You could’ve invented \( \text{tmf} \).

Aaron Mazel-Gee

University of California, Berkeley

aaron@math.berkeley.edu

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Overview

You could've invented \textit{tmf}.
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1. The finite stable homotopy category

You could've invented $tmf$. 
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2. Chromatic homotopy theory
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3. Topological modular forms

You could’ve invented \( tmf \).
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2. Chromatic homotopy theory
3. Topological modular forms
4. Fun with $tmf$
1. The finite stable homotopy category
The category $\text{SHC}^{\text{fin}}$

Let’s do some topology!

So, what is topology?

Topology is the study of topological spaces.

But this is hard.

Let’s make things a bit easier for ourselves, and study the homotopy category of topological spaces.

$\text{CW complexes}$
$\text{finite}$

This is called the finite stable homotopy category, denoted $\text{SHC}^{\text{fin}}$.

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This is called the finite stable homotopy category, denoted \( \text{SHC}^{\text{fin}} \).
What does it mean to stabilize?

Recall that the suspension of a space $X$ is given by

$$\Sigma X = [0, 1] \times X / (0, x_1) \sim (0, x_2) (1, x_1) \sim (1, x_2).$$

Given a map $X \to Y$, we can suspend it to a map $\Sigma X \to \Sigma Y$.

This gives us a system

$$[X, Y] \to [\Sigma X, \Sigma Y] \to [\Sigma^2 X, \Sigma^2 Y] \to [\Sigma^3 X, \Sigma^3 Y] \to \cdots$$

The Freudenthal suspension theorem tells us that this system always stabilizes.

So for two finite CW complexes $X$ and $Y$, we define

$$\text{Hom}_{\text{SHC}^\text{fin}}(X, Y) = \lim_{n \to \infty} [\Sigma^n X, \Sigma^n Y].$$
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The set
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(Recall that \([\Sigma X, Z]\) is always a group (for the same reason that \(\pi_1\) is a group), and that \([\Sigma^n X, Z]\) is always an abelian group for \(n \geq 2\) (for the same reason that \(\pi_{\geq 2}\) is an abelian group).)
And as long as we only need to be able to define maps between arbitrarily high suspensions of our finite CW complexes, we may as well allow *desuspensions* too.
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Then, for example,

\[ \text{Hom}_{\text{SHC}^{\text{fin}}}(\Sigma^{-i}X, \Sigma^{-j}Y) = \lim_{n \to \infty} [\Sigma^{n-i}X, \Sigma^{n-j}Y]. \]
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So, the objects of $\text{SHC}^{\text{fin}}$ are the finite CW complexes and their formal desuspensions.
The geometry of $\text{SHC}^{\text{fin}}$

We would like to study the global structure of the finite stable homotopy category. What structure does it carry?

A subcategory is called thick if it is closed under mapping cones, retracts, and weak equivalences. So, the “kernel” of any co/homology theory is thick.

We can use the smash product to define ideal subcategories and prime ideal subcategories, exactly as in ring theory.

Given a category $C$ with this structure, we can define a space $\text{Spec}(C)$ in much the same way an algebraic geometer defines the prime spectrum of a ring:

$$\text{Spec}(C) = \{ P \subset C : P \text{ a thick prime ideal subcategory} \}.$$ 

This packages a lot of information really cleanly.

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So, what is $\text{Spec}(\text{SHC}^{\text{fin}})$?
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There is a beautiful answer, given by the *nilpotence theorem* of Devinatz–Hopkins–Smith.
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$$
\begin{array}{cccc}
\mathcal{P}_\infty, (2) & \mathcal{P}_\infty, (3) & \mathcal{P}_\infty, (5) & \mathcal{P}_\infty, (7) \\
\vdots & \vdots & \vdots & \vdots \\
\mathcal{P}_3, (2) & \mathcal{P}_3, (3) & \mathcal{P}_3, (5) & \mathcal{P}_3, (7) \\
\vdots & \vdots & \vdots & \vdots \\
\mathcal{P}_2, (2) & \mathcal{P}_2, (3) & \mathcal{P}_2, (5) & \mathcal{P}_2, (7) \\
\vdots & \vdots & \vdots & \vdots \\
\mathcal{P}_1, (2) & \mathcal{P}_1, (3) & \mathcal{P}_1, (5) & \mathcal{P}_1, (7) \\
\vdots & \vdots & \vdots & \vdots \\
\mathcal{P}_0 & \mathcal{P}_0, (3) & \mathcal{P}_0, (5) & \mathcal{P}_0, (7) \\
\end{array}
$$

$$
\mathbb{N}_0 \cup \{\infty\} \quad \text{Spec}(\mathbb{Z})
$$

This is exciting! But to explain what the subcategories $\mathcal{P}_n, (p)$ are, we’ll have to talk about...

