Dieudonné modules and the classification of formal groups

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Abstract

Just as Lie groups can be understood through their Lie algebras, formal groups can be understood through their associated *Dieudonné modules*. We'd like to mimic Lie theory, but over an arbitrary ring we don't have anything like the Baker-Campbell-Hausdorff formula at our disposal. Instead, we study the full group of formal curves through the origin of our formal group, along with the actions of some geometrically-flavored endomorphisms as well as some purely algebraic Frobenius endomorphisms. This turns out to be enough structure to give us an equivalence of categories between formal groups and Dieudonné modules.

In this talk, I'll remind you why topologists care about formal groups, introduce the various algebraic objects at play, and illustrate some rather striking classification results.

1 Introduction

1.1 Formal groups and Dieudonné modules

Lie theory teaches us that there is a lot of power in linearization. Namely, there is a diagram

$$\mathbf{LieGrp} \xrightarrow{H \mapsto \widetilde{H}} \mathbf{LieGrp}_{\text{s.c.}} \underset{\text{BCH}}{\overset{\text{Lie}}{\rightleftarrows}} \mathbf{LieAlg}$$

in which the functors Lie and BCH give inverse equivalences of categories. So given a connected Lie group, once we strip away the global topology by passing to the simply connected cover, all the remaining information is encoded in the linearization at the identity.

This is not so for abelian varieties over an arbitrary base ring R. The first problem is that their Lie algebras are always commutative, so there's no information left besides the dimension! Moreover, BCH requires denominators that may not be available to us. Instead, *completing* an abelian variety at the identity yields what is known as a *formal group*. This retains algebraic information while stripping away the global structure; every formal group is isomorphic as a formal variety to formal affine space.

Now, recall that tangent vectors can be thought of as equivalence classes of curves. Rather than looking at tangent vectors on a formal group G, then, we will study CG, the entire *group of curves*. This comes with various endomorphisms – homotheties $[a]$ for all $a \in R$, Verschiebungs V_n for $n \geq 0$, and Frobenii (Frobeniuses?) F_n for $n \geq 1$ – which are natural and hence collectively determine endomorphisms of the functor C.

There is a subgroup $DG \subset CG$ of *p-typical curves*, and when our ground ring is a $\mathbb{Z}_{(p)}$-algebra, CG splits as an infinite product of copies of DG. The only remaining endomorphisms of DG are generated by F_p and V_p along with the homotheties, and this structure makes DG into a *Dieudonné module* over R. In this p-local setting, we then have an analogous diagram

$$\mathbf{AbVar}_R \xrightarrow{\Gamma \mapsto \Gamma_e^\wedge} \mathbf{FGrp}_R \underset{\mathcal{G}}{\overset{D}{\rightleftarrows}} \mathbf{DieuMod}_R$$

in which the functors D and $\mathcal{G}$ give inverse equivalences of categories.

This provides serious traction on the category of formal groups. Dieudonné modules pave the way for *Dieudonné crystals*, which control deformations of formal groups away from characteristic p, and Dieudonné theory plays a role in Wiles' proof of Fermat's last theorem.

1.2 Formal groups in topology

Recall that $\mathbb{C}P^\infty$ carries the universal complex line bundle. This means that the universal pair of line bundles lives over $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$; their tensor product, being a line bundle, is then classified by a map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$. When we apply a *complex-orientable* cohomology theory E^*, $E^*\mathbb{C}P^\infty \cong E_*[[c]]$ and the above map induces a diagonal $E^*\mathbb{C}P^\infty \to E^*\mathbb{C}P^\infty \otimes_{E_*} E^*\mathbb{C}P^\infty$ that makes Spf $E^*\mathbb{C}P^\infty$ into a 1-dimensional commutative formal group. (In these cases, c is known as a *generalized first Chern class*.) For example, integral cohomology yields the additive formal group since $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$, and complex K-theory yields the multiplicative formal group.

