The Steenrod algebra and its applications – talk 2

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We continue our foray into the world of Steenrod squares and related notions. This lecture covers a number of topics which are of independent interest but will be necessary for computing homotopy groups of spheres.

The Bockstein homomorphisms

Given any sexseq of coefficients, we get a lexseq in cohomology. For example, $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$ gives rise to

$$\ldots \to H^n(X;\mathbb{Z}) \to H^n(X;\mathbb{Z}) \to H^n(X;\mathbb{Z}_2) \to H^{n+1}(X;\mathbb{Z}) \to \ldots.$$

The connecting map for this lexseq is called the *Bockstein homomorphism*, which we denote $\beta : H^n(X;\mathbb{Z}_2) \to H^{n+1}(X;\mathbb{Z})$.

On cochains, this can be realized as follows. Any $\mathbb{Z}_2$-cohomology class $\overline{x} \in H^n(X;\mathbb{Z}_2)$ is a represented by an integral n-cochain x with the property that $\delta x \equiv 0 \pmod 2$, i.e. for any $(n+1)$-chain c, $\delta x(c) = x(\partial c) \equiv 0 \pmod 2$. (This is what it means for x to be a *cocycle mod 2*.) So it must be that $\delta x = 2y$ for some (integral) $(n+1)$-cochain y. Then $\beta(\overline{x}) = y$.

The *reduced Bockstein homomorphism* $d_1 : H^n(X;\mathbb{Z}_2) \to H^{n+1}(X;\mathbb{Z}_2)$ (which we will henceforth simply refer to as the Bockstein homomorphism) is obtained by following β by reduction mod 2. (For experts: observe that, as usual, we are just skipping the internal map in the Bockstein exact couple.) Of course, using the same notation as before, $d_1(\overline{x}) = \overline{y}$.

Note that $\ker(d_1)$ consists of exactly those $\mathbb{Z}_2$-cohomology classes $\overline{x}$ such that $\delta x/2$ still evaluates to 0 mod 2 on all boundaries. Thus we may write $\delta x = 4y'$, and we define the second Bockstein homomorphism by $d_2(\overline{x}) = \overline{y'}$. More generally, we define the r^{th} Bockstein homomorphism d_r on $\ker(d_{r-1})$ by

$$d_r(\overline{x}) = \overline{\left(\frac{\delta x}{2^r}\right)}.$$

From these, we can obtain information about $\mathbb{Z}$-cohomology from $\mathbb{Z}_2$-cohomology. If x generates a copy of $\mathbb{Z}$ in $H^n(X;\mathbb{Z})$, then $d_i(\overline{x}) = 0$ for all i. If on the other hand x generates a copy of $\mathbb{Z}_{2^r}$ in $H^{n+1}(X;\mathbb{Z})$, then x gives rise to $\chi \in H^n(X;\mathbb{Z}_2)$ and $\overline{x} \in H^{n+1}(X;\mathbb{Z})$ (via the universal coefficient theorem) such that $d_i(\chi) = 0$ for $i < r$, $d_i(\overline{x}) = 0$ for $i < r$, and $d_r(\chi) = \overline{x}$.

Fibrations

A *fibration* (which for us will always be "in the sense of Serre", for those who care) is a map $p : E \to B$ such that, for any finite complex K, we have the *covering homotopy property*:

$$\begin{array}{ccc} K & \longrightarrow & E \\ {\scriptstyle i_0}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ K \times I & \longrightarrow & B. \end{array}$$

Assuming B is connected, every fiber $p^{-1}(b) \subseteq E$ is homotopy equivalent. We denote by F the fiber over the basepoint, and we write the fibration as $F \hookrightarrow E \twoheadrightarrow B$.

A fibration gives rise to a *lexseq in homotopy*

$$\ldots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \ldots.$$

Example. A covering space is a fibration with a discrete fiber. In the case of $\mathbb{Z} \hookrightarrow \mathbb{R} \twoheadrightarrow S^1$, since $\mathbb{R}$ is contractible the lexseq gives us that $\pi_1(S^1) = \mathbb{Z}$ and $\pi_i(S^1) = 0$ for $i > 1$. Thus $S^1 = K(\mathbb{Z}, 1)$.