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![Diagram of Spec(\text{SHC}^{\text{fin}})]

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2. Chromatic homotopy theory
Formal group laws in topology

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The story of chromatic homotopy theory begins with formal group laws, and the story of formal group laws begins with $\mathbb{CP}^\infty$.

Recall that $\mathbb{CP}^\infty$ is a *classifying space* for complex line bundles; that is, it carries a *universal* line bundle $\mathcal{L}_{univ} \downarrow \mathbb{CP}^\infty$, where $\mathcal{L}_{univ}$ is a universal line bundle over $\mathbb{CP}^\infty$. You could've invented $\text{tmf}$. 
The story of chromatic homotopy theory begins with formal group laws, and the story of formal group laws begins with $\mathbb{CP}^\infty$.

Recall that $\mathbb{CP}^\infty$ is a classifying space for complex line bundles; that is, it carries a universal line bundle $\mathcal{L}_{univ} \downarrow \mathbb{CP}^\infty$, and there is a natural isomorphism

$$\left\{ \text{line bundles over } X \right\} \cong \left[ X, \mathbb{CP}^\infty \right].$$

$$f^* \mathcal{L}_{univ} \leftrightarrow f$$
By Yoneda’s lemma, the natural operation of tensor product of two line bundles is classified by a map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{\mu} \mathbb{C}P^\infty$. You could've invented tmf.
By **Yoneda’s lemma**, the natural operation of *tensor product* of two line bundles is classified by a map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{\mu} \mathbb{CP}^\infty$.

(A pair of line bundles $\mathcal{L}_1, \mathcal{L}_2 \downarrow X$ is classified by a map

$$X \xrightarrow{f} \mathbb{CP}^\infty \times \mathbb{CP}^\infty$$

such that $\mathcal{L}_1 \cong f^* \mathcal{L}_{univ, 1}$ and $\mathcal{L}_2 \cong f^* \mathcal{L}_{univ, 2}$. 

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such that $L_1 \cong f^* L_{univ,1}$ and $L_2 \cong f^* L_{univ,2}$. By assumption,

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{\mu} \mathbb{C}P^\infty$$

classifies $L_{univ,1} \otimes L_{univ,2} \cong \mu^* L_{univ}$.)
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classifies $\mathcal{L}_{univ,1} \otimes \mathcal{L}_{univ,2} \cong \mu^* \mathcal{L}_{univ}$, so the composite

$$X \xrightarrow{f} \mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{\mu} \mathbb{CP}^\infty$$

classifies $\mathcal{L}_1 \otimes \mathcal{L}_2 \cong (\mu f)^* \mathcal{L}_{univ}$.)

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Recall that $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[[t]]$; by the Künneth formula, $H^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong \mathbb{Z}[[x, y]]$. You could’ve invented $tmf$. 

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$$H^*(\mathbb{CP}^\infty) \to H^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)$$

is entirely described by

$$t \mapsto F(x, y) \in \mathbb{Z}[\![x, y]\!]$$.
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What is $F(x, y)$?
To determine the map

\[ H^*(\mathbb{CP}^\infty) \to H^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \]
\[ t \mapsto F(x, y), \]

we need to remember that the element \( t \) also goes by another name: the \textit{first Chern class}.
To determine the map

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$$t \mapsto F(x, y),$$

we need to remember that the element $t$ also goes by another name: the \textit{first Chern class}. We also need to remember that for any two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$,

$$c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2).$$
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Since \( F(x, y) \) just encodes how the first Chern class behaves under tensor product, it follows that \( F(x, y) = x + y \).
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A ring-valued cohomology theory $E^*$ is called complex-oriented if we have an isomorphism

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We’ll always have $E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong E_*[[x, y]]$, and so once again the map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ induces a map $E_*[[t]] \to E_*[[x, y]]$, and once again this is determined by $t \mapsto F_E(x, y) \in E_*[[x, y]]$.

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What can we say about $F_E(x, y)$?
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The power series $F_E(x, y) \in E_*[[x, y]]$ enjoys certain properties coming from analogous properties of the tensor product of line bundles.

\[ \mathcal{L} \otimes \mathbb{C} \cong \mathcal{L} \quad \Rightarrow \quad F_E(x, 0) = x \quad \text{(unitality)} \]

\[ \mathcal{L}_1 \otimes \mathcal{L}_2 \cong \mathcal{L}_2 \otimes \mathcal{L}_1 \quad \Rightarrow \quad F_E(x, y) = F_E(y, x) \quad \text{(commutativity)} \]
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(\mathcal{L}_1 \otimes \mathcal{L}_2) \otimes \mathcal{L}_3 &\cong \mathcal{L}_1 \otimes (\mathcal{L}_2 \otimes \mathcal{L}_3) \quad \Rightarrow \quad F_E(F_E(x, y), z) \\
&= F_E(x, F_E(y, z)) \quad \text{(associativity)}
\end{align*}
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The power series $F_E(x, y) \in E_*[[x, y]]$ enjoys certain properties coming from analogous properties of the tensor product of line bundles.

