The Steenrod algebra and its applications – talk 1

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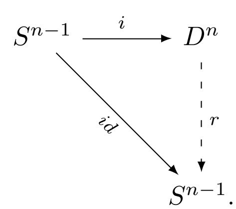
These talks have been lifted shamelessly from the excellent book *Cohomology Operations and Applications in Homotopy Theory* by Mosher & Tangora. Pretty much everything here has been copied from that book, except that I added a bunch of mistakes.

The idea of algebraic topology

Question. *Is there a retraction $r : D^n \to S^{n-1}$?*

We can write this as



By applying the functor $H_{n-1}(\ \ ;\mathbb{Z})$, we can answer this question in the negative.

Question. *For $n > m$, is there a map $\mathbb{R}P^n \to \mathbb{R}P^m$ which is an isomorphism on π_1?*

The answer is again no, but now we need to use cohomology: $H^*(\mathbb{R}P^n;\mathbb{Z}_2) = \mathbb{Z}_2[\alpha_n]/\alpha_n^{n+1} = 0$, and the condition on π_1 implies that the induced map $H^*(\mathbb{R}P^m;\mathbb{Z}_2) \to H^*(\mathbb{R}P^n;\mathbb{Z}_2)$ has $\alpha_m \mapsto \alpha_n$. But this is impossible, since the induced map must be a ring homomorphism.

We will employ a particular collection of much finer structures, called *cohomology operations*, to prove fancier things and answer much harder questions. (Hopefully by the third talk we'll be able to calculate a bunch of homotopy groups of spheres, and our end goal is the Adams spectral sequence.) A cohomology operation of type $(\pi, n; G, m)$ is a natural transformation of functors

$$H^n(\ \ ;\pi) \rightsquigarrow H^m(\ \ ;G).$$

Background

For any space X, we have the *Hurewicz homomorphism* $h : \pi_i(X) \to H_i(X;\mathbb{Z})$, obtained by considering the image of a sphere as an integral chain. We then have

Theorem (Hurewicz)**.** *For a connected space X:*

- $h : \pi_1(X) \to H_1(X;\mathbb{Z})$ *is abelianization, i.e.* $H_1(X;\mathbb{Z}) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$*;*
- *if* $\pi_1(X) = 0$*, then* $h : \pi_2(X) \to H_2(X;\mathbb{Z})$ *is an isomorphism;*
- *if* $\pi_1(X) = \pi_2(X) = 0$*, then* $h : \pi_3(X) \to H_3(X;\mathbb{Z})$ *is an isomorphism;*
- *if* $\pi_1(X) = \pi_2(X) = \pi_3(X) = 0$*, then* $h : \pi_4(X) \to H_4(X;\mathbb{Z})$ *is an isomorphism;*
- *etc.*

Loosely, we can summarize this as saying that the lowest-dimension Hurewicz homomorphism which is nontrivial is an isomorphism, except if $\pi_1(X)$ is nonabelian in which case it's as close to an isomorphism as it can get.

For any abelian group π and any natural number n, the *Eilenberg-Maclane space* $K(\pi, n)$ is characterized (up to homotopy equivalence) by the property that $\pi_n(K(\pi, n)) = G$ and $\pi_i(K(\pi, n)) = 0$ for $i \neq n$. These spaces represent cohomology; that is, there is a bijection

$$H^n(X; \pi) \cong [X, K(\pi, n)],$$

where the brackets denote homotopy classes of maps. If we are careful, this can be made into a group isomorphism.

The *universal coefficient theorem* for cohomology with coefficients in a group π can be expressed as a sexseq

$$0 \to \text{Ext}(H_{n-1}(X; \mathbb{Z}), \pi) \to H^n(X; \pi) \to \text{Hom}(H_n(X; \mathbb{Z}), \pi) \to 0.$$

This says that while we'd expect $H^n(X; \pi)$ to just be $\text{Hom}(H_n(X; \mathbb{Z}), \pi)$ (since to get cohomology you just dualize your chain complexes), in fact this is almost true but there's an extra correction term.