Example. The Hopf map $\eta : S^3 \to S^2$ is a fibration with fiber S^1. So the homotopy lexseq is

$$\ldots \to 0 \to \pi_n(S^3) \to \pi_n(S^2) \to 0 \to \ldots \to \pi_3(S^3) \to \pi_3(S^2) \to 0 \to \pi_2(S^3) \to \pi_2(S^2) \to \mathbb{Z} \to 0 \to \ldots.$$

Since $\pi_i(S^i) = \mathbb{Z}$ for any i, this tells us that $\pi_2(S^3) = 0$ (which we already knew by cellular approximation), that $\pi_3(S^2) = \mathbb{Z}$ and this is generated by η, and more generally that $\pi_n(S^3) = \pi_n(S^2)$ for $n \geq 3$.

Example. Let E be the space of based paths $\gamma : I \to (B, b_0)$, and let $p(\gamma) = \gamma(1)$. Clearly E is contractible, so applying the homotopy lexseq to the fibration $\Omega B \hookrightarrow E \twoheadrightarrow B$ gives that $\pi_n(\Omega B) = \pi_{n+1}(B)$ for all n.

Example. Considering $S^{2n-1} \subseteq \mathbb{C}^n$, we have (for $n > 1$) the fibration $S^1 \hookrightarrow S^{2n-1} \twoheadrightarrow \mathbb{CP}^{n-1}$. So $\pi_2(\mathbb{CP}^{n-1}) = \pi_1(S^1) = \mathbb{Z}$. Furthermore, $\pi_i(\mathbb{CP}^{n-1}) = \pi_i(S^{2n-1})$ for $i > 2$, and this is 0 up until $2n-1$. Taking the (co)limit as $n \to \infty$, we get that $\mathbb{CP}^\infty = K(\mathbb{Z}, 2)$.

The Serre spectral sequence

For a fibration $F \hookrightarrow E \twoheadrightarrow B$, we have *Serre's spectral sequence in cohomology*

$$E_2^{p,q} = H^p(B; \mathcal{H}^q(F; R)) \Rightarrow H^*(E; R),$$

where R is any (commutative) ring and the coefficients may be twisted by $\pi_1(B)$. This has the following properties:

- E_r is a bigraded ring for all r;
- d_r is an antiderivation, i.e. $d_r(a \cdot b) = d_r(a) \cdot b + (-1)^{|a|} a \cdot d_r(b)$;
- the product in E_{r+1} is induced by the product in E_r;
- if R is a field, then $E_2 = H^*(B; R) \otimes H^*(F; R)$ by the Künneth theorem.

(It will always be easy to distinguish these differentials from the Bockstein homomorphisms from the context.)

If B and F are $(p-1)$- and $(q-1)$-connected, resp., then the spectral sequence degenerates to *Serre's lexseq in cohomology*:

$$\ldots \longrightarrow H^{p+q-2}(F) \xrightarrow{\tau} H^{p+q-1}(B) \xrightarrow{p^*} H^{p+q-1}(E) \xrightarrow{j^*} H^{p+q-1}(F),$$

where $p : E \twoheadrightarrow B$ and $j : F \hookrightarrow E$.

More generally, we call any element $x \in H^{n-1}(F)$ *transgressive* if in Serre's spectral sequence $d_i(x) = 0$ for all $i < n$. Then we have $d_n(x) \in E_n^{n,0}$, which is a subquotient of $H^n(B)$. In this case, we write $\tau(x) = d_n(x)$.

Proposition. *If x is transgressive, then so is $Sq^i x$ for all i, and $\tau(Sq^i x) = Sq^i(\tau(x))$.*

Cohomology of Eilenberg-Maclane spaces

In this section, all cohomology will have $\mathbb{Z}_2$ coefficients. To ease notation, we write $H^*(\pi, n)$ for $H^*(K(\pi, n))$. Recall that we have the fundamental class $\iota_n \in H^n(\pi, n; \pi)$. Using the above proposition and Serre's spectral sequence, one can calculate:

- $H^*(\mathbb{Z}, 2)$ is the $\mathbb{Z}_2$-polynomial ring generated by $\iota_2 \in H^2(\mathbb{Z}, 2)$ (where ι_2 is really the reduction mod 2 of the original fundamental class);
- for $q > 2$, $H^*(\mathbb{Z}, q)$ is the $\mathbb{Z}_2$-polynomial ring generated by $\{Sq^I(\iota_q) : I \text{ admissible}, e(I) < q, i_r \neq 1\}$.
- $H^*(\mathbb{Z}_2, q)$ is the $\mathbb{Z}_2$-polynomial ring generated by $\{Sq^I(\iota_q) : I \text{ admissible}, e(I) < q\}$;
- for $q > 2$, $H^*(\mathbb{Z}_{2^m}, q)$ is the $\mathbb{Z}_2$-polynomial ring generated by $\{Sq^{I_m}(\iota_q) : I \text{ admissible}, e(I) < q\}$, where $Sq^{I_m} = Sq^I$ if $i_r > 1$, while if $i_r = 1$ then we replace Sq^1 with the Bockstein homomorphism d_m;

Note that the first calculation is actually a special case of the second, and the third is actually a special case of the fourth. (Recall that $d_1 = Sq^1$.)