One can ask whether a formal group over a ring R comes from a complex-oriented cohomology theory; a satisfactory answer is provided by the *Landweber exact functor theorem*, which essentially says that R must additionally be a flat comodule for the Hopf algebroid of universal homology cooperations for complex-oriented theories. (This should at least have a ring of truth to it, since $\mathbf{Comod}_{(MU_*, MU_*MU)}$ and $QCoh_{\mathscr{M}_{FG}}$ are 2-categorically equivalent.) So, the question becomes: What formal groups are there, anyways?

2 Formal geometry

We quickly cover a few preliminaries. Throughout, let R be a ring.

An *adic R-algebra* is an augmented R-algebra $\varepsilon : A \to R$. The powers of the augmentation ideal $\mathfrak{J}(A) = \ker \varepsilon$ define a complete and separated topology on A, and the category $\mathbf{Adic}_R$ consists of adic R-algebras and continuous maps. We write Spf $A = \mathrm{Hom}_{\mathbf{Adic}_R}(A, -)$ for "formal Spec". In particular,

$$\mathrm{Hom}_{\mathbf{Adic}_R}(R[[x_1, \ldots, x_n]], A) = \mathrm{Hom}_R^{\mathrm{cts}}(R[[x_1, \ldots, x_n]], A) \cong \mathfrak{J}(A)^n$$

and so we call Spf $R[[x_1, \ldots, x_n]]$ the *formal affine n-plane* $\hat{\mathbb{A}}^n_R$; we also define $\hat{\mathbb{A}}^\infty_R = \mathrm{colim}\ \hat{\mathbb{A}}^n_R$. Then, the category $\mathbf{FVar}_R$ of formal varieties over R has for objects the functors $V : \mathbf{Adic}_R \to \mathbf{Set}_*$ isomorphic to $\hat{\mathbb{A}}^n_R$ for some $n \in [1, \infty]$, and has for morphisms the natural transformations between these functors. The number n is called the *dimension* of V. An isomorphism $\hat{\mathbb{A}}^n_R \to V$ is called a *parametrization*, and an isomorphism $V \to \hat{\mathbb{A}}^n_R$ is called a *coordinate*. Via parametrizations and coordinates, morphisms between finite-dimensional formal varieties are given by tuples of multivariate power series with no constant terms.

A *formal group* over R is a group object $G \in \mathbf{FVar}_R$. A *formal group law* is a formal group G along with a specified isomorphism $G \cong \hat{\mathbb{A}}^n_R$ as formal varieties. Via parametrizations and coordinates, a formal group law of dimension n determines a morphism $F \in \mathrm{Hom}_{\mathbf{FVar}_R}(\hat{\mathbb{A}}^{2n}_R, \hat{\mathbb{A}}^n_R)$, that is, an n-tuple of $2n$-variate power series. We often write $F(\underline{x}, \underline{y}) = \underline{x} +_F \underline{y}$.

Example 1. The *additive formal group* $\hat{\mathbb{G}}_a$ is given by $\hat{\mathbb{G}}_a(A) = \mathfrak{J}(A)$, with addition coming from addition in A. The identity $\mathfrak{J}(A) \to \mathfrak{J}(A)$ determines a parametrization $\gamma : \hat{\mathbb{A}}^1_R \to \hat{\mathbb{G}}_a$. Then, $x +_{\hat{\mathbb{G}}_a} y = x + y$.

Example 2. The *multiplicative formal group* $\hat{\mathbb{G}}_m$ is given by $\hat{\mathbb{G}}_m(A) = (1 + \mathfrak{J}(A))^\times$, with addition coming from multiplication in A. We can define a parametrization $\gamma : \hat{\mathbb{A}}^1_R \to \hat{\mathbb{G}}_m$ by $\gamma(t) = 1 - t$. Then, $x +_{\hat{\mathbb{G}}_m} y \leftrightarrow (1-x)\cdot(1-y) = (1 - x - y + xy) \leftrightarrow x + y - xy$.

All our formal groups will be commutative.

3 The group of formal curves

3.1 The functor of formal curves and its endomorphisms

Suppose we have a formal group G over a ring R. We define the *group of formal curves* on G by $CG = C_R G = \mathrm{Hom}_{\mathbf{FVar}_R}(\hat{\mathbb{A}}^1_R, G)$. This defines a functor $C_R : \mathbf{FGrp}_R \to \mathbf{Grp}$ via addition on the target.