However, if X is $(n-1)$-connected and $\pi = \pi_n(X)$, then by the Hurewicz theorem $H_{n-1}(X; \mathbb{Z}) = 0$, and so in fact we do get an isomorphism $H^n(X; \pi) \cong \text{Hom}(H_n(X; \mathbb{Z}), \pi)$. In this case, the Hurewicz homomorphism $h : \pi_n(X) \to H_n(X; \mathbb{Z})$ is an isomorphism and we have $h^{-1} \in \text{Hom}(H_n(X; \mathbb{Z}), \pi)$. This corresponds (under the isomorphism above) to what we call the *fundamental class* $\iota_X \in H^n(X; \pi)$ of X. In particular, we have a fundamental class $\iota_n \in H^n(K(\pi, n); \pi)$, and in fact a map $f : X \to K(\pi, n)$ corresponds under the bijection $H^n(X; \pi) \cong [X, K(\pi, n)]$ to the cohomology class $f^*(\iota_n) \in H^n(X; \pi)$.

Since cohomology operations are natural, an operation of type $(\pi, n; G, m)$ is represented as a map $K(\pi, n) \to K(G, m)$. (We can see this by applying the operation to the universal case $X = K(\pi, n)$.) But this is just an element of $H^m(K(\pi, n); G)$. Thus, *knowing cohomology operations is the same as knowing the cohomology of Eilenberg-Maclane spaces.*

The Steenrod squares

We will focus our attention on the cohomology operations

$$Sq^i : H^n(\ \ ; \mathbb{Z}_2) \rightsquigarrow H^{n+i}(\ \ ; \mathbb{Z}_2)$$

(for $i \geq 0$) known as the *Steenrod squares*. These are group homomorphisms which can be applied to either absolute or relative cohomology. They have this name because of the second of their

Characterizing Properties:

1. if $i > n$ and $x \in H^n$, then $Sq^i x = 0$;
2. if $x \in H^n$, then $Sq^n x = x^2 \in H^{2n}$;
3. Sq^0 is the identity;
4. Sq^1 is the Bockstein connecting homomorphism associated to the sexseq of coefficients $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$ (to be discussed more next time);
5. $\delta^* Sq^i = Sq^i \delta^*$, where $\delta^* : H^n(A) \to H^{n+1}(X, A)$ is the connecting homomorphism of the cohomology lexseq of the pair (X, A);
6. the *Cartan formula*:
$$Sq^i(xy) = \sum_{j=0}^{i} (Sq^j x)(Sq^{i-j} y);$$
7. the *Adem relations*: if $a < 2b$, then
$$Sq^a \circ Sq^b = \sum_c \binom{b-c-1}{a-2c} Sq^{a+b-c} \circ Sq^c.$$

Remember that we're working mod 2, so the coefficients in the Adem relations are all either 0 or 1. The Adem relations are very important, because they give relationships between the squares that must hold *in any* $\mathbb{Z}_2$-*cohomology ring*. For example, $Sq^1Sq^1 = 0$, $Sq^1Sq^2 = Sq^3$, etc. There is an Adem relations calculator at `http://math.berkeley.edu/~aaron/adem`.

The *Steenrod algebra*, denoted $\mathcal{A}$, is the $\mathbb{Z}_2$-algebra on the Steenrod squares; addition is given by addition of functions and multiplication is given by composition.

We often collect all the squares into a single map $Sq : H^*(X;\mathbb{Z}_2) \to H^*(X;\mathbb{Z}_2)$ given by

$$Sq = \sum_{i=0}^{\infty} Sq^i.$$

This has the advantage of being a *ring* homomorphism. Note that when evaluated on any actual element of cohomology, the sum is finite (by property 1). As an example of the power of Sq, we easily prove

Proposition. *For any* $x \in H^1$, $Sq^i(x^j) = \binom{j}{i} x^{i+j}$.

Proof. Since $Sq(x) = Sq^0x + Sq^1x = x + x^2$, then

$$Sq(x^j) = (Sq(x))^j = (x + x^2)^j = \sum_i \binom{j}{i} x^{i+j}.$$

□

For a sequence of positive integers $I = \{i_1, \ldots, i_r\}$, we abbreviate $Sq^I = Sq^{i_1} \cdot \ldots \cdot Sq^{i_r}$. (If $I = \emptyset$, then by convention $Sq^I = Sq^0 = id$.) We say that I is *admissible* if $i_j \geq 2i_{j+1}$ for all j. The *length* of I is $l(I) = r$, the *degree* of I is $d(I) = \sum i_j$, and the *excess* of I is $e(I) = (i_1 - 2i_2) + (i_2 - 2i_3) + \ldots + (i_r)$.

Theorem. $\{Sq^I : I \text{ admissible}\}$ *forms a* $\mathbb{Z}_2$*-vector space basis for* $\mathcal{A}$.