Classes of abelian groups

A *class of abelian groups* $\mathcal{C}$ is a collection of abelian groups which is closed under taking subgroups, quotients, and group extensions. A group homomorphism is a *$\mathcal{C}$-monomorphism* if its kernel is contained in $\mathcal{C}$, a *$\mathcal{C}$-epimorphism* if its cokernel is contained in $\mathcal{C}$, and a *$\mathcal{C}$-isomorphism* if it is both a $\mathcal{C}$-monomorphism and a $\mathcal{C}$-epimorphism.

We will be interested in $\mathcal{C}_2$, the class of abelian torsion groups of finite exponent such that the order of every element is prime to 2 (i.e., odd). In other words, we will be interested in "ignoring odd torsion". This class is contagious under tensor product (i.e., if $A \in \mathcal{C}_2$ then $A \otimes B \in \mathcal{C}_2$ for any abelian group B), and if $A \in \mathcal{C}_2$, then $H_n(A, 1; \mathbb{Z}) \in \mathcal{C}_2$ for every $n > 0$.

In general, we have the *Hurewicz theorem mod* $\mathcal{C}$, the *relative Hurewicz theorem mod* $\mathcal{C}$, and *Whitehead's theorem mod* $\mathcal{C}$. However, what will be most important for us is

Theorem ($\mathcal{C}_2$-approximation)**.** *Suppose $f : A \to X$, $\pi_1(A) = \pi_1(X) = 0$, in each degree the homology of A and X is finitely generated, and $f_\# : \pi_2(A) \to \pi_2(X)$ is epimorphic. Then any one of the equivalent conditions*

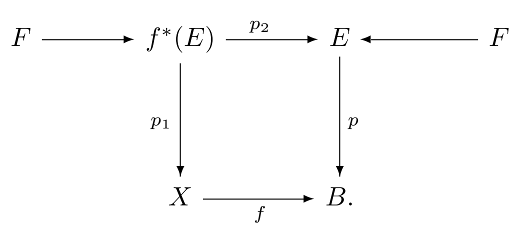
1. $f^* : H^i(X; \mathbb{Z}_2) \to H^i(A; \mathbb{Z}_2)$ *is isomorphic for* $i < n$ *and monomorphic for* $i = n$;
2. $f_* : H_i(A; \mathbb{Z}_2) \to H_i(X; \mathbb{Z}_2)$ *is isomorphic for* $i < n$ *and epimorphic for* $i = n$;
3. $H_i(X, A; \mathbb{Z}_2) = 0$ *for* $i \leq n$;
4. $H_i(X, A; \mathbb{Z}) \equiv 0 \pmod{\mathcal{C}_2}$ *for* $i \leq n$;
5. $\pi_i(X, A) \equiv 0 \pmod{\mathcal{C}_2}$ *for* $i \leq n$;
6. $f_\# : \pi_i(A) \to \pi_i(X)$ *is* $\mathcal{C}_2$*-isomorphic for* $i < n$ *and* $\mathcal{C}_2$*-epimorphic for* $i = n$

implies that $\pi_i(A) \equiv \pi_i(X) \pmod{\mathcal{C}_2}$ *for* $i < n$.

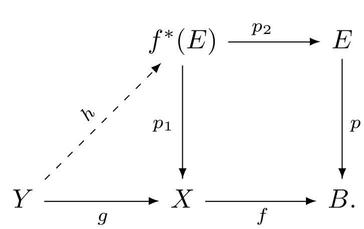
Now we can broadly state our method for computing the homotopy groups of S^n. We begin with $K(\mathbb{Z}, n)$, which has the same cohomology as S^n up through dimension n. Then, we successively kill its higher $\mathbb{Z}_2$-cohomology groups, so that the homotopy groups of the resulting space will agree (mod $\mathcal{C}_2$) with those of S^n. However, to do this we will need to know a bit more about fibrations.

