^*[\![x]\!] = \mathfrak{m}_{\mathbb{G}_E,0} \to \widetilde{E}^*(\mathbb{C}P^1) = (x)/(x^2) = \mathfrak{m}_{\mathbb{G}_E,0}/\mathfrak{m}^2_{\mathbb{G}_E,0} = \omega_{\mathbb{G}_E/E_*} = \omega_{C/E_*}$$

    (which may be interpreted either as the relative cotangent space at the identity section or as the module of invariant 1-forms). Thus, sectionwise we have that $\pi_{2n}\mathcal{O}^{top} \cong \omega^{\otimes n}$. Since modular forms of weight n are by definition global sections of the line bundle $\omega^{\otimes n}$ over $\overline{\mathcal{M}}_{ell}$, we might therefore expect $\pi_{2n} Tmf$ to agree with MF_n. But in fact, $\pi_* Tmf$ is computed via the *descent spectral sequence* (which will be discussed in the final few talks of this seminar), which takes the form $H^s(\overline{\mathcal{M}}_{ell}, \pi_t^\dagger \mathcal{O}^{top}) \Rightarrow \pi_{t-s} Tmf$ (where the dagger denotes sheafification). Note that we have $E_2^{0,*} = MF_*$ by definition, and then the natural map $E_\infty^{0,*} \to E_2^{0,*}$ induces a map $\pi_{2*} Tmf \to MF_*$, which is an isomorphism away from 6 (in nonnegative degrees).[19] One might therefore call $\pi_* tmf$ the ring of *derived modular forms*. (Note that this is no longer even-concentrated.)
  - The derived stack $(\overline{\mathcal{M}}_{ell}, \mathcal{O}^{top})$ is essentially uniquely characterized by the requirements that it enhances $(\overline{\mathcal{M}}_{ell}, \mathcal{O}_{\overline{\mathcal{M}}_{ell}})$ and that its sectionwise homotopy groups recover the tensor powers of the module of invariant 1-forms.

- For X a Deligne-Mumford stack, $i : X_{\text{ét},aff} \to X_{\text{ét}}$ induces a Quillen equivalence $i^* : \mathtt{Pre}_{X_{\text{ét}}}(\mathtt{Sp}) \leftrightarrows \mathtt{Pre}_{X_{\text{ét},aff}}(\mathtt{Sp}) : i_*$ (using the Jardine model structure), and every fibrant presheaf over $X_{\text{ét}}$ (which in particular must satisfy descent for hypercovers) is the pushforward of a fibrant presheaf over $X_{\text{ét},aff}$. And $\mathcal{O}^{top}$ will be constructed as a fibrant presheaf, so the homotopy limit won't need to be corrected, so all you model category nerds can cool your jets.[20]

- We restrict our attention to étale maps because then we get what we want over $\mathcal{M}_{ell}^{ss}$, which as it turns out can be recovered via $K(2)$-localization.[21] Also, this will make the obstruction theory manageable: objects of the étale site will basically look just like pieces of $\overline{\mathcal{M}}_{ell}$, and pullbacks will not be so far from honest intersections. (Recall that an étale morphism should be thought of as a local-on-the-source isomorphism, and an étale cover is a set of étale maps which are jointly surjective on geometric points.) To wit...

6.2 The "easy" part: construction of $\mathcal{O}_{K(2)}^{top}$.

Throughout this section, R will be a p-complete ring, and C/R will be an elliptic curve.

Recall that the *scheme of n-torsion* of C is by definition the pullback of $C \xrightarrow{[n]} C \xleftarrow{0} \mathrm{Spec}(R)$ in the category of schemes. The *p-divisible group* of C is then defined to be $C[p^\infty] = \mathrm{colim}_n\, C[p^n]$. This should be thought of as an algebraic (as opposed to naive) intersection.

Example 1. Consider the group scheme $(\mathbb{G}_m)_{\mathbb{F}_p}$; recall that this can be presented as $(\mathbb{G}_m)_{\mathbb{F}_p} \cong \mathbb{F}_p[t^\pm]$, with comultiplication determined by $\Delta(t) = t \otimes t$. On geometric points, there are ℓ distinct ℓ^{th} roots of unity, but there is

[19] Things always go screwy when the geometry (in this case, the orders of automorphisms of elliptic curves) lines up with the characteristic.

[20] Or you can again cf. Douglas's *Sheaves in homotopy theory*.