The group $C_R G$ comes with a number of endomorphisms, which are natural and hence induce endomorphisms of the functor C_R.

1. **homothety**: Given $a \in R$, the curve $[a]\gamma$ is given by precomposition with $\hat{\mathbb{A}}^1_R \xrightarrow{t \mapsto at} \hat{\mathbb{A}}^1_R$.

2. **Verschiebung**: For each $n \geq 0$, the curve $V_n \gamma$ is given by precomposition with $\hat{\mathbb{A}}^1_R \xrightarrow{t \mapsto t^n} \hat{\mathbb{A}}^1_R$.

3. **Frobenius**: For each $n \geq 1$, if we let ζ denote a primitive n^{th} root of unity, then the curve $F_n \gamma$ is given by the expression

$$(F_n \gamma)(t) = \sum_{j=0}^{n-1} {}^G \gamma(\zeta^j t^{1/n})$$

(where Σ^G denotes summation with respect to the group operation of G). A priori this is a power series in $t^{1/n}$ and involves ζ, but the group axioms and the basic arithmetic of roots of unity will imply that this is again an element of CG.

Example 3. If $G = \hat{\mathbb{G}}_a$ is the additive formal group and $\gamma(t) = \sum_{k=1}^{\infty} b_k t^k$ (in the canonical coordinate), then

$$(F_n\gamma)(t) = \sum_{j=0}^{n-1}{}^{\mathbb{G}_a} \gamma(\zeta^j t^{1/n}) = \sum_{j=0}^{n-1}\sum_{k=1}^{\infty} b_k \zeta^{jk} t^{k/n} = \sum_{k=1}^{\infty}\left(\sum_{j=0}^{n-1}\zeta^{jk}\right) b_k t^{k/n} = \sum_{d=1}^{\infty} n b_{dn} t^d$$

since the sum of powers of ζ vanishes unless $k = dn$, in which case it equals n.

Proposition 1. *The endomorphisms of $C_R G$ obey the following relations:*

1. $F_m \circ V_n = V_n \circ F_m$ *whenever* $(m, n) = 1$.
2. $F_n \circ V_n = n_G$ *(i.e. n-fold formal addition on G).*
3. $F_m \circ F_n = F_{mn}$.
4. $V_m \circ V_n = V_{mn}$.
5. $F_n \circ [a] = [a^n] \circ F_n$.
6. $[a] \circ V_n = V_n \circ [a^n]$.

Checking these is routine. For instance, $[a] \circ V_n$ is given by precomposition with $\hat{\mathbb{A}}^1_R \xrightarrow{t \mapsto at} \hat{\mathbb{A}}^1_R \xrightarrow{t \mapsto t^n} \hat{\mathbb{A}}^1_R$, which equals the composition $\hat{\mathbb{A}}^1_R \xrightarrow{t \mapsto t^n} \hat{\mathbb{A}}^1_R \xrightarrow{t \mapsto a^n t} \hat{\mathbb{A}}^1_R$, and precomposition with this gives $V_n \circ [a^n]$.

3.2 Witticism

The *big Witt scheme* $\mathbb{W}$ is a ring scheme which is isomorphic as a scheme to Spec $\mathbb{Z}[x_1, x_2, \ldots]$. A sequence $(\underline{a}) = (a_1, a_2, \ldots) \in \mathbb{W}(R)$, called a *big Witt vector*, corresponds to the power series $\prod_{k=1}^{\infty}(1 - a_k x^k)$; under this correspondence, addition of Witt vectors corresponds to multiplication of power series. Thus by definition,

$$\prod_{k=1}^{\infty}(1 - (\underline{a} +_{\mathbb{W}(R)} \underline{b})_k x^k) = \left(\prod_{k=1}^{\infty}(1 - a_k x^k)\right) \cdot \left(\prod_{k=1}^{\infty}(1 - b_k x^k)\right).$$

(There is a bijective correspondence between power series in this form and power series in the usual form that have constant term 1.) The multiplication is more difficult to describe, but it's also naturally defined. This is a generalization of the construction of the p-adics: $\mathbb{W}(\mathbb{F}_p) = \mathbb{Z}_p$.

