Every love story is a GHOsT story: Goerss–Hopkins obstruction theory for ∞-categories

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1. Introduction

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obstructions live in ***André–Quillen cohomology*** in $\mathcal{A}$:

- to ***existence*** in $H^{n+2}_{AQ}(A, \Omega^n A)$ for $n \geq 1$,
- to ***uniqueness*** in $H^{n+1}_{AQ}(A, \Omega^n A)$ for $n \geq 1$.

higher-categorical perspective: write

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GHOsT: can (try to) compute all $\pi_n(\mathscr{M}(A))$!

spectral sequence $H^*_{AQ}(A, \Omega^* A) \Rightarrow \pi_*(\mathscr{M}(A))$

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GHOsT for ∞-cats $\overset{?}{\rightsquigarrow}$ "naive" theory of DAG in other ∞-cats

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dogmatic: ∞-cat : model cat :: manifold : atlas

Goerss–Hopkins worked very, very hard to get just the right 'atlases' when they set up GHOsT, but this shouldn't be necessary.

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$\rightsquigarrow$ GHOsT should be *model-independent*, i.e. construction itself should descend to the *underlying* ∞*-category* of spectra.

pragmatic: then, may as well do it for *all* ∞-categories, to get:

- GHOsT in
  - equivariant / motivic homotopy theory
  - logarithmic E_∞-ring spectra
  - cxes of qcoh sheaves ($\rightsquigarrow$ coeffs for factoriz^n homology)
- ∞-categorical Rognes–Galois correspondence
- et al.

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- E^2***-model structure*** of Dwyer–Kan–Stover on $s\mathrm{Top}_*$

  a/k/a *resolution model structure*: generalizes the notion of "projective resolutions" to nonabelian setting

2. Obstruction theory

setup

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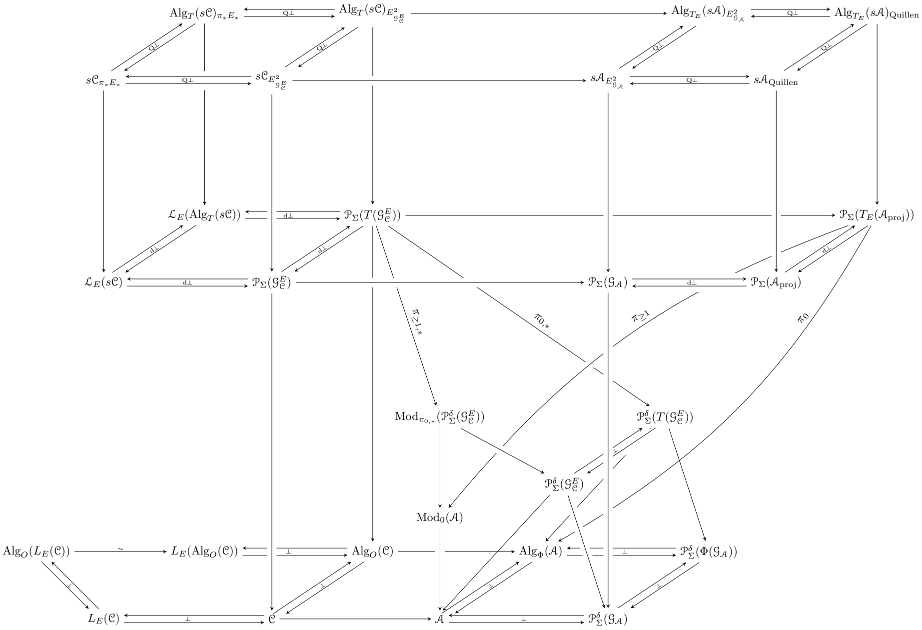

$\mathrm{Alg}_T(s\mathcal{C})_{\pi_*E_*}$
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π_0
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$\sim$
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Q⊥
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⊥

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- $\mathcal{C}$ a presentable homotopy theory;
- $\mathcal{G}$ a set of generators;
- define "homotopy" functor

$$\mathcal{C} \xrightarrow{\pi_*} \mathcal{A}$$

by

$$\pi_* X = \{[S^\beta, X]\}_{S^\beta \in \mathcal{G}}.$$

(by def^{n} of "generators", π_* detects equivalences.)

example: $\mathcal{C} = \mathrm{Top}_*^{\geq 1}$, $\mathcal{G} = \{S^n\}_{n \geq 1}$.

example: $\mathcal{C} = \mathrm{Spectra}$, $\mathcal{G} = \{S^n\}_{n \in \mathbb{Z}}$.

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goal: given $A \in \mathcal{A}$, want to understand $\mathscr{M}(A) \subset \mathcal{C}$.

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solution: "flip $\mathbb{Z}$'s worth of π_* on its side" and resolve "upwards" in a new simplicial direction.