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L_1 \otimes L_2 \cong L_2 \otimes L_1 \quad \Rightarrow \quad F_E(x, y) = F_E(y, x) \quad \text{(commutativity)}
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\[
(L_1 \otimes L_2) \otimes L_3 \cong L_1 \otimes (L_2 \otimes L_3) \quad \Rightarrow \quad F_E(F_E(x, y), z) = F_E(x, F_E(y, z)) \quad \text{(associativity)}
\]

These are the three defining properties for $F_E$ to be a (1-dimensional commutative) formal group law over the ring $E_*$. 

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One can obtain *formal group laws* as germs of *algebraic groups*, much as one can obtain Lie algebras as germs of Lie groups.

The formal group law $F_{H\mathbb{Z}}(x, y) = x + y$ associated to singular cohomology is called the *additive formal group law*, denoted $\widehat{\mathbb{G}}_a$, which is the germ of the *additive group*, denoted $\mathbb{G}_a$. 
Another example: complex $K$-theory.
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Recall that $KU^0(X)$ is the group-completion of the monoid $(\text{Vect}_\mathbb{C}(X), \oplus)$, with multiplication given by $\otimes$.

$K$-theory is complex-oriented: $KU^*(\mathbb{C}P^\infty) \cong KU_*[[t]]$, where $t = [L_{\text{univ}}] - [\mathbb{C}] = [L_{\text{univ}}] - 1$. 

You could’ve invented $tmf$. 
Another example: complex $K$-theory.

Recall that $KU^0(X)$ is the group-completion of the monoid $(\text{Vect}_\mathbb{C}(X), \oplus)$, with multiplication given by $\otimes$.

$K$-theory is complex-oriented: $KU^*(\mathbb{CP}^\infty) \cong KU_*[[t]]$, where $t = [\mathcal{L}_{\text{univ}}] - [\mathbb{C}] = [\mathcal{L}_{\text{univ}}] - 1$. So with $x = [\mathcal{L}_1] - 1$ and $y = [\mathcal{L}_2] - 1$, since $\mu^* \mathcal{L}_{\text{univ}} = \mathcal{L}_1 \otimes \mathcal{L}_2$ we compute that $F_{KU}(x, y) = [\mathcal{L}_1 \otimes \mathcal{L}_2] - 1$.
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$$F_{KU}(x, y) = [\mathcal{L}_1 \otimes \mathcal{L}_2] - 1 = [\mathcal{L}_1] \cdot [\mathcal{L}_2] - 1 = ([\mathcal{L}_1] \cdot [\mathcal{L}_2] - [\mathcal{L}_1] - [\mathcal{L}_2] + 1) + [\mathcal{L}_1] - 1 + [\mathcal{L}_2] - 1 = ([\mathcal{L}_1] - 1) \cdot ([\mathcal{L}_2] - 1) + ([\mathcal{L}_1] - 1) + ([\mathcal{L}_2] - 1) = xy + x + y.$$
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This is called the multiplicative formal group law, denoted $\hat{G}_m$, which is the germ of the multiplicative group, denoted $\mathbb{G}_m$. We’ll come back to this.
Returning to the general theory, we have a functor

\[ E \mapsto F_E(x, y) \in E_*[[x, y]] \]

\[ \{ \text{complex-oriented cohomology theories} \} \rightarrow \{ \text{formal group laws} \} \]

You could've invented \( tmf \).
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In fact, there is a partial inverse (i.e. it's not defined on all formal group laws) given by the \textit{Landweber exact functor theorem}.
The finite stable homotopy category
Chromatic homotopy theory
Topological modular forms
Fun with tmf

Formal group laws in topology
The Morava $K$-theories
The Morava $E$-theories

The Morava $K$-theories

To define the Morava $K$-theories, we first need to define the height of a formal group law. Given a formal group law $F(x, y) \in R[[x, y]]$, we define its $n$-series $\left[ n \right] F(x) \in R[[x]]$ inductively by:

1. $\left[ 1 \right] F(x) = x$
2. $\left[ n \right] F(x) = F(x, \left[ n-1 \right] F(x))$

This classifies “$n$-fold addition.” From the axioms, we know that $F(x, y) = x + y + \cdots$, and so $\left[ n \right] F(x) = nx + \cdots$. If $R = k$ is a field of characteristic $p$, then the first term of $\left[ p \right] F(x)$ vanishes. In fact, we’ll always have $\left[ p \right] F(x) = ux^ph + \cdots$ for $u \in k^\times$ and $h \geq 1$, and this integer $h$ is called the height of $F$. You could’ve invented tmf.
The Morava $K$-theories

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In fact, we'll always have $[p]_F(x) = ux^p + \cdots$ for $u \in k \times$ and $h \geq 1$, and this integer $h$ is called the *height* of $F$.