More on fibrations

Given a fibration $p : E \to B$ and a map $f : X \to B$, we have the *induced fiber space* $f^*(E) = \{(x, e) : f(x) = p(e)\}$, topologized as a subspace of $X \times E$. The first projection $p_1 : f^*(E) \to X$ is a fibration with the same fiber as $p : E \to B$, and we also have the second projection $p_2 : f^*(E) \to E$. We can summarize the situation in the following diagram:



Proposition. *Suppose that in addition, Y is a finite complex and we have a map $g : Y \to X$. Then if $fg : Y \to B$ is nullhomotopic, then there is a lifting $h : Y \to f^*(E)$. If E is contractible, then the converse holds: i.e., if there is a lifting $h : Y \to f^*(E)$ making the diagram commute, then $fg : Y \to B$ is nullhomotopic.*



Proposition. *Suppose that F is $(n-1)$-connected. Then the fundamental class $\iota_F \in H^n(F;\pi_n(F))$ is transgressive. In particular, for the path fibration $K(\pi,n) \hookrightarrow * \twoheadrightarrow K(\pi,n+1)$ we have an isomorphism $\tau : H^n(\pi,n;\pi) \to H^{n+1}(\pi,n+1;\pi)$, and $\tau(\iota_n) = \iota_{n+1}$.*

Theorem. *Suppose we represent a cohomology class $x \in H^{n+1}(X;\pi)$ by a map $f : X \to K(\pi,n+1)$. Then for the induced fibration $K(\pi,n) \hookrightarrow X_1 \twoheadrightarrow X$, $\tau(\iota_n) = f^*(\iota_{n+1}) = x$.*

The big picture

We can now state more precisely our plan of attack. (All our cohomology with be with $\mathbb{Z}_2$ coefficients from now on.) In order to calculate the homotopy groups of S^n (aside from odd torsion), we will begin with $K(\mathbb{Z},n)$ and work our way up killing cohomology so that the cohomology of the resulting space agrees more and more with that of S^n. Then by the mod $\mathcal{C}_2$-approximation theorem, the resulting space will have homotopy groups which are isomorphic mod $\mathcal{C}_2$ to those of S^n in (approximately) those same dimensions.

Now from this last theorem, we see roughly how to do this. If we don't like the cohomology class $x \in H^{n+1}(X)$, we'd like to represent it by a map $f : X \to K(\mathbb{Z}_2, n+1)$. Then (assuming $n \geq 2$) we will have for the induced fibration $K(\pi,n) \hookrightarrow X_1 \twoheadrightarrow X$ that

$$H^n(\mathbb{Z}_2,n) \xrightarrow{\tau} H^{n+1}(X) \xrightarrow{p^*} H^{n+1}(X_1) \xrightarrow{j^*} H^{n+1}(\mathbb{Z}_2,n) \xrightarrow{\tau} H^{n+1}(X)$$

by Serre's lexseq. Hence $\tau : H^n(\mathbb{Z}_2,n) \to H^{n+1}(X)$ is epimorphic. However, *we need $\tau : H^{n+1}(\mathbb{Z}_2,n) \to H^{n+2}(X)$ to be monomorphic in order to conclude that $H^{n+1}(X_1) = 0$.* This will not be true in general. It all depends on what's going on in integral cohomology. To fix this, we may need to replace $\mathbb{Z}_2$ with $\mathbb{Z}_{2^m}$ or even $\mathbb{Z}$, according to the following crucial

Lemma (Bockstein)**.** *Let $F \hookrightarrow E \twoheadrightarrow B$ be a fibration. Write $j : F \to E$ for the inclusion and $p : E \to B$ for the projection. Suppose $u \in H^n(F)$ is transgressive, and suppose that there is some class $v \in H^n(B)$ such that $d_i v = \tau(u)$ (for some $i \geq 1$). Then $d_{i+1}p^*v$ is defined, and $j^* d_{i+1} p^* v = d_i u$.*

Recall that after the fundamental class $\iota_n \in H^n(\mathbb{Z}/2,n)$, the next cohomology class of $H^*(\mathbb{Z}/2,n)$ is $Sq^1\iota_n = d_1\iota_n \in H^{n+1}(\mathbb{Z}_2,n)$. So if $\tau(d_1\iota_n) = 0$, we can't use this fibration to kill $H^{n+1}(X_1)$. In this case we make a new fibration with fiber $K(\mathbb{Z}_4,n)$, whose next cohomology class after $\iota_n \in H^n(\mathbb{Z}_4,n)$ is $d_2\iota_n \in H^{n+1}(\mathbb{Z}_4,n)$. If for this fibration $\tau(d_2\iota_n) = 0$ then this doesn't work either, so we try $K(\mathbb{Z}_8,n)$, and on up as far as we need to go. And if $\tau(d_m\iota_n)$ for all m, then we use $K(\mathbb{Z},n)$, which has $H^{n+1}(\mathbb{Z},n) = 0$.

And now, we're ready to compute homotopy groups of spheres!