So, work in $s\mathcal{C}$. We have a ***homotopy spectral sequence***

$$E^2 = \pi_*(\pi_*^{\mathrm{lw}} X) \Rightarrow \pi_*|X|.$$

question: When does $X \in s\mathcal{C}$ have $|X| \in \mathscr{M}(A)$?

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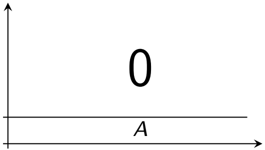
easiest answer: When the spectral sequence collapses!

$E^2 =$



$$\pi_i(\pi_*^{\mathrm{lw}} X) \cong \begin{cases} A, & i = 0 \\ 0, & i > 0 \end{cases}$$

call such an $X \in s\mathcal{C}$ an ∞-***stage*** for A.

$$\begin{array}{ccc} s\mathcal{C} & \xrightarrow{\quad |-| \quad} & \mathcal{C} \\ \uparrow & & \uparrow \\ (\infty\text{-stages}) & \dashrightarrow & \mathscr{M}(A) \end{array}$$

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obviously false as stated: a map $X \to Y$ of ∞-stages can be an iso on $E^2 = \pi_*(\pi_*^{\mathrm{lw}}(-))$ (so that $|X| \xrightarrow{\sim} |Y|$) even if it's not a *levelwise* equivalence, i.e. an iso on $E^1 = \pi_*^{\mathrm{lw}}(-)$.

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$\rightsquigarrow$ invert such "E^2-equivalences" $\rightsquigarrow$ E^2-***model structure*** on $s\mathcal{C}$

$\rightsquigarrow$ moduli space $\mathscr{M}_\infty(A) \subset s\mathcal{C}_{E^2}$ of ∞-stages for A

It turns out that this is just what we need:

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moral reason: ∞-stages only have $\pi_i \pi_*^{\mathrm{lw}}$ at $i = 0$

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step 2: find a Postnikov decomposition of $\mathscr{M}_\infty(A)$.

"global" version of Postnikov tower

Define an n-***stage*** for A to be $X \in s\mathcal{C}$ with

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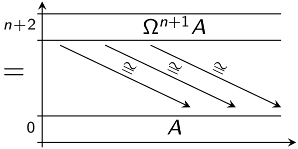

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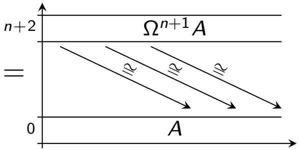

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have "truncation" functors $\mathscr{M}_n(A) \xrightarrow{P_{n-1}} \mathscr{M}_{n-1}(A)$, and

$$\mathscr{M}(A) \xleftarrow{\sim} \mathscr{M}_\infty(A) \xrightarrow{\lim} \cdots \to \mathscr{M}_2(A) \xrightarrow{P_1} \mathscr{M}_1(A) \xrightarrow{P_0} \mathscr{M}_0(A).$$

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however, for Y to extend to an n-stage, actually need to have a *weak equivalence* (!) $\pi_*^{\mathrm{lw}} Y \simeq A \ltimes (\Omega^n A)[n+1]$ in $s\mathcal{A}$.

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1. a theory of ***model ∞-categories***.

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(a/k/a the *nonabelian derived ∞-category* of $\mathcal{C}$)

problem: The setup of GHOsT uses the actual 1-category $s\mathbf{M}$ – and its E^2-model structure – to make computations in $s\mathbf{M}[\mathbf{W}_{E^2}^{-1}]$. How can we do this in the ∞-categorical context?

solution: We need an "E^2-model structure" on the ∞-category $s\mathcal{C}$ which presents the ∞-category $\mathcal{P}_\Sigma(\mathcal{G})$!

$\rightsquigarrow$ two ingredients in generalizing GHOsT to ∞-categories:

1. a theory of ***model ∞-categories***.
2. plagiarism.

3. Model ∞-categories

A ***model structure*** on a category $\mathbf{M}$ allows us to effectively compute the hom-sets

$$\hom_{\mathbf{M}[\mathbf{W}^{-1}]}(x, y).$$

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N.B.: ∞-categories are already ***homotopically well-behaved***.

$\rightsquigarrow$ has more to do with *interesting mathematical structures* (namely, with ***resolutions***) than with *eliminating pathologies* (e.g. replacing spaces with CW-cxes).

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$\rightsquigarrow$ **another perspective**: model structures on ∞-categories can compute the composition of total derived functors of (classical) left and right Quillen functors.