You could've invented $tmf$. 

Aaron Mazel-Gee
The finite stable homotopy category
Chromatic homotopy theory
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---

Aaron Mazel-Gee

You could’ve invented tmf.
Over $\mathbb{F}_p$ itself, for each height $n \in [1, \infty]$ we have the $n^{\text{th}}$ Honda formal group law, denoted $H_{n,p}$, with $p$-series $[p]_{H_{n,p}}(x) = x^{p^n}$. 
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Aaron Mazel-Gee
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- $H_{1,p} = (\hat{\mathbb{G}}_m)_{\mathbb{F}_p}$, and so $K(1, p) = KU/p$ (mod $p$ complex $K$-theory).
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- Even though there’s no $H_{0,p}$, it turns out that we can reasonably define $K(0, p) = H\mathbb{Q}$ (rational singular cohomology) for any prime $p$. 

You could’ve invented $tmf$. 

Aaron Mazel-Gee
The Morava $K$-theories represent essentially all\footnote{The set $\{K(n,p)\}_{n,p}$ plays the same role for ring-valued cohomology theories as the set $\{\mathbb{Q}\} \cup \{\mathbb{F}_p\}$ plays for ordinary rings: it is the set of \textit{prime fields}.} of the homology theories that have K"unneth \textit{isomorphisms} (instead of just a K"unneth short exact sequence, or worse).
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That is, they are essentially all the homology theories whose kernel is not just a \textit{thick} subcategory, but is also a \textit{prime ideal} subcategory.
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That is, they are essentially all the homology theories whose kernel is not just a *thick* subcategory, but is also a *prime ideal* subcategory.

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Recall that the points of the space $\text{Spec}(\text{SHC}^{\text{fin}})$ are the *thick prime ideal* subcategories of $\text{SHC}^{\text{fin}}$. 

You could've invented $tmf$. 
Recall that the points of the space $\text{Spec}(\text{SHC}^{\text{fin}})$ are the *thick prime ideal* subcategories of $\text{SHC}^{\text{fin}}$. In fact, the point $\mathfrak{P}_{n,(p)}$ is nothing more or less than the kernel of $K(n,p)_*$!
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![Diagram](attachment:image.png)

However, as fantastic as the Morava $K$-theories are for giving us information at the various points of $\text{Spec}(\text{SHC}^{\text{fin}})$, \textit{they do not tell us how to stitch that information back together}. 

Aaron Mazel-Gee You could’ve invented $\text{tmf}$.
To define the Morava $E$-theories, we first need to define a deformation of a formal group law. Let $k$ be a perfect field of characteristic $p$, let $F$ be a formal group law over $k$, and let $(A, m)$ be a complete local ring with projection $A \to A/m$ to its residue field. A deformation of $F$ from $k$ to $A$ is a formal group law $\tilde{F}$ over $A$ and a map $k \to A/m$ such that $\pi_* \tilde{F} = i_* F$ over $A/m$. In algebraic geometry, the picture looks like this: [picture] These collect into $\text{Def}_{F/k}(A)$.

You could've invented $tmf$. 

Aaron Mazel-Gee
The Morava $E$-theories

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A deformation of $F$ from $k$ to $A$ is a formal group law $\bar{F}$ over $A$ and a map $k \xrightarrow{i} A/m$ such that $\pi^* \bar{F} = i^* F$ over $A/m$. 

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The finite stable homotopy category
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A deformation of $F$ from $k$ to $A$ is a formal group law $F'$ over $A$ and a map $k \xrightarrow{i} A/\mathfrak{m}$ such that $\pi^*F' = i^*F$ over $A/\mathfrak{m}$. In algebraic geometry, the picture looks like this:

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These collect into $\text{Def}_{F/k}(A)$.
In fact, there is a *universal deformation*, which is a formal group law \( \tilde{F} \) living over the *Lubin–Tate ring* \( LT_{F/k} \) such that

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For any space $X$, $(E_n,p)_*X = 0$ if and only if $K(i,p)_*X = 0$ for $i \in [0, n]$. 

You could’ve invented $tmf$. 