Completing $\mathbb{W}$ at the origin gives us $\hat{\mathbb{W}}$, the *Witt formal group*, which is isomorphic as a scheme to Spf $\mathbb{Z}[[x_1, x_2, \ldots]]$ and has addition defined in the analogous way. So $\hat{\mathbb{W}}$ is an infinite-dimensional formal group over $\mathbb{Z}$, and base-changing to a ring R gives us a formal group over R that we will call $\hat{\mathbb{W}}_R$.

There is a distinguished curve $\gamma_1(t) = (t, 0, 0, \ldots) \in C_R\hat{\mathbb{W}}_R$, and we readily compute that

$$\begin{aligned}(F_n\gamma_1)(t) &= (t^{1/n}, 0, \ldots) +_{\hat{\mathbb{W}}_R} (\zeta t^{1/n}, 0, \ldots) +_{\hat{\mathbb{W}}_R} \cdots +_{\hat{\mathbb{W}}_R} (\zeta^{n-1} t^{1/n}, 0, \ldots) \\ &\leftrightarrow (1 - t^{1/n}x) \cdot (1 - \zeta t^{1/n} x) \cdot \ldots \cdot (1 - \zeta^{n-1} t^{1/n} x) \\ &= 1 - t x^n \\ &\leftrightarrow (0, \ldots, 0, t, 0, \ldots)\end{aligned}$$

(where the t is in the n^{th} slot).

Theorem 1 (Dieudonné)**.** *If $G \in \mathbf{FGrp}_R$, then there is an isomorphism of groups $\mathrm{Hom}_{\mathbf{FGrp}_R}(\hat{\mathbb{W}}_R, G) \to C_R G$ given by $f \mapsto f \circ \gamma_1$. That is, $\mathrm{Hom}_{\mathbf{FGrp}_R}(\hat{\mathbb{W}}_R, G) \cong \mathrm{Hom}_{\mathbf{FVar}_R}(\hat{\mathbb{A}}^1_R, G)$; one says that $\hat{\mathbb{W}}_R$ is the* free formal group *on the formal line $\hat{\mathbb{A}}^1_R$. The inverse is given by taking a curve $\gamma : \hat{\mathbb{A}}^1_R \to G$ to the homomorphism $f : \hat{\mathbb{W}}_R \to G$ given by*

$$f = \sum_{k=1}^{\infty}{}^{G} (F_k\gamma) \circ \pi_k,$$

where $\pi_k : \hat{\mathbb{W}}_R \to \hat{\mathbb{A}}^1_R$ is the k^{th} projection.

Since $C_R G \cong \mathrm{Hom}_{\mathbf{FGrp}_R}(\hat{\mathbb{W}}_R, G)$, it follows that

$$\mathrm{End}_{Fun(\mathbf{FGrp}_R,\mathbf{Grp})}(C_R) \cong \mathrm{End}_{\mathbf{FGrp}_R}(\hat{\mathbb{W}}_R) = \mathrm{Hom}_{\mathbf{FGrp}_R}(\hat{\mathbb{W}}_R, \hat{\mathbb{W}}_R) \cong C_R\hat{\mathbb{W}}_R.$$

That is, the endomorphisms of the functor C_R are precisely given by the curves on $\hat{\mathbb{W}}_R$. It takes a little of work (and I haven't even told you the multiplication on $\hat{\mathbb{W}}$!), but this ends up naturally endowing $C_R G$ with the structure of a $\mathbb{W}(R)$-module. Explicitly, extending the R-action via homotheties is a ring homomorphism $E : \mathbb{W}(R) \to \mathrm{End}(C_R)$ given by $(\underline{a}) \mapsto \sum_{k=1}^{\infty} V_k \circ [a_k] \circ F_k$.