(e.g. $s\mathcal{C} \to s\mathcal{C}[\mathbf{W}_{E^2}^{-1}] \simeq \mathcal{P}_{\Sigma}(\mathcal{G})$ is a right adjoint followed by a left adjoint)

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recall: in a model 1-category, a ***cylinder*** for $x \in \mathbf{M}$ is a factoriz$^{\mathsf{n}}$

$$x \sqcup x \rightarrowtail \mathrm{cyl}(x) \xrightarrow{\approx} x,$$

and a ***path*** for $y \in \mathbf{M}$ is a factoriz$^{\mathsf{n}}$

$$y \xrightarrow{\approx} \mathrm{path}(y) \twoheadrightarrow y \times y.$$

model ∞-categories: don't truncate these co/simplicial objects!
$\rightsquigarrow$ a ***cylinder obj*** is $\mathrm{cyl}^\bullet(x) \in c\mathcal{M}$, a ***path obj*** is $\mathrm{path}_\bullet(y) \in s\mathcal{M}$.
("cofib$^{\mathsf{t}}$ **W**-cohypercover" and "fib$^{\mathsf{t}}$ **W**-hypercover", resp.)

1-topos theory : ***quotient*** by an equiv$^{\text{ce}}$ rel$^{\text{n}}$ ::

∞-topos theory : ***geom realiz^{n}*** of a simplicial object

$\rightsquigarrow$ define ***space of left htpy classes of maps*** by

$$\hom^{\overset{l}{\sim}}_{\mathcal{M}}(x,y) = \left| \hom^{\text{lw}}_{\mathcal{M}}(\text{cyl}^{\bullet}(x), y) \right|$$

and ***space of right htpy classes of maps*** by

$$\hom^{\overset{r}{\sim}}_{\mathcal{M}}(x,y) = \left| \hom^{\text{lw}}_{\mathcal{M}}(x, \text{path}_{\bullet}(y)) \right| .$$

fundamental theorem of model ∞-categories

if x cofib$^{\text{t}}$ and y fib$^{\text{t}}$, then for *any* cylinder/path obj's,

$$\hom^{\overset{l}{\sim}}_{\mathcal{M}}(x,y) \xrightarrow{\sim} \hom_{\mathcal{M}[\mathbf{W}^{-1}]}(x,y) \xleftarrow{\sim} \hom^{\overset{r}{\sim}}_{\mathcal{M}}(x,y).$$

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proof uses model two important model ∞-categories:

- the ***Quillen model structure*** on $s\mathcal{S}$,
- the ***Thomason model structure*** on $\mathcal{C}\mathrm{at}_\infty$.

(can't use fund thm here: must prove things in these model ∞-cats by hand!)

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model 1-cats enriched in *sets* $\rightsquigarrow$ $s\mathcal{S}\mathrm{et}_{\mathsf{Q}}$ plays a distinguished role.
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$$\pi_0 : s\mathcal{S} \rightleftarrows s\mathcal{S}\mathrm{et} : \delta, \quad I_{\mathrm{Q}} = \{\partial\Delta^n \to \Delta^n\}_{n\geq 0}, \quad J_{\mathrm{Q}} = \{\Lambda^n_i \to \Delta^n\}_{0\leq i\leq n>0}.$$

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$s\mathcal{S}_{\mathrm{Q}} \rightleftarrows s\mathcal{S}\mathrm{et}_{\mathrm{Q}}$ a Quillen equiv$^{\mathrm{ce}}$! (derived adjunction is $\mathcal{S} \overset{\mathrm{id}}{\rightleftarrows} \mathcal{S}$.)

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$I_{\mathrm{Moer}} = \{\partial\Delta^n \boxtimes \partial\Delta^n \to \Delta^n \boxtimes \Delta^n\}$.

$\rightsquigarrow \cdots \rightsquigarrow$ in $s\mathcal{S}$, $\quad \mathrm{rlp}\left(\{\partial\Delta^n \boxtimes S^{n-1} \to \Delta^n \boxtimes \mathrm{pt}\}\right) \subset \mathbf{W}_{\mathrm{Q}} = \mathbf{W}_{\mathrm{colim}}$.

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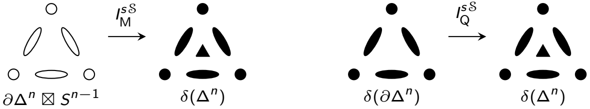
These maps have serious geometric content!

$n = 2$:



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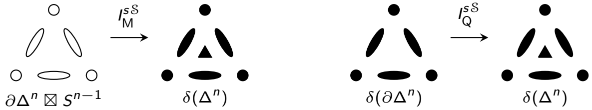

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moral: working with model ∞-cats allows us to replace *maps in from* ***spheres*** with *homotopy-coherent maps in from* ***points***.

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cofib$^{\mathrm{tly}}$ gen$^{\mathrm{d}}$, lifted *directly* along $s\mathcal{S}_{\mathrm{Q}} \rightleftarrows \mathcal{CSS} \simeq \mathcal{C}\mathrm{at}_\infty$, which is a **Quillen equiv$^{\mathbf{ce}}$** (so this model ∞-cat also presents $\mathcal{S}$).

