The Steenrod algebra and its applications – talk 1
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These talks have been lifted shamelessly from the excellent book *Cohomology Operations and Applications in Homotopy Theory* by Mosher & Tangora. Pretty much everything here has been copied from that book, except that I added a bunch of mistakes.

The idea of algebraic topology

**Question.** Is there a retraction \( r : D^n \to S^{n-1} \)?

We can write this as

\[
\begin{array}{ccc}
S^{n-1} & \xrightarrow{i} & D^n \\
\downarrow{\omega} & & \downarrow{r} \\
S^{n-1} & & 
\end{array}
\]

By applying the functor \( H_{n-1}(\cdot; \mathbb{Z}) \), we can answer this question in the negative.

**Question.** For \( n > m \), is there a map \( \mathbb{R}P^n \to \mathbb{R}P^m \) which is an isomorphism on \( \pi_1 \)?

The answer is again no, but now we need to use cohomology: \( H^*(\mathbb{R}P^n; \mathbb{Z}_2) = \mathbb{Z}_2[\alpha_n]/\alpha_n^{n+1} = 0 \), and the condition on \( \pi_1 \) implies that the induced map \( H^*(\mathbb{R}P^m; \mathbb{Z}_2) \to H^*(\mathbb{R}P^n; \mathbb{Z}_2) \) has \( \alpha_m \mapsto \alpha_n \). But this is impossible, since the induced map must be a ring homomorphism.

We will employ a particular collection of much finer structures, called *cohomology operations*, to prove fancier things and answer much harder questions. (Hopefully by the third talk we’ll be able to calculate a bunch of homotopy groups of spheres, and our end goal is the Adams spectral sequence.) A cohomology operation of type \((\pi, n; G, m)\) is a natural transformation of functors

\[ H^n(\cdot; \pi) \to H^m(\cdot; G). \]

**Background**

For any space \( X \), we have the *Hurewicz homomorphism* \( h : \pi_1(X) \to H_1(X; \mathbb{Z}) \), obtained by considering the image of a sphere as an integral chain. We then have

**Theorem** (Hurewicz). For a connected space \( X \):

- \( h : \pi_1(X) \to H_1(X; \mathbb{Z}) \) is abelianization, i.e. \( H_1(X; \mathbb{Z}) = \pi_1(X)/[[\pi_1(X), \pi_1(X)] \);
- if \( \pi_1(X) = 0 \), then \( h : \pi_2(X) \to H_2(X; \mathbb{Z}) \) is an isomorphism;
- if \( \pi_1(X) = \pi_2(X) = 0 \), then \( h : \pi_3(X) \to H_3(X; \mathbb{Z}) \) is an isomorphism;
- if \( \pi_1(X) = \pi_2(X) = \pi_3(X) = 0 \), then \( h : \pi_4(X) \to H_4(X; \mathbb{Z}) \) is an isomorphism;
- etc.
Loosely, we can summarize this as saying that the lowest-dimension Hurewicz homomorphism which is nontrivial is an isomorphism, except if \( \pi_1(X) \) is nonabelian in which case it’s as close to an isomorphism as it can get.

For any abelian group \( \pi \) and any natural number \( n \), the Eilenberg-Maclane space \( K(\pi, n) \) is characterized (up to homotopy equivalence) by the property that \( \pi_n(K(\pi, n)) = G \) and \( \pi_i(K(\pi, n)) = 0 \) for \( i \neq n \). These spaces represent cohomology; that is, there is a bijection

\[
H^n(X; \pi) \cong [X, K(\pi, n)],
\]

where the brackets denote homotopy classes of maps. If we are careful, this can be made into a group isomorphism.

The universal coefficient theorem for cohomology with coefficients in a group \( \pi \) can be expressed as a sexseq

\[
0 \to \text{Ext}(H_{n-1}(X; \pi), \pi) \to H^n(X; \pi) \to \text{Hom}(H_n(X; \pi), \pi) \to 0.
\]

This says that while we’d expect \( H^n(X; \pi) \) to just be \( \text{Hom}(H_n(X; \pi), \pi) \) (since to get cohomology you just dualize your chain complexes), in fact this is almost true but there’s an extra correction term.