Aaron Mazel-Gee
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For any space $X$, $(E_{n,p})_*X = 0$ if and only if $K(i, p)_*X = 0$ for $i \in [0, n]$. So, the kernel of $E_{n,p}$ is $\{\mathcal{P}_0, \mathcal{P}_{1,(p)}, \ldots, \mathcal{P}_{n,(p)}\}$. You could’ve invented $tmf$. 
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Thus, the Morava $E$-theories afford us a notion of *chromatic globalization*, i.e. of stitching together information in the chromatic direction.
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But what about arithmetic globalization?
3. Topological modular forms

You could’ve invented $tmf$. 
Towards \textit{tmf}

So what is this sheaf, anyways? ...and how is it constructed?

We can see the $E_n(p)$ as telling us how to globalize in the chromatic direction; what's much more subtle is the question of how to globalize in the arithmetic direction. A global height-$n$ theory should allow us to recover the $E_n(p)$ at all primes $p$.

$$\begin{array}{c}
P_0 \cup \{\infty\} \\
P_1, (2) \\
P_1, (3) \\
P_1, (5) \\
P_1, (7) \\
P_2, (2) \\
P_2, (3) \\
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p \rightarrow \cdots \\
\end{array}$$

Spec($\mathbb{Z}$)

You could've invented \textit{tmf}. 

Aaron Mazel-Gee
Towards $tmf$

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Towards \( \text{tmf} \)

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Towards \textit{tmf}

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Towards \textit{tmf}

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![Diagram](image-url)
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Well, let’s just not $p$-complete the darn thing! We can take $KU$ as a global height-1 theory.
In order to go further, let’s reimagine this a bit.
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We define a *quasicoherent sheaf of cohomology theories* over $X = \text{Spec} \mathbb{Z}$, which we denote $KU$: 

*You could’ve invented $tmf$.***
In order to go further, let’s reimagine this a bit.

We define a quasicoherent sheaf of cohomology theories over $X = \text{Spec}(\mathbb{Z})$, which we denote $\mathcal{KU}$: we take the global sections to be

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and then by quasicoherence, evaluation on $X_p^\wedge \subset X$ yields

$$\mathcal{KU}(X_p^\wedge) = KU \hat{\otimes} \mathbb{Z}_p = KU_p^\wedge.$$

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This corresponds to the sheaf of formal groups over $X$ determined by $\hat{\mathbb{G}}_m$, which evaluates as $\hat{\mathbb{G}}_m(X^\wedge_p) = (\hat{\mathbb{G}}_m)_{\mathbb{Z}_p} = \hat{\mathbb{H}}_{1,p}$.
So, to get a global height-2 theory, we should look for some object with a sheaf of formal groups which contains as sections the $\widetilde{H}_{2,p}$ for all primes $p$. 
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However, we have a clue: Recall that we can obtain a formal group as the germ of (1-dimensional commutative) algebraic group.

Besides $\mathbb{G}_a$ and $\mathbb{G}_m$, what other 1-dimensional commutative algebraic groups *are* there, anyways?
As it turns out, there is only one other sort of 1-dimensional commutative algebraic group besides $\mathbb{G}_a$ and $\mathbb{G}_m$: the elliptic curves.
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Very luckily, some elliptic curves over $\mathbb{F}_p$ have formal group of height 2! Such elliptic curves are called *supersingular*. 

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For each prime $p$, there is a moduli of deformations of supersingular elliptic curves, denoted $\mathcal{M}_{\text{ell}, p}^{ss}$, and its canonical sheaf of formal groups does indeed contain $\tilde{H}_{2, p}$.
As it turns out, there is only one other sort of 1-dimensional commutative algebraic group besides $\mathbb{G}_a$ and $\mathbb{G}_m$: the *elliptic curves*.

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For each prime $p$, there is a *moduli of deformations* of supersingular elliptic curves, denoted $\mathcal{M}^{ss}_{\text{ell},p}$, and its canonical sheaf of formal groups does indeed contain $\widetilde{H}_{2,p}$.

So, we might hope to bring these all together and define a sheaf of cohomology theories over $\coprod_p \mathcal{M}^{ss}_{\text{ell},p}$.

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Unfortunately, there is a problem: $\coprod_p M_{\text{ell},p}^{ss}$ is extremely disconnected.
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So, the question becomes: Is there some \textit{connected} object \( \mathcal{M} \) admitting an embedding

\[
\bigsqcup_p M^{ss}_{\text{ell},p} \hookrightarrow \mathcal{M}
\]

and with a sheaf of formal groups extending that of \( \bigsqcup_p M^{ss}_{\text{ell},p} \)?
And the answer is:

...and how is it constructed?