[21] Actually we won't recover the E_2, but rather their homotopy fixedpoints along the automorphism groups of the associated supersingular elliptic curves. These are called *higher real K-theories* and denoted EO_2, although this notation is somewhat ambiguous since there's some mess regarding precisely which finite subgroup of the Morava stabilizer group we're using to take homotopy fixedpoints. To partially fix this, we can work away from a fixed prime p and study elliptic curves with p^k-level structure for sufficiently large k; this extra marking will kill off the automorphisms, although to counterbalance we'll end up with more supersingular points. In any case, by analogy the global sections of the resulting sheaves might be called *higher complex K-theories*.

only the trivial p^{th} root of unity: we can ask for roots of the polynomial $t^p - 1$, but already we have $t^p - 1 = (t-1)^p$. So in characteristic p, taking the set-theoretic p-torsion may not recover the rank of the group. However, passing to schemes of torsion always gives the correct rank. On the one hand, $(\mathbb{G}_m)_{\mathbb{F}_p}[\ell]$ is a constant group scheme on $\mathbb{Z}/\ell$, so this still works out fine. On the other hand, $(\mathbb{G}_m)_{\mathbb{F}_p}[p] \cong \mathbb{F}_p[t^{\pm}]/(t^p - 1) = \mathbb{F}_p[t]/(t-1)^p$ is unreduced, but it still has rank p. These both reflect that this group has rank 1.

Now, (assuming $\mathrm{ht}(\widehat{C}_0)$ is constant over all mod-p reductions C_0 of C) we have a short exact sequence $0 \to \widehat{C} \to C[p^\infty] \to C_{\text{ét}} \to 0$; this is analogous to the situation where G is a Lie group, and then we have the short exact sequence $0 \to G_0 \to G \to \pi_0 G \to 0$. Heights are additive over short exact sequences, and so we have the defining dichotomy

type	$\mathrm{ht}(\widehat{C})$	$\mathrm{ht}(C[p^\infty])$	$\mathrm{ht}(C_{\text{ét}})$
ordinary	1	2	1
supersingular	2	2	0

for elliptic curves.

Here is the theorem that inspired us to invent *tmf* in the first place.

Theorem 1 (Serre-Tate)**.** *If k is a field of characteristic p, C_0/k is an elliptic curve, and Def_k denotes deformations to complete local rings with residue field k, then $\mathrm{Def}_k(C_0) \to \mathrm{Def}_k(C_0[p^\infty])$ is an equivalence of categories. In other words, if A is such a ring, then*

$$\begin{tikzcd} \mathtt{Ell}_A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\, h", very near start] & p\text{-}\mathtt{div}_A \arrow[d] \\ \mathtt{Ell}_k \arrow[r] & p\text{-}\mathtt{div}_k \end{tikzcd}$$

is a pullback diagram.

(Note that we consider deformations as a groupoid.)

In general, deformations of a p-divisible group $\mathbb{G}$ yield deformations of its splitting sequence $0 \to \widehat{\mathbb{G}} \to \mathbb{G} \to \mathbb{G}_{\text{ét}} \to 0$; however, étale groups have no deformations (by definition), and so the deformations of $\mathbb{G}$ are determined by deformations of $\widehat{\mathbb{G}}$ along with an extension class. But if C_0/k is a supersingular elliptic curve, then $\mathrm{ht}((C_0)_{\text{ét}}) = 0$, and so there is no extension class to consider. So in this case, $\mathrm{Def}_k(C_0) \cong \mathrm{Def}_k(C_0[p^\infty]) \cong \mathrm{Def}_k(\widehat{C}_0)$.

Next, we have an identification of these formal moduli spaces.

Theorem 2 (Lubin-Tate)**.** *If k is a perfect field of characteristic p and $\mathbb{G}/k$ is a formal group of height $n < \infty$, then $\mathrm{Def}_k(\mathbb{G}) \cong \mathrm{Spf}(\mathbb{W}(k)[\![u_1, \ldots, u_{n-1}]\!])$.*

Here $\mathbb{W}(k)$ denotes the ring of *Witt vectors* of k, which is the initial complete local ring with residue field k; for instance, $\mathbb{W}(\mathbb{F}_p) = \mathbb{Z}_p$ and $\mathbb{W}(\mathbb{F}_{p^n}) = \mathbb{Z}_p[\zeta_{p^n-1}]$. Again, this result tells us that we can deform $\mathbb{G}$ into any height $\leq n$. It also tells us that the functor in question is homotopically discrete: deformations of formal groups admit no nontrivial automorphisms.