However, if \( X \) is \((n-1)\)-connected and \( \pi = \pi_n(X) \), then by the Hurewicz theorem \( H_{n-1}(X; \pi) = 0 \), and so in fact we do get an isomorphism \( H^n(X; \pi) \cong \text{Hom}(H_n(X; \pi), \pi) \). In this case, the Hurewicz homomorphism \( h : \pi_n(X) \to H_n(X; \pi) \) is an isomorphism and we have \( h^{-1} \in \text{Hom}(H_n(X; \pi), \pi) \). This corresponds (under the isomorphism above) to what we call the fundamental class \( i_X \in H^n(X; \pi) \) of \( X \). In particular, we have a fundamental class \( i_n \in H^n(K(\pi, n); \pi) \), and in fact a map \( f : X \to K(\pi, n) \) corresponds under the bijection \( H^n(X; \pi) \cong [X, K(\pi, n)] \) to the cohomology class \( f^*(i_n) \in H^n(X; \pi) \).

Since cohomology operations are natural, an operation of type \( (\pi, n; G, m) \) is represented as a map \( K(\pi, n) \to K(G, m) \). (We can see this by applying the operation to the universal case \( X = K(\pi, n) \).) But this is just an element of \( H^n(K(\pi, n); G) \). Thus, knowing cohomology operations is the same as knowing the cohomology of Eilenberg-Maclane spaces.

The Steenrod squares

We will focus our attention on the cohomology operations

\[
Sq^i : H^n(\cdot; \mathbb{Z}_2) \to H^{n+i}(\cdot; \mathbb{Z}_2)
\]

(for \( i \geq 0 \)) known as the Steenrod squares. These are group homomorphisms which can be applied to either absolute or relative cohomology. They have this name because of the second of their

**Characterizing Properties:****

1. if \( i > n \) and \( x \in H^n \), then \( Sq^i x = 0 \);
2. if \( x \in H^n \), then \( Sq^n x = x^2 \in H^{2n} \);
3. \( Sq^0 \) is the identity;
4. \( Sq^1 \) is the Bockstein connecting homomorphism associated to the sexseq of coefficients \( 0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0 \) (to be discussed more next time);
5. \( \delta^* Sq^i = Sq^i \delta^* \), where \( \delta^* : H^n(A) \to H^{n+1}(X, A) \) is the connecting homomorphism of the cohomology lexseq of the pair \( (X, A) \);
6. the Cartan formula:

\[
Sq^i(xy) = \sum_{j=0}^{i} (Sq^j x)(Sq^{i-j} y);
\]
7. the Adem relations: if \( a < 2b \), then

\[
Sq^a \circ Sq^b = \sum_{c} \binom{a-b-c}{a-2c} Sq^{a+b-c} \circ Sq^c.
\]
Remember that we’re working mod 2, so the coefficients in the Adem relations are all either 0 or 1. The Adem relations are very important, because they give relationships between the squares that must hold in any $\mathbb{Z}_2$-cohomology ring. For example, $Sq^1Sq^1 = 0$, $Sq^1Sq^2 = Sq^3$, etc. There is an Adem relations calculator at http://math.berkeley.edu/~aaron/adem.

The Steenrod algebra, denoted $A$, is the $\mathbb{Z}_2$-algebra on the Steenrod squares; addition is given by addition of functions and multiplication is given by composition.

We often collect all the squares into a single map $Sq : H^*(X; \mathbb{Z}_2) \to H^*(X; \mathbb{Z}_2)$ given by

$$Sq = \sum_{i=0}^{\infty} Sq^i.$$  

This has the advantage of being a ring homomorphism. Note that when evaluated on any actual element of cohomology, the sum is finite (by property 1). As an example of the power of $Sq$, we easily prove

**Proposition.** For any $x \in H^1$, $Sq^i(x^j) = \binom{j}{i} x^{i+j}$.

**Proof.** Since $Sq(x) = Sq^0 x + Sq^1 x = x + x^2$, then

$$Sq(x^i) = (Sq(x))^j = (x + x^2)^j = \sum_{i} \binom{j}{i} x^{i+j}. \quad \square$$

For a sequence of positive integers $I = \{i_1, \ldots, i_r\}$, we abbreviate $Sq^I = Sq^{i_1} \cdots \cdot Sq^{i_r}$. (If $I = \emptyset$, then by convention $Sq^\emptyset = Sq^0 = id$.) We say that $I$ is admissible if $i_j \geq 2i_{j+1}$ for all $j$. The length of $I$ is $l(I) = r$, the degree of $I$ is $d(I) = \sum i_j$, and the excess of $I$ is $e(I) = (i_1 - 2i_2) + (i_2 - 2i_3) + \ldots + (i_r)$.