Yes!

We can use the embedding $\bigcup_p \mathcal{M}_{\text{ss},p} \hookrightarrow \mathcal{M}_{\text{ell}}$ into the moduli of all elliptic curves. Actually, following our friends in number theory, we instead use the embedding $\bigcup_p \mathcal{M}_{\text{ss},p} \hookrightarrow \mathcal{M}_{\text{ell}}$ into its Deligne–Mumford compactification. The moral is: We use the ordinary locus to interpolate between the supersingular neighborhoods.

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We use the ordinary locus to interpolate between the supersingular neighborhoods.
So what the heck is this sheaf, anyways?

Recall that $\mathcal{M}_{\text{ell}}$ is the moduli of generalized elliptic curves (i.e. of smooth elliptic curves and their nodal degenerations). We endow $\mathcal{M}_{\text{ell}}$ with a sheaf $\mathcal{O}_{\text{top}}$ of ring-valued cohomology theories. If $\text{Spec}(R) \subset \mathcal{M}_{\text{ell}}$ carries the generalized elliptic curve $C$ over the ring $R$, then $E = \mathcal{O}_{\text{top}}(\text{Spec}(R))$ is a complex-oriented cohomology theory whose formal group law $F_E$ coincides with $\hat{C}$. This is called the elliptic cohomology theory associated to $C$.
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If $E$ is an elliptic cohomology theory associated to $C$, then $E_{2n} \cong \omega(C) \otimes n$ and $E_{2n+1} = 0$ for all $n \in \mathbb{Z}$. 

Towards $tmf$

So what is this sheaf, anyways? ...and how is it constructed?
Any generalized elliptic curve $C$ has a \textit{cotangent space} at the identity, denoted $\omega(C)$. (This is also the module of invariant 1-forms.) These assemble into a line bundle $\omega \downarrow \overline{M}_{ell}$.

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Thus, it is reasonable to call the global sections of our sheaf

$$Tmf = \mathcal{O}^{\text{top}}(\overline{M}_{ell}),$$

the cohomology theory of \textit{topological modular forms}. 

You could've invented \textit{tmf}. 

Aaron Mazel-Gee
What is the coefficient ring $Tmf_*$?
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Instead, there is a descent spectral sequence (essentially a Serre spectral sequence, if you squint hard enough) running

$$H^s(\bar{M}_{\text{ell}}, \omega^\otimes t) \Rightarrow Tmf_{2t-s}$$

which accounts for their interchange.
Recall that by definition,

$$MF_t = \omega^t(\mathcal{M}_{ell}) = H^0(\mathcal{M}_{ell}, \omega^t).$$
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The theory of spectral sequences tells us that the inclusion

$$H^0(\mathcal{M}_{ell}, \omega \otimes t) \hookrightarrow H^s(\mathcal{M}_{ell}, \omega \otimes t) \Rightarrow Tmf_{2t-s}$$

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To mimic number theory as closely as possible, we actually usually work with $tmf = \tau_{\geq 0} Tmf$, which is also called *topological modular forms*. 

Aaron Mazel-Gee  You could've invented $tmf$. 
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Very, very carefully.
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Instead, we use the category of $\mathcal{E}_\infty$-ring spectra; these are much more rigid, and give rise to ring-valued cohomology theories with lots of extra structure.
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$$(\text{Open}(\overline{\mathcal{M}_{\text{ell}}}))^{\text{op}} \to \left\{ \text{ring-valued cohomology theories} \right\} \cong \{\text{ring spectra}\}$$

We have the presheaf of ring-valued cohomology theories represented by the bottom arrow thanks to the Landweber exact functor theorem.
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We have the presheaf of ring-valued cohomology theories represented by the bottom arrow thanks to the Landweber exact functor theorem. We would like to lift this to a presheaf of $\mathcal{E}_\infty$-ring spectra, since there we have a good notion of sheaves and sheafification.

You could've invented $tmf$. 

Aaron Mazel-Gee
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“Although it is much harder to write down an $\mathcal{E}_\infty$-ring than a spectrum, it is also much harder to write down a map between $\mathcal{E}_\infty$-rings than a map between spectra.
The Goerss–Hopkins–Miller obstruction theory for $\mathcal{E}_\infty$-ring spectra guarantees that there is indeed such a lift, and moreover that it is essentially unique. In the immortal words of Lurie:

“Although it is much harder to write down an $\mathcal{E}_\infty$-ring than a spectrum, it is also much harder to write down a map between $\mathcal{E}_\infty$-rings than a map between spectra. The practical effect of this, in our situation, is that it is much harder to write down the wrong maps between $\mathcal{E}_\infty$-rings and much easier to find the right ones.”
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There is also a construction of $\mathcal{O}^{\text{top}}$ due to Lurie, which was the original motivation for his theory of derived algebraic geometry.
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There is also a construction of $\mathcal{O}^{top}$ due to Lurie, which was the original motivation for his theory of derived algebraic geometry. This ultimately relies on the Goerss–Hopkins–Miller obstruction theory, too.
Here is a result, modulo some minor loose ends.
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**Pre-theorem (M-G–Spitzweck)**

*There is a Goerss–Hopkins–Miller obstruction theory in the setting of motivic homotopy theory.*
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(Schemes represent functors-of-points. So, we just redefine “point” to mean “scheme” and then proceed from there: a simplex is just a fattened-up point, and this suggests the definition for “motivic simplicial complexes”.)
This result should eventually yield a motivic version of \( tmf \), though there are still a number of substantial hurdles to overcome.
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It should also tell us about the space of $\mathcal{E}_\infty$-endomorphisms of \textit{algebraic} $K$-\textit{theory}, the motivic analog of complex $K$-theory. This would tell us about the \textit{power operations} on algebraic $K$-theory.
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It should also tell us about the space of $\mathcal{E}_\infty$-endomorphisms of \textit{algebraic $K$-theory}, the motivic analog of complex $K$-theory. This would tell us about the \textit{power operations} on algebraic $K$-theory.

Power operations are the “extra structure” present on cohomology theories represented by $\mathcal{E}_\infty$-ring spectra referred to earlier.
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It should also tell us about the space of $\mathcal{E}_\infty$-endomorphisms of algebraic $K$-theory, the motivic analog of complex $K$-theory. This would tell us about the power operations on algebraic $K$-theory.

Power operations are the “extra structure” present on cohomology theories represented by $\mathcal{E}_\infty$-ring spectra referred to earlier. (This refinement is analogous to enriching ordinary cohomology from a graded group to a graded ring.)
Curiously enough, the above result requires no real algebraic geometry!
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Work in progress (M-G)

There is a Goerss–Hopkins–Miller obstruction theory in the setting of $\infty$-categories.
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It would also re-prove the original obstruction theory in a much cleaner and more streamlined way, although of course it would still rely on the very heavy machinery of $\infty$-categories.
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(The original obstruction theory is the result of six hefty papers, together over 500 pages, which use extremely technical results in the theory of *model categories*.)
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(The original obstruction theory is the result of six hefty papers, together over 500 pages, which use extremely technical results in the theory of \textit{model categories}. An $\infty$-category is to a model category as a manifold is to an atlas.)
4. Fun with $tmf$!
The Witten genus and the String orientation

Using arguments from physics, Witten defined a genus for String manifolds, which associates to each String manifold a modular form.

What is a genus? Given some structure group $G$ (e.g. unoriented, oriented, Spin, String, etc.), the $G$ bordism ring is the graded ring $\Omega_G^*$ whose elements are cobordism classes of $G$ manifolds, with addition given by disjoint union and multiplication given by Cartesian product. Then, a genus is just a homomorphism $\Omega_G^* \to R^*$ of graded rings.

So, the Witten genus is a homomorphism $\Omega_{\text{String}}^* \to \text{MF}^*$.

You could’ve invented $tmf$. 

Aaron Mazel-Gee
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Given some structure group $G$ (e.g. unoriented, oriented, Spin, String, etc.), the $G$ bordism ring is the graded ring $\Omega^*_G$ whose elements are cobordism classes of $G$ manifolds, with addition given by disjoint union and multiplication given by Cartesian product.
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Then, a genus is just a homomorphism $\Omega^G_* \to R_*$ of graded rings. So, the Witten genus is a homomorphism

$$\Omega^\text{String}_* \to MF_*.$$
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Indeed, Ando–Hopkins–Rezk–Strickland construct the $\sigma$-orientation

$$M\text{String} \to tmf.$$
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The *sphere spectrum*, denoted $S$, is the initial ring spectrum (just as $\mathbb{Z}$ is the initial ring). But $S$ also gives the bordism homology theory for *framed* manifolds. The $\sigma$-orientation is a factorization of the unit map

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S = M_{\text{Framed}} \xrightarrow{} M_{\text{String}} \xrightarrow{} \text{tmf}
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through the natural “forgetful” map from Framed bordism to String bordism. (A Framed structure gives a String structure, just like a Spin structure gives an orientation.)
This fits into a whole tower of factorizations of unit maps through various bordism homology theories.
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\mathbb{S} = \text{Mframed} \xrightarrow{\text{lim}} \cdots \rightarrow M\text{String} \rightarrow M\text{Spin} \rightarrow M\text{SO} \rightarrow M\text{O}
\]

\[
\begin{align*}
\text{tmf} & \quad \text{ko} & \quad H\mathbb{Z} & \quad H\mathbb{F}_2
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The finite stable homotopy category
Chromatic homotopy theory
Topological modular forms
Fun with $tmf$

The Witten genus and the String orientation
Transchromatic detection

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It is a classical result that a Spin-manifold is Spin-nullcobordant if (and only if) its \( ko \)-Pontrjagin and Stiefel–Whitney classes vanish.