Proof. This follows directly from the Adem relations. □

Application: the Hopf invariant

Let $[f] \in \pi_{2n-1}(S^n)$, i.e. $f : S^{2n-1} \to S^n$. We obtain a complex called the *mapping cone* given by

$$K = C(f) = S^n \cup_f e^{2n}.$$

(That is, we attach the cell e^{2n} to S^n along its boundary $\partial e^{2n} = S^{2n-1}$.) Assuming $n \geq 2$, then by cellular cohomology,

$$\tilde{H}^i(K;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = n, 2n \\ 0 & \text{otherwise.} \end{cases}$$

If we choose generators σ of $H^n(K;\mathbb{Z})$ and τ of $H^{2n}(K;\mathbb{Z})$, then the ring structure on $H^*(K;\mathbb{Z})$ given by cup product gives us $\sigma^2 = H(f) \cdot \tau$. Here, $H(f) \in \mathbb{Z}$ is called the *Hopf invariant* of f. (It is only well-defined up to sign until we fix generators.)

Proposition. $H : \pi_{2n-1}(S^n) \to \mathbb{Z}$ *is a group homomorphism.*

The addition structure on $\pi_{2n-1}(S^n)$ is defined as follows. Write Δ' for the "pinching" map $S^{2n-1} \to S^{2n-1} \vee S^{2n-1}$ which collapses the equator to a point. Then for any $[f], [g] \in \pi_{2n-1}(S^n)$, we represent $[f] + [g]$ by the composite map

$$S^{2n-1} \xrightarrow{\Delta'} S^{2n-1} \vee S^{2n-1} \xrightarrow{f \vee g} S^n.$$

Proposition. *If* n *is odd, then* $H = 0$.

Proof. By the skew-commutativity of the cup product, if n is odd then $\sigma^2 = -\sigma^2$ so $\sigma^2 = 0$. □

Proposition. *If* n *is even, then there is an element* $[f] \in \pi_{2n-1}(S^n)$ *with* $H(f) = 2$.

For experts: we can take $f = [\iota_n, \iota_n]$, where $\iota_n \in \pi_n(S^n)$ is the class of the identity map and the bracket is the *Whitehead bracket* $\pi_j \times \pi_k \to \pi_{j+k-1}$.

Corollary. *If n is even, then $\pi_{2n-1}(S^n)$ contains $\mathbb{Z}$ as a direct summand.*

As it turns out, this along with the fact that $\pi_n(S^n) = \mathbb{Z}$ for all n describes all of the torsion-free part of *any* $\pi_k(S^n)$! (The proof uses a Serre spectral sequence over $\mathbb{Q}$.)

We call Sq^i *decomposable* if we can write it as a polynomial in $\mathcal{A}$ of strictly lower degree; otherwise it is *indecomposable*.

Lemma. *Sq^i is indecomposable iff $i = 2^k$.*

Proof. First, suppose that $i = 2^k$. Write $H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]$, with $|\alpha| = 1$. Then $Sq(\alpha) = Sq^0\alpha + Sq^1\alpha = \alpha + \alpha^2$, so $Sq(\alpha^i) = (Sq(\alpha))^i = (\alpha + \alpha^2)^i = \alpha^i + \alpha^{2i}$ (since i is a power of 2, and we are working mod 2). So $Sq^t\alpha = 0$ for $0 < t < i$, and $Sq^i(\alpha^i) = \alpha^{2i} \neq 0$. Therefore Sq^i could not possibly be decomposed as a composition of lower-degree squares (or as sums of such compositions). Hence Sq^i is indecomposable.

On the other hand, suppose that $i = a + 2^k$ for $0 < a < 2^k$. Then certainly $a < 2 \cdot 2^k$, so by the Adem relations,

$$Sq^a Sq^{2^k} = \binom{2^k - 1}{a} Sq^{a+2^k} + (\text{other terms}).$$

Using basic number theory, it is easy to see that

$$\binom{2^k - 1}{a} \equiv 1 \pmod{2},$$

and hence

$$Sq^a Sq^{2^k} = Sq^i + (\text{other terms}).$$

So Sq^i is decomposable. □

Note that we proved a *natural* statement (i.e., one that holds in all cohomology rings) by looking at the action of $\mathcal{A}$ on a *specific* cohomology ring.