$\rightsquigarrow$ $\mathbf{W}_{\mathrm{Th}}$ created by $\mathcal{C}\mathrm{at}_\infty \xrightarrow{CSS} s\mathcal{S} \xrightarrow{|-|} \mathcal{S}$, i.e. by $\mathcal{C}\mathrm{at}_\infty \xrightarrow{(-)^{\mathrm{gpd}}} \mathcal{S}$.

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image of $\mathcal{C} \in \mathcal{C}at \xrightarrow{\mathsf{N}} s\mathcal{S}et_{\mathsf{Q}}$ or $\mathcal{C} \in \mathcal{C}at_\infty \xrightarrow{CSS} s\mathcal{S}_{\mathsf{Q}}$ is fibrant iff $\mathcal{C}$ is a ***groupoid***.

note: 1-gpds only model 1-types, but ∞-gpds model *all* spaces.

$\rightsquigarrow$ $\mathcal{C}at_{\mathsf{Th}}$ can only be lifted along

$$\mathsf{ho} \circ \mathsf{sd}^2 : s\mathcal{S}et_{\mathsf{Q}} \rightleftarrows s\mathcal{S}et \rightleftarrows s\mathcal{S}et \rightleftarrows \mathcal{C}at : \mathsf{Ex}^2 \circ \mathsf{N},$$

at least if we want this to be a Quillen *equivalence*.

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localiz[n] thm for model ∞-cats: define $CSS(\mathcal{M}, \mathbf{W})_\bullet \in s\mathcal{S}$ by

$$CSS(\mathcal{M}, \mathbf{W})_n = \left(\mathrm{Fun}([n], \mathcal{M})^{\mathbf{W}}\right)^{\mathrm{gpd}}, \qquad \begin{array}{ccccccc} \bullet & \longrightarrow & \bullet & \longrightarrow \cdots & \longrightarrow & \bullet \\ \downarrow\wr & & \downarrow\wr & & & \downarrow\wr \\ \bullet & \longrightarrow & \bullet & \longrightarrow \cdots & \longrightarrow & \bullet \end{array}$$

(following Rezk's "classification diagram" functor $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \to s(s\mathcal{S}\mathrm{et})$).

then: $CSS(\mathcal{M}, \mathbf{W})_\bullet$ is actually a complete Segal space, and

$$(CSS(\mathcal{M})_\bullet \to CSS(\mathcal{M}, \mathbf{W})_\bullet) \in \mathcal{CSS} \quad \leftrightsquigarrow \quad (\mathcal{M} \to \mathcal{M}[\mathbf{W}^{-1}]) \in \mathcal{C}\mathrm{at}_\infty.$$

proof of fundamental theorem of model ∞-cats

set up $s\mathcal{S}_{\mathsf{Q}}$, then prove string of equiv$^{\text{ces}}$ in $\mathcal{S}$ following Dwyer–Kan.

eventually, need to access hom-space $\hom_{\mathcal{M}[\mathbf{W}^{-1}]}(x,y)$. do so using

localiz$^{\mathbf{n}}$ thm for model ∞-cats: define $CSS(\mathcal{M},\mathbf{W})_\bullet \in s\mathcal{S}$ by

$$CSS(\mathcal{M},\mathbf{W})_n = \left(\mathrm{Fun}([n],\mathcal{M})^{\mathbf{W}}\right)^{\mathrm{gpd}}, \qquad \begin{array}{ccccccc} \bullet & \longrightarrow & \bullet & \longrightarrow \cdots & \longrightarrow & \bullet \\ \downarrow\wr & & \downarrow\wr & & & \downarrow\wr \\ \bullet & \longrightarrow & \bullet & \longrightarrow \cdots & \longrightarrow & \bullet \end{array}$$

(following Rezk's "classification diagram" functor $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \to s(s\mathcal{S}\mathrm{et})$).

then: $CSS(\mathcal{M},\mathbf{W})_\bullet$ is actually a complete Segal space, and

$$(CSS(\mathcal{M})_\bullet \to CSS(\mathcal{M},\mathbf{W})_\bullet) \in \mathcal{CSS} \quad \leftrightsquigarrow \quad (\mathcal{M} \to \mathcal{M}[\mathbf{W}^{-1}]) \in \mathcal{C}\mathrm{at}_\infty.$$

proof: set up $(\mathcal{C}\mathrm{at}_\infty)_{\mathsf{Thomason}}$, then follow Barwick–Kan.

Thanks for listening!

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references:

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- Barwick–Kan, *From partial model categories to* ∞*-categories*.

this talk:

`http://math.berkeley.edu/~aaron/writing/ytm-cghost-beamer.pdf`

greatly expanded version:

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