**Theorem.** $\{Sq^I : I \text{ admissible}\}$ forms a $\mathbb{Z}_2$-vector space basis for $A$.

**Proof.** This follows directly from the Adem relations. \square

**Application: the Hopf invariant**

Let $[f] \in \pi_{2n-1}(S^n)$, i.e. $f : S^{2n-1} \to S^n$. We obtain a complex called the mapping cone given by

$$K = C(f) = S^n \cup_f e^{2n}.$$  

(That is, we attach the cell $e^{2n}$ to $S^n$ along its boundary $\partial e^{2n} = S^{2n-1}$.) Assuming $n \geq 2$, then by cellular cohomology,

$$H^i(K; \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = n, 2n \\ 0 & \text{otherwise.} \end{cases}$$

If we choose generators $\sigma$ of $H^n(K; \mathbb{Z})$ and $\tau$ of $H^{2n}(K; \mathbb{Z})$, then the ring structure on $H^*(K; \mathbb{Z})$ given by cup product gives us $\sigma^2 = H(f) \cdot \tau$. Here, $H(f) \in \mathbb{Z}$ is called the Hopf invariant of $f$. (It is only well-defined up to sign until we fix generators.)

**Proposition.** $H : \pi_{2n-1}(S^n) \to \mathbb{Z}$ is a group homomorphism.

The addition structure on $\pi_{2n-1}(S^n)$ is defined as follows. Write $\Delta'$ for the “pinching” map $S^{2n-1} \to S^{2n-1} \cup S^{2n-1}$ which collapses the equator to a point. Then for any $[f], [g] \in \pi_{2n-1}(S^n)$, we represent $[f] + [g]$ by the composite map

$$S^{2n-1} \xrightarrow{\Delta'} S^{2n-1} \cup S^{2n-1} \xrightarrow{f \vee g} S^n.$$  

**Proposition.** If $n$ is odd, then $H = 0$.  

**Proof.** By the skew-commutativity of the cup product, if $n$ is odd then $\sigma^2 = -\sigma^2$ so $\sigma^2 = 0$. \square

**Proposition.** If $n$ is even, then there is an element $[f] \in \pi_{2n-1}(S^n)$ with $H(f) = 2$.  

3
For experts: we can take $f = [t_n, t_n]$, where $t_n \in \pi_n(S^n)$ is the class of the identity map and the bracket is the Whitehead bracket $\pi_j \times \pi_k \to \pi_{j+k-1}$.

**Corollary.** If $n$ is even, then $\pi_{2n-1}(S^n)$ contains \( \mathbb{Z} \) as a direct summand.

As it turns out, this along with the fact that $\pi_n(S^n) = \mathbb{Z}$ for all $n$ describes all of the torsion-free part of any $\pi_k(S^n)$! (The proof uses a Serre spectral sequence over \( \mathbb{Q} \)).

We call $Sq^i$ **decomposable** if we can write it as a polynomial in $A$ of strictly lower degree; otherwise it is **indecomposable**.

**Lemma.** $Sq^i$ is indecomposable iff $i = 2^k$.

**Proof.** First, suppose that $i = 2^k$. Write $H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]$, with $|\alpha| = 1$. Then $Sq(\alpha) = Sq^0\alpha + Sq^1\alpha = \alpha + \alpha^2$, so $Sq(a^i) = (Sq(\alpha))^i = (\alpha + \alpha^2)^i = \alpha^i + \alpha^{2i}$ (since $i$ is a power of 2, and we are working mod 2). So $Sq^i\alpha = 0$ for $0 < i < k$, and $Sq^i(\alpha') = \alpha^{2i} \neq 0$. Therefore $Sq^i$ could not possibly be decomposed as a composition of lower-degree squares (or as sums of such compositions). Hence $Sq^i$ is indecomposable.

On the other hand, suppose that $i = a + 2^k$ for $0 < a < 2^k$. Then certainly $a < 2 \cdot 2^k$, so by the Adem relations,

$$Sq^aSq^{2^k} = \binom{2^k - 1}{a} Sq^{a+2^k} + \text{(other terms)}.$$  

Using basic number theory, it is easy to see that

$$\binom{2^k - 1}{a} \equiv 1 \pmod{2},$$

and hence

$$Sq^aSq^{2^k} = Sq^i + \text{(other terms)}.$$  

So $Sq^i$ is decomposable. \( \square \)

Note that we proved a natural statement (i.e., one that holds in all cohomology rings) by looking at the action of $A$ on a specific cohomology ring.