4 The group of p-typical formal curves

4.1 The functor of p-typical formal curves and its endomorphisms

Let us from now on focus on the situation where R is a $\mathbb{Z}_{(p)}$-algebra, which will simplify the story considerably. We call a curve $\gamma \in CG$ *p-typical* if $F_n\gamma = 0$ whenever n is not a power of p. These form a subgroup $DG \subset CG$. Clearly the Frobenius endomorphisms $F_n : CG \to CG$ descend to $F_n : DG \to DG$, but they're all zero by definition besides F_{p^k}. Since $F_{p^k} = (F_p)^k$, we simply write F for F_p and call this "the" Frobenius endomorphism on DG. Next, by the relations on endomorphisms, homotheties preserve p-typicality too. Lastly, we retain the Verschiebung endomorphism V_p (and its iterates): if $n = mp^k$ with $(m,p) = 1$ and $m > 1$, then $F_n(V_p\gamma) = F_{p^k}F_mV_p\gamma = F_{p^k}V_pF_m\gamma = 0$. We simply call this V.

4.2 p-typical Witticism

The *p-typical Witt scheme* $\mathbb{W}_p$ is also a ring scheme which is isomorphic as a scheme to Spec $\mathbb{Z}[x_0, x_1, \ldots]$, but of course with a different ring structure from $\mathbb{W}$. As before, the *p-typical Witt formal group* $\hat{\mathbb{W}}_p$ is the completion of $\mathbb{W}_p$ at the origin, and we can base-change to obtain a formal group $(\hat{\mathbb{W}}_p)_R$ over R. The ring scheme $\mathbb{W}_p$ (and hence the formal group $\hat{\mathbb{W}}_p$) comes with two special endomorphisms, the Verschiebung $V : \mathbb{W}_p \to \mathbb{W}_p$ and the Frobenius $\varphi : \mathbb{W}_p \to \mathbb{W}_p$.

There is always an epimorphism of ring schemes $\rho : \mathbb{W} \to \mathbb{W}_p$ given by $(a_1, a_2, \ldots) \mapsto (a_1, a_p, a_{p^2}, \ldots)$. When R is a $\mathbb{Z}_{(p)}$-algebra, this actually admits a splitting $\theta : \mathbb{W}_p \to \mathbb{W}$, which gives us an isomorphism $\mathbb{W}_{\mathbb{Z}_{(p)}} \cong \prod_{(n,p)=1}(\mathbb{W}_p)_{\mathbb{Z}_{(p)}}$. Thus by Dieudonné's theorem, $C_R G \cong \prod_{(n,p)=1} \mathrm{Hom}_{\mathbf{FGrp}_R}((\hat{\mathbb{W}}_p)_R, G)$. So $C_R G$ is determined by $\mathrm{Hom}_{\mathbf{FGrp}_R}((\hat{\mathbb{W}}_p)_R, G)$.

There is a distinguished p-typical curve $\gamma_0 = \rho \circ \gamma_1 \in D(\hat{\mathbb{W}}_p)_R$.

Theorem 2 (p-typical Dieudonné)**.** *If R is a $\mathbb{Z}_{(p)}$-algebra and $G \in \mathbf{FGrp}_R$, then there is an isomorphism of groups $\mathrm{Hom}_{\mathbf{FGrp}_R}((\hat{\mathbb{W}}_p)_R, G) \to DG$ given by $g \mapsto g \circ \gamma_0$. Moreover, F on curves corresponds to V on $\hat{\mathbb{W}}_p$, and if $p = 0 \in R$ then V on curves corresponds to φ on $\mathbb{W}$.*

Much like before, this all implies that DG is a $\mathbb{W}_p(R)$-module determined by a ring homomorphism $E_p : \mathbb{W}_p(R) \to \mathrm{End}(D)$ given by $(\underline{a}) \mapsto \sum_{n=0}^{\infty} V^n \circ [a_n] \circ F^n$. Unfortunately, F and V are not $\mathbb{W}_p(R)$-linear; however, in the case that R is an $\mathbb{F}_p$-algebra, $F \circ E_p(\underline{a}) = E_p(\varphi(\underline{a})) \circ F$ and $V \circ E_p(\varphi(\underline{a})) = E_p(\underline{a}) \circ V$. Thus, what we in fact have are $\mathbb{W}_p(R)$-module homomorphisms $F : DG \to \varphi_* DG$ (or equivalently $F : \varphi^* DG \to DG$) and $V : \varphi_* DG \to DG$.