Thus, it is natural to hope that \( tmf \)-characteristic classes allow us to completely detect String-cobordism.
Transchromatic detection

Let's work at a fixed prime $p$.

Recall that elliptic curves can either have height 1, in which case they are called ordinary, or height 2, in which case they are called supersingular.

Over the moduli $M_{\text{ord}}\text{ell}, p$ of ordinary elliptic curves over $p$-complete rings, there is a covering space $M_{\text{ord}}\text{ell}, p \to M_{\text{ord}}\text{ell}, p$, which is associated to Katz's ring $V$ of $p$-adic modular forms.
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So, the fiber over a point is a copy of \( \text{Aut}_{\mathbb{Z}_p}(\hat{\mathbb{G}}_m) \cong \mathbb{Z}_p^\times \).

We should think of this as the group of *deck transformations*. 
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- The covering space $\mathcal{M}_{\text{ell},p}^{\text{ord}}(p^\infty) \downarrow \mathcal{M}_{\text{ell},p}^{\text{ord}}$ is connected.

- In any (punctured) neighborhood of a *supersingular* point, i.e. a point in the complement of

$$\mathcal{M}_{\text{ell},p}^{\text{ord}} \subset \mathcal{M}_{\text{ell},p},$$

the covering space remains connected.
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In broad terms, it’s somehow saying that *despite living at height 1, the ring of p-adic modular forms “knows about” the points of height 2.*

(Connected covering spaces correspond to quotient groups of the fundamental group. So, this covering space “sees the missing points”.)

How can we interpret this result in topology?
The study of ramified coverings in arithmetic geometry goes by the name of *class field theory*, which gives results analogous to the Riemann–Hurwitz theorem for ramified coverings of Riemann surfaces.
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(For example, \( \mathbb{Z}[i] = \mathbb{Z}[x]/(x^2 + 1) \) ramifies at \( (2) \in \text{Spec}(\mathbb{Z}) \), but defines a unramified extension when we localize away from \( (2) \), i.e. if we restrict to \( \text{Spec}(\mathbb{Z}[2^{-1}]) \subset \text{Spec}(\mathbb{Z}) \).

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For this reason, we say that \( \mathbb{Z} \) is *separably closed*: it has no nontrivial connected finite Galois extensions.
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However, when we take its localization $L_{K(n,p)}S$ with respect to the Morava $K$-theory $K(n,p)$, it ceases to be separably closed;
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Since $\mathcal{O}^{\text{top}}(\mathcal{M}^{ss}_{\text{ell}, p})$ is essentially $E_{2,p}$ and since $K(1, p)$-localization models restriction to the ordinary locus, Igusa’s theorem suggests the following.
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- *The covering space of $L_{K(1,p)}\text{tmf}$ coming from the ring $V$ is a connected Galois extension.*
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- The covering space of $L_{K(1,p)}tmf$ coming from the ring $V$ is a connected Galois extension.
- The restriction of this covering space to $L_{K(1,p)}E_{2,p}$ is also a connected Galois extension. (In particular, $E_{2,p}$ is no longer separably closed after $K(1,p)$-localization.)
Thank you!
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Further reading, in order of appearance.

- M-G, *An introduction to spectra*.¹
- Peterson, *The geometry of formal varieties in algebraic topology I and II*.²
- M-G, *Dieudonné modules and the classification of formal groups*.¹
- Hopkins, *Complex oriented cohomology theories and the language of stacks (a/k/a COCTALOS)*.³
- Lurie, *A survey of elliptic cohomology*.³
- M-G, *What are $\mathcal{E}_\infty$-rings*?⁴
- M-G, *Model categories for algebraists, or: What’s really going on with injective and projective resolutions, anyways*?¹
- Katz, *p-adic $L$-functions via moduli of elliptic curves*.

¹ [http://math.berkeley.edu/~aaron/writing/](http://math.berkeley.edu/~aaron/writing/)
² [http://math.berkeley.edu/~aaron/xkcd/fall2010.html](http://math.berkeley.edu/~aaron/xkcd/fall2010.html)
³ [googleable](http://math.berkeley.edu/~aaron/writing/)
⁴ math.stackexchange answer; googleable via the string “what are e-infty rings”