**Corollary.** \( \{Sq^{2^k}\} \) generates $A$ (nonfreely) as an algebra.

**Theorem.** If $[f] \in \pi_{2n-1}(S^n)$ has $H(f) = 1$, then $n = 2^k$.

**Proof.** In $H^*(K; \mathbb{Z})$, $\sigma^2 = \tau$, so this is true in $H^*(K; \mathbb{Z}_2)$ as well. But also $\sigma^2 = Sq^n\sigma$, and since $\tilde{H}^*(K; \mathbb{Z}_2)$ is also supported in degrees $n$ and $2n$, if $Sq^n$ were decomposable then $Sq^n\sigma$ would have to be zero. So it must be that $n = 2^k$. \( \square \)

**Application of application: vector fields on spheres**

An $H$-space structure on a based space $(X, x_0)$ is a continuous (based) multiplication map $\mu : X \times X \to X$ with the property that $x_0$ is a 2-sided identity.

**Proposition.** If $\mu : S^{n-1} \times S^{n-1} \to S^{n-1}$ is an $H$-space structure, then there exists $[f] \in \pi_{2n-1}(S^n)$ with $H(f) = 1$.

**Proof.** For any map $g : S^{n-1} \times S^{n-1}$, we define the bidegree $(\alpha, \beta)$ of $g$ to be the pair of degrees of the restriction of $g$ to $S^{n-1} \times S_0$ and $S_0 \times S^{n-1}$. We carry out the Hopf construction on $g$ to obtain a map $h(g) : S^{2n-1} \to S^n$ as follows. We consider

$$S^{2n-1} = S^{n-1} \ast S^{n-1} = S^{n-1} \times I \times S^{n-1}/S^{n-1} \times \{0\} \times S_0, S_0 \times \{1\} \times S^{n-1};$$

this is the join of $S^{n-1}$ with $S^{n-1}$. We also consider

$$S^n = SS^{n-1} = S^{n-1} \times I/S^{n-1} \times \{0,1\} \cup S_0 \times I;$$

this is the (reduced) suspension of $S^{n-1}$. Then, we put

$$h(g)(a, t, b) = (g(a, b), t).$$

It is routine to verify that $H(h(g)) = \alpha \beta$. If $\mu$ is an $H$-space structure on $S^{n-1}$, then the existence of a 2-sided identity implies that its bidegree is $(1, 1)$. \( \square \)
In general, if we have a $k$-plane field on $S^{n-1}$, we may assume that our vector fields are orthonormal by the Gram-Schmidt process. Since all tangent vectors to a point $x \in S^{n-1}$ are perpendicular to $x$, we may therefore consider a $k$-plane field on $S^{n-1}$ as being equivalent to a section of the fiber bundle over $S^{n-1}$ whose total space is the Stiefel manifold
\[
V_{n,k+1} = \{ n \times (k+1) \text{ matrices with orthonormal columns} \}.
\]
Here, $V_{n,k+1} \rightarrow S^{n-1}$ is the projection onto the first column.

Now, suppose that $S^{n-1}$ is parallelizable. This means that we have an $(n-1)$-plane field on $S^{n-1}$, which is equivalent to a section $v$ of $V_{n,n} = O(n) \rightarrow S^{n-1}$. This gives us a map
\[
S^{n-1} \times S^{n-1} \xrightarrow{(v,\text{id})} O(n) \times S^{n-1} \xrightarrow{\alpha} S^{n-1},
\]
where $\alpha : O(n) \times S^{n-1} \rightarrow S^{n-1}$ is the obvious action of $O(n)$ on $S^{n-1}$. Write $e_1 \in S^{n-1} \subseteq \mathbb{R}^n$. Then $(x,e_1) \mapsto (v(x), e_1) \mapsto x$, so $e_1$ is a right identity. It is not hard to “straighten” this map so that $e_1$ is also a left identity. Then this new map is an H-space structure on $S^{n-1}$. As we have seen, this implies that $n = 2^k$!

To reiterate:

\[
\begin{align*}
S^{n-1} \text{ is parallelizable} & \Rightarrow S^{n-1} \text{ is an H-space} \\
& \Rightarrow \exists [f] \in \pi_{2n-1}(S^n) \text{ with } H(f) = 1 \\
& \Rightarrow n = 2^k
\end{align*}
\]

As it turns out, $S^{n-1}$ is parallelizable if and only if $\mathbb{R}^n$ can be made into a division algebra. In low dimensions, we have the division ring structures $\mathbb{R}^1 \cong \mathbb{R}$, $\mathbb{R}^2 \cong \mathbb{C}$, $\mathbb{R}^4 \cong \mathbb{H}$, $\mathbb{R}^8 \cong \mathbb{O}$. (Here $\mathbb{H}$ denotes the quaternions and $\mathbb{O}$ denotes the octonions.)

**Theorem** (Adams). *These are the only possibilities.*

**Proof.** K-theory.

\[
\square
\]