4.3 Dieudonné modules

We are now nearly equipped to finally define Dieudonné modules. We first need two definitions. Let M be an abelian group and $V : M \to M$ be a homomorphism. Then M is called *reduced* (with respect to V) if

$$M \cong \lim_{\leftarrow} M/V^k M,$$

and M is called *uniform* (with respect to V) if

$$V^k : M/VM \to V^k M/V^{k+1}M$$

is an isomorphism for all $k \geq 1$. It is a fact that for any formal group G over a $\mathbb{Z}_{(p)}$-algebra R, DG is uniform and reduced with respect to the Verschiebung.

Now, if $R = K$ is actually a perfect field of characteristic p, then for any $\mathbb{W}_p(K)$-module M we have an isomorphism $M \cong \varphi^* M$, which implies that $\mathrm{Hom}_{\mathbb{W}_p(K)}(\varphi_* M, M) \cong \mathrm{Hom}_{\mathbb{W}_p(K)}(M, \varphi^* M)$. Moreover, by the universal relations we always have $FV = p_G$, and it turns out that $VF = p_G$ iff $p = 0 \in R$. Thus we finally define a *Dieudonné module* over a perfect field K of characteristic p to be a $\mathbb{W}_p(K)$-module M equipped with $\mathbb{W}_p(K)$-linear maps $F : \varphi^* M \leftrightarrows M : V$ such that $FV : M \to M$ and $VF : \varphi^* M \to \varphi^* M$ are both multiplication by p and such that M is uniform and reduced with respect to V.

However, if R is only a $\mathbb{Z}_{(p)}$-algebra then the situation is slightly trickier. We define the *Cartier algebra* $\mathrm{Cart}_p(R)$ of R to be the ring generated by the symbols V, F, and $[a]$ for $a \in R$, subject to the universal relations spelled out above (as well as one more that won't matter to us). This admits embeddings $R \to \mathbb{W}_p(R) \to \mathrm{Cart}_p(R)$. We define a *Dieudonné module* over R to be a $\mathrm{Cart}_p(R)$-module which is reduced and uniform with respect to $V \in \mathrm{Cart}_p(R)$ and such that M/VM (the "tangent space" $T_e G$) is a free R-module. This of course agrees with the previous definition.

We now come to the main theorem.

Theorem 3. *Let R be a $\mathbb{Z}_{(p)}$-algebra. Then $D : \mathbf{FGrp}_R \to \mathbf{DieuMod}_R$ is an equivalence of categories.*

Idea of proof. In the slightly simpler case that $R = K$ is a perfect field of characteristic p, we define a functor $\mathcal{G} : \mathbf{DieuMod}_K \to \mathbf{FGrp}_K$ by assigning to any $M \in \mathbf{DieuMod}_K$ the functor $\mathcal{G}(M)$ taking $A \in \mathbf{Adic}_K$ to the abelian group

$$\mathcal{G}(M)(A) = \hat{\mathbb{W}}_p(A) \otimes_{\mathbb{W}_p(K)} M / (Va \otimes m - a \otimes Fm, \varphi a \otimes m - a \otimes Vm).$$

(This respects the correspondence given in the p-typical Dieudonné theorem.) □

We will see that in the special case that $R = K$ is a perfect field of characteristic p, this reduces to a particularly nice classification theorem. First, we'll run a few warm-up examples to get a sense of what's going on.

Example 4 (the additive group)**.** Recall that $\hat{\mathbb{G}}_a(A) = \mathfrak{J}(A)$ with addition coming from addition in A. We can canonically choose the identity $\gamma : \hat{\mathbb{A}}^1_R \to \hat{\mathbb{G}}_a$ as a parameter. We compute that for $n > 1$,

$$(F_n \gamma)(t) = \sum_{j=0}^{n-1}{}^{\hat{\mathbb{G}}_a} \zeta^j t^{1/n} = 0,$$

so γ is p-typical for any p. The parameter γ is represented by

$$\sum_{k=1}^{\infty}{}^{\hat{\mathbb{G}}_a} (F_k \gamma) \circ \pi_k = \pi_1 : \hat{\mathbb{W}} \to \hat{\mathbb{G}}_a,$$