Write $B(k, \mathbb{G}) = \mathbb{W}(k)[\![u_1, \ldots, u_{n-1}]\!]$, and let $\widetilde{\mathbb{G}}/B(k,\mathbb{G})$ denote the universal deformation of $\mathbb{G}$. The following result lifts this whole story to topology.

Theorem 3 (Goerss-Hopkins-Miller)**.** *There is a contravariant functor taking the pair $(k, \mathbb{G})$ to the E_∞-ring spectrum $E(k,\mathbb{G})$, where $E(k,\mathbb{G})$ is Landweber exact and even periodic, $\pi_0 E(k,\mathbb{G}) \cong B(k,\mathbb{G})$, and $\mathbb{G}_{E(k,\mathbb{G})} \cong \widetilde{\mathbb{G}}$. This is an equivalence of topological categories onto its essential image.*

This theorem requires a ridiculous amount of work to prove, but we'll take it as a black box for the moment. It actually follows from the more general Goerss-Hopkins obstruction theory, which we'll talk about in a bit.

With this in hand, we can now construct $\mathcal{O}^{top}_{K(2)}$. So suppose that C/R is the elliptic curve classified by the étale map $f : \mathrm{Spf}(R, I) \to \mathcal{M}^{ss}_{ell}$. This induces an étale map on special fibers $f_0 : \mathrm{Spec}(R/I) \to (\mathcal{M}^{ss}_{ell})_{\mathbb{F}_p}$, which classifies the reduction $C_0/(R/I)$. Observe that the target is zero-dimensional; this is because for any formal group $\mathbb{G}/k$ of height 2, by our identification of its p-series we know that the height-2 locus of $\widetilde{\mathbb{G}}$ in $\mathrm{Spf}(B(k,\mathbb{G})) \cong \mathrm{Spf}(\mathbb{W}(k)[\![u_1]\!])$

is precisely $V(p, u_1) \cong \mathrm{Spec}(k)$. This means that $\mathrm{Spec}(R/I)$ is étale over $\mathrm{Spec}(\mathbb{F}_p)$, so we have $R/I \cong \prod_i k_i$ for some finite (i.e. étale) field extensions $k_i/\mathbb{F}_p$. This induces a decomposition $C_0 \cong \coprod_i C_{0,i}$. Since f is étale, then C is a universal deformation of C_0, so $R \cong \prod_i B(k_i, \widehat{C}_{0,i})$. Thus we set $\mathcal{O}^{top}_{K(2)}(f : \mathrm{Spf}(R) \to \mathcal{M}^{ss}_{ell}) = \prod_i E(k_i, \widehat{C}_{0,i})$. This is even periodic, and by construction its formal group is isomorphic to $\widehat{C}$. So it is indeed an elliptic E_∞-ring associated to C/R.

6.3 The not "easy" part: outline of the rest of the construction.

For the remainder of this talk, we'll give a sweeping overview of the rest of the construction.

6.3.1 Talk 7: The Igusa tower.

We have a moduli stack $\mathcal{M}^{ord}_{ell}(p^k)$ of generalized elliptic curves C/R (with R p-complete) with ordinary reduction, equipped with *p^k-level structure*, i.e. an isomorphism $\mathbb{G}_m[p^k] \xrightarrow{\cong} \widehat{C}[p^k]$.[22] These assemble into the *Igusa tower*

$$\mathrm{Spf}(V^\wedge_\infty) = \mathcal{M}^{ord}_{ell}(p^\infty) \xrightarrow{\lim} \cdots \longrightarrow \mathcal{M}^{ord}_{ell}(p^2) \to \mathcal{M}^{ord}_{ell}(p^1) \to \mathcal{M}^{ord}_{ell}(p^0) = \mathcal{M}^{ord}_{ell}$$

Here, $V^\wedge_\infty$ is the ring of *p-adic modular functions*, i.e. the universal invariants for generalized elliptic curves C/R (necessarily with no supersingular fibers) equipped with a trivialization $\mathbb{G}_m \xrightarrow{\cong} \widehat{C}$.[23]