Corollary. *$\{Sq^{2^k}\}$ generates $\mathcal{A}$ (nonfreely) as an algebra.*

Theorem. *If $[f] \in \pi_{2n-1}(S^n)$ has $H(f) = 1$, then $n = 2^k$.*

Proof. In $H^*(K; \mathbb{Z})$, $\sigma^2 = \tau$, so this is true in $H^*(K; \mathbb{Z}_2)$ as well. But also $\sigma^2 = Sq^n\sigma$, and since $\tilde{H}^*(K; \mathbb{Z}_2)$ is also supported in degrees n and $2n$, if Sq^n were decomposable then $Sq^n\sigma$ would have to be zero. So it must be that $n = 2^k$. □

Application of application: vector fields on spheres

An *H-space structure* on a based space (X, x_0) is a continuous (based) multiplication map $\mu : X \times X \to X$ with the property that x_0 is a 2-sided identity.

Proposition. *If $\mu : S^{n-1} \times S^{n-1} \to S^{n-1}$ is an H-space structure, then there exists $[f] \in \pi_{2n-1}(S^n)$ with $H(f) = 1$.*

Proof. For any map $g : S^{n-1} \times S^{n-1}$, we define the *bidegree* (α, β) of g to be the pair of degrees of the restriction of g to $S^{n-1} \times s_0$ and $s_0 \times S^{n-1}$. We carry out the *Hopf construction* on g to obtain a map $h(g) : S^{2n-1} \to S^n$ as follows. We consider

$$S^{2n-1} = S^{n-1} * S^{n-1} = S^{n-1} \times I \times S^{n-1} / S^{n-1} \times \{0\} \times s_0, s_0 \times \{1\} \times S^{n-1};$$

this is the *join* of S^{n-1} with S^{n-1}. We also consider

$$S^n = \Sigma S^{n-1} = S^{n-1} \times I / S^{n-1} \times \{0, 1\} \cup s_0 \times I;$$

this is the *(reduced) suspension* of S^{n-1}. Then, we put

$$h(g)(a, t, b) = (g(a, b), t).$$

It is routine to verify that $H(h(g)) = \alpha\beta$. If μ is an H-space structure on S^{n-1}, then the existence of a 2-sided identity implies that its bidegree is $(1, 1)$. □

In general, if we have a k-plane field on S^{n-1}, we may assume that our vector fields are orthonormal by the Gram-Schmidt process. Since all tangent vectors to a point $x \in S^{n-1}$ are perpendicular to x, we may therefore consider a k-plane field on S^{n-1} as being equivalent to a section of the fiber bundle over S^{n-1} whose total space is the *Stiefel manifold*

$$V_{n,k+1} = \{n \times (k+1) \text{ matrices with orthonormal columns}\}.$$

Here, $V_{n,k+1} \twoheadrightarrow S^{n-1}$ is the projection onto the first column.

Now, suppose that S^{n-1} is parellelizable. This means that we have an $(n-1)$-plane field on S^{n-1}, which is equivalent to a section v of $V_{n,n} = O(n) \twoheadrightarrow S^{n-1}$. This gives us a map

$$S^{n-1} \times S^{n-1} \xrightarrow{(v,id)} O(n) \times S^{n-1} \xrightarrow{\alpha} S^{n-1},$$

where $\alpha : O(n) \times S^{n-1} \to S^{n-1}$ is the obvious action of $O(n)$ on S^{n-1}. Write $e_1 \in S^{n-1} \subseteq \mathbb{R}^n$. Then $(x, e_1) \mapsto (v(x), e_1) \mapsto x$, so e_1 is a right identity. It is not hard to "straighten" this map so that e_1 is also a left identity. Then this new map is an H-space structure on S^{n-1}. As we have seen, this implies that $n = 2^k$!

To reiterate:

$$\boxed{S^{n-1} \text{ is parallelizable}} \Rightarrow \boxed{S^{n-1} \text{ is an H-space}} \Rightarrow \boxed{\exists\ [f] \in \pi_{2n-1}(S^n) \text{ with } H(f) = 1} \Rightarrow \boxed{n = 2^k}$$

As it turns out, S^{n-1} is parallelizable if and only if $\mathbb{R}^n$ can be made into a division algebra. In low dimensions, we have the division ring structures $\mathbb{R}^1 \cong \mathbb{R}$, $\mathbb{R}^2 \cong \mathbb{C}$, $\mathbb{R}^4 \cong \mathbb{H}$, $\mathbb{R}^8 \cong \mathbb{O}$. (Here $\mathbb{H}$ denotes the quaternions and $\mathbb{O}$ denotes the octonions.)

Theorem (Adams)**.** *These are the only possibilities.*

Proof. K-theory. ∎