and $(V\gamma)(t) = t^p$. Thus

$$D\hat{\mathbb{G}}_a \cong \left\{ \sum_{k=0}^{\infty}{}^{\hat{\mathbb{G}}_a} (V^k \circ [a_k])(\gamma) \right\} = \left\{ \sum_{k=0}^{\infty} a_k t^{p^k} \right\}$$

as sets. If $R = K$ is a perfect field of characteristic p, then

$$D\hat{\mathbb{G}}_a \cong \prod_{k=0}^{\infty} V^k \cdot K$$

as Dieudonné modules, where F acts trivially and Witt vectors act through the projection $\pi_1 : \mathbb{W}_p(K) \to K$.

Example 5 (the multiplicative group)**.** Recall that $\hat{\mathbb{G}}_m(A) = (1 + \mathfrak{J}(A))^\times$ with addition coming from multiplication in A. If we choose the parameter $\gamma : \hat{\mathbb{A}}^1_R \to \hat{\mathbb{G}}_m$ given by $\gamma(t) = 1 - t$ and let ζ denote a primitive n^{th} root of unity, then

$$(F_n \gamma)(t) = \sum_{j=0}^{n-1}{}^{\hat{\mathbb{G}}_m} \gamma(\zeta^j t^{1/n}) = \prod_{j=0}^{n-1} (1 - \zeta^j t^{1/n}) = 1 - t = \gamma(t).$$

So the homomorphism $f : \hat{\mathbb{W}} \to \hat{\mathbb{G}}_m$ representing γ is given by

$$f(\underline{a}) = \sum_{k=1}^{\infty}{}^{\hat{\mathbb{G}}_m} (F_k \gamma) \circ \pi_k(\underline{a}) = \prod_{k=1}^{\infty} \gamma(a_k) = \prod_{k=1}^{\infty} (1 - a_k);$$

if we write $q_{(\underline{a})}(x) = \prod_{k=1}^{\infty}(1 - a_k x^k)$ for the power series associated to $(\underline{a})$, then $f(\underline{a}) = q_{(\underline{a})}(1)$. This parameter γ isn't p-typical, but we can obtain a p-typical parameter $\tilde{\gamma}$ as

$$\hat{\mathbb{A}}^1 \xrightarrow{\gamma_0} \hat{\mathbb{W}}_p \xrightarrow{\theta} \hat{\mathbb{W}} \xrightarrow{f} \hat{\mathbb{G}}_m.$$

This will have $F\tilde{\gamma} = \tilde{\gamma}$, which implies that $V\tilde{\gamma} = V \circ F\tilde{\gamma} = p\tilde{\gamma}$. So, over any $\mathbb{Z}_{(p)}$-algebra R, $D\hat{\mathbb{G}}_m$ is free of rank 1 over $\mathbb{W}_p(R)$ generated by $\tilde{\gamma}$ with $F\tilde{\gamma} = \tilde{\gamma}$ and $V\tilde{\gamma} = p\tilde{\gamma}$.

In particular, $\hat{\mathbb{G}}_a \not\cong \hat{\mathbb{G}}_m$ over a perfect field of characteristic p, and hence over any $\mathbb{Z}_{(p)}$-algebra which projects down to such a field. This is a nontrivial fact! (As it turns out, $\hat{\mathbb{G}}_a$ and $\hat{\mathbb{G}}_m$ are isomorphic iff we're over a $\mathbb{Q}$-algebra.) The *height* of a 1-dimensional formal group G is by definition the ("total") rank (i.e. the minimal number of generators) of DG as a $\mathbb{W}_p(R)$-module, so $\mathrm{ht}(\hat{\mathbb{G}}_a) = \mathrm{rk}_{\mathbb{W}_p(R)} D\hat{\mathbb{G}}_a = \infty$ and $\mathrm{ht}(\hat{\mathbb{G}}_m) = \mathrm{rk}_{\mathbb{W}_p(R)} D\hat{\mathbb{G}}_m = 1$.

4.4 Classification theorems

First, we describe the possible types of Dieudonné modules over a perfect field of characteristic p.