Every map $\mathcal{M}^{ord}_{ell}(p^{k+1}) \to \mathcal{M}^{ord}_{ell}(p^k)$ is an étale $\mathbb{Z}/p$-torsor, except at $k = 0$ when this is an étale $(\mathbb{Z}/p)^\times$-torsor; these compose to make the map $\mathrm{Spf}(V^\wedge_\infty) \to \mathcal{M}^{ord}_{ell}$ into an ind-étale $\mathbb{Z}_p^\times$-torsor (via the identification $\mathrm{Aut}_{\mathbb{Z}_p}(\widehat{\mathbb{G}}_m) \cong \mathbb{Z}_p^\times$). In fact, $V^\wedge_\infty$ is canonically a *θ-algebra*, which is the structure naturally present on the p-adic K-theory of an E_∞-ring. Roughly, this is the data of an action of $\mathbb{Z}_p^\times$, called the *Adams operations*, along with a commuting Frobenius lift. Here, the Adams operations are defined via precomposition of the trivialization with elements of $\mathrm{Aut}((\widehat{\mathbb{G}}_m)_{\mathbb{Z}_p}) \cong \mathbb{Z}_p^\times$, and the Frobenius lift is defined by taking $(C, \phi : \widehat{\mathbb{G}}_m \xrightarrow{\cong} \widehat{C})$ to $(C/C[p], \overline{\phi} : \widehat{\mathbb{G}}_m \cong \widehat{\mathbb{G}}_m/\widehat{\mathbb{G}}_m[p] \xrightarrow{\cong} \widehat{C/C[p]})$.

6.3.2 Talk 8: θ-algebras and E_∞-rings.

Lifts of maps of θ-algebras to maps of E_∞-rings are governed by *Goerss-Hopkins obstruction theory.* Goerss-Hopkins obstruction theory is an extremely general framework for realizing an algebraic map (of algebras over a monad) as a topological map (of algebras in spectra over some operad). They construct a theory of functorial Postnikov towers with respect to the appropriate notion of homotopy groups, which allows them to construct a tower of moduli spaces of maps of simplicial spectra whose realizations yield better and better approximations to the given algebraic map. The Eilenberg-MacLane objects for this theory of Postnikov towers (i.e. the targets of the k-invariants) naturally represent André-Quillen cohomology; thus the obstructions to lifting a vertex through the tower live in André-Quillen cohomology groups in the appropriate algebraic category. Fantastically, this means that the obstructions are given entirely at the level of algebra.

6.3.3 Talk 9: $K(1)$-local elliptic spectra.

Suppose R is p-complete, and suppose E is a $K(1)$-local E_∞ elliptic spectrum associated to a generalized elliptic curve C/R. (That E is $K(1)$-local implies that C has ordinary reduction.) Then the p-adic K-theory of E is given by the pullback diagram

[22] For $k > 0$, the existence of such a level structure implies that C cannot have any supersingular fibers anyways.

[23] This is immediate by Yoneda's lemma, once we know that $\mathcal{M}^{ord}_{ell}(p^\infty)$ is formally affine. In fact, $\mathcal{M}^{ord}_{ell}(p^k)$ is formally affine for all $k \geq 2$, and is formally affine for $k = 1$ whenever $p > 2$.

$$\begin{tikzcd} \mathrm{Spf}((K_p^\wedge)_0 E) \arrow[r] \arrow[d] \arrow[dr, phantom, "h"] & \mathcal{M}_{ell}^{ord}(p^\infty) \arrow[d] \\ \mathrm{Spf}(R) \arrow[r, "f"'] & \mathcal{M}_{ell}^{ord} \end{tikzcd}$$

This is always $\mathbb{Z}_p^\times$-equivariant, i.e. the Adams operations on $(K_p^\wedge)_0E$ coincide with the torsor structure induced from that of $\mathrm{Spf}(V_\infty^\wedge) \to \mathcal{M}_{ell}^{ord}$. When f is étale, this also induces a θ-algebra structure on $(K_p^\wedge)_0E$. This may not coincide with the one already there, but it will coincide on the sections of $\mathcal{O}_{K(1)}^{top}$ by construction.

6.3.4 Talk 10: Construction of $\mathcal{O}_{K(1)}^{top}$.

The construction of $\mathcal{O}_{K(1)}^{top}$ proceeds in two steps, both using Goerss-Hopkins obstruction theory.