Theorem 4. *Suppose K is a perfect field of characteristic p, and let M be a Dieudonné module over K of dimension 1 (so that M corresponds to a 1-dimensional formal group). Then either*

- *there is a non-zero element $\gamma \in M$ such that $p\gamma = 0$, in which case $M \cong D\hat{\mathbb{G}}_a$ and $\mathrm{ht}(M) = \infty$, or*
- *M is free of finite rank h over $\mathbb{W}_p(K)$, so $\mathrm{ht}(M) = h$.*

Then, there is the following existence result.

Theorem 5. *Over a perfect field K of characteristic p, there is a 1-dimensional Dieudonné module M_h of each height $h \in [1, \infty]$.*

Even better, there is the following partial uniqueness result.

Theorem 6. *If $K = \overline{K}$, then height is a complete isomorphism invariant of one-dimensional formal groups over K: for any $M \in \mathbf{DieuMod}_K$, $M \cong M_{\mathrm{ht}(M)}$.*

This last theorem is proved by beginning with an arbitrary formal group G of height h with an arbitrary parameter $\gamma : \hat{\mathbb{A}}^1_K \to G$ and attempting to define a new parameter $\tilde{\gamma} : \hat{\mathbb{A}}^1_K \to G$ with respect to which $DG \cong M_h$. The parameter γ is inductively improved towards $\tilde{\gamma}$ by solving polynomials in K, so the fact that $K = \overline{K}$ guarantees that the process runs without a hitch.

On the other hand, if $K \neq \overline{K}$, then this total classification won't hold in general. Suppose we have a formal group G over K of height h. If we denote by Γ_h the formal group associated to M_h, then we have an isomorphism $G \cong \Gamma_h$ over $\overline{K}$, and this isomorphism will descend to K iff it is invariant under $\mathrm{Gal}(\overline{K}/K)$. This leads to a bijection

$$\mathbf{FGrp}_K^{\mathrm{ht}=h}/\mathrm{iso.} \cong H^1(\mathrm{Gal}(\overline{K}/K), \mathrm{Aut}(\Gamma_h)),$$

which is a tantalizing classification indeed.

5 Topological conclusions

There are a great many connections back to topology. First of all, p-typical formal group laws over a $\mathbb{Z}_{(p)}$-algebra are also classified by their p-series $p_F(x) = x +_F \ldots +_F x$. Writing L for the Lazard ring, the coefficients of the universal p-series over $L \otimes \mathbb{Z}_{(p)}$ are exactly the (Kudo-Araki) generators v_n of $BP_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$. This is no accident. The moduli stack of formal groups $\mathcal{M}_{FG}$ is the stackification of the groupoid scheme (Spec MU_*, Spec MU_*MU), and the p-localization of MU splits as a wedge sum of copies of BP.

Moreover, equivalences of fibered categories yield equivalent stacks. This is an important technique. For instance, Zink defined a certain generalization of Dieudonné modules, called *displays*, which classify p-divisible groups: if we write $\mathscr{M}_p(h)$ for the moduli stack of p-divisible groups of height h, there is a Hopf algebroid $(\mathcal{A}_h, \Gamma_h)$ with stackification $\mathscr{M}_{(\mathcal{A}_h,\Gamma_h)} \simeq \mathscr{M}_p(h)$. Meanwhile, for a map Spec $R \to \mathscr{M}_p(h)$, a theorem of Lurie gives sufficient conditions to produce a sheaf of E_∞ ring specta $\mathcal{E}$ over Spec R realizing the formal group classified by the composition Spec $R \to \mathscr{M}_p(h) \to \mathscr{M}_{FG}$ (where the latter map is completion at the identity). Unfortunately, these conditions are in general quite difficult to verify. However, Lawson combined these two to give a much more tractable condition: for $h \geq 2$ there is a canonical map Spec $\mathcal{A}_h \to \mathbb{P}^{h-1}$, and given a map Spec $R \to$ Spec $\mathcal{A}_h$ classifying a nilpotent display, the conditions of Lurie's theorem are satisfied iff the composition Spec $R \to$ Spec $\mathcal{A}_h \to \mathbb{P}^{h-1}$ is formally étale. Lawson successfully exploited this framework to construct ring spectra of chromatic height 2 and above.