1. We construct $tmf_{K(1)} = \mathcal{O}_{K(1)}^{top}(\mathcal{M}_{ell}^{ord})$ as follows.
   (a) If $p > 2$ then $\mathcal{M}_{ell}^{ord}(p)$ is formally affine, and so we can relatively easily construct $tmf(p)^{ord} = \mathcal{O}_{K(1)}^{top}(\mathcal{M}_{ell}^{ord}(p))$ along with an action of $(\mathbb{Z}/p)^\times$ through E_∞-ring maps. Then, we set $tmf_{K(1)} = (tmf(p)^{ord})^{h(\mathbb{Z}/p)^\times}$.
   (b) At $p = 2$ we only have that $\mathcal{M}_{ell}^{ord}(4)$ is formally affine. We might like mimic the previous setup and try to construct a spectrum $tmf(4)^{ord}$ with a $(\mathbb{Z}/4)^\times$-action, but this group has order 2 and so the obstructions don't vanish. Instead, we replace K with KO; the obstruction theory doesn't carry over entirely, but it carries over enough for us to be able to produce $tmf_{K(1)}$ directly.[24]
2. We construct the sheaf $\mathcal{O}_{K(1)}^{top}$ in the category of $tmf_{K(1)}$-algebras.

The second step uses crucially the isomorphism $(K_p^\wedge)_0 tmf_{K(1)} \cong V_\infty^\wedge$, which reflects the fact that when E and F are Landweber exact, then $\mathrm{Spec}(\pi_0(E \wedge F)) = \underline{\mathrm{Iso}}(\mathbb{G}_E, \mathbb{G}_F)$.

6.3.5 Not a talk: The chromatic attaching map.

The sheaves $\mathcal{O}_{K(1)}^{top}$ and $\mathcal{O}_{K(2)}^{top}$ interrelate as follows. Write $B = \mathbb{W}(k)[[u_1]]$ with C/B a universal deformation of a supersingular elliptic curve, and let $E = \mathcal{O}_{K(2)}^{top}(\mathrm{Spf}(B))$. Then C restricts to an *ordinary* elliptic curve C^{ord} over the punctured formal disk $\mathrm{Spf}(B^{ord})$, where $B^{ord} = B[u_1^{-1}]_p^\wedge$, and moreover it turns out that $E_{K(1)}$ is an appropriate corresponding elliptic spectrum. This will be the object that receives the *chromatic attaching map* $\alpha_{\mathrm{chrom}} : (\iota_{ord})_*\mathcal{O}_{K(1)}^{top} \to \left((\iota_{ss})_*\mathcal{O}_{K(2)}^{top}\right)_{K(1)}$. This map is also constructed in two steps: we construct the map $tmf_{K(1)} \to (tmf_{K(2)})_{K(1)}$ of global sections, and then we use obstruction theory for $tmf_{K(1)}$-algebras to extend this to a map of sheaves as desired.

6.3.6 Not a talk: Construction of $\mathcal{O}_{\mathbb{Q}}^{top}$ and the arithmetic attaching map.

Note that over a $\mathbb{Q}$-algebra, every formal group is isomorphic to $\widehat{\mathbb{G}}_a$; thus, the sections of $\mathcal{O}_{\mathbb{Q}}^{top}$ are all essentially rational Eilenberg-MacLane spectra. Then, we construct the *arithmetic attaching map* $\alpha_{\mathrm{arith}} : (\iota_{\mathbb{Q}})_*\mathcal{O}_{\mathbb{Q}}^{top} \to \left(\prod_p (\iota_p)_*\mathcal{O}_p^{top}\right)_{\mathbb{Q}}$ as follows. First, we observe that $(\overline{\mathcal{M}}_{ell})_{\mathbb{Q}}$ is covered by $(\overline{\mathcal{M}}_{ell}[\Delta^{-1}])_{\mathbb{Q}}$ and $(\overline{\mathcal{M}}_{ell}[c_4^{-1}])_{\mathbb{Q}}$; note that on singular Weierstraß curves, c_4 is invertible precisely if there are no cuspidal singularities. So, we construct α_{arith} on the sections over these substacks and over their intersection, and we verify that they are compatible; by descent, this induces the desired map of presheaves.

[24]Alternatively, Laures gives a construction of $tmf_{K(1)}$ at $p = 2$ by attaching two $K(1)$-local E_∞-cells to the $K(1)$-local sphere, cf. *$K(1)$-local topological modular forms.*