Math 256A: Algebraic Geometry

Martin Olsson

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1 Introduction

Consider the equation $X^n + Y^n + Z^n = 0$ for some $n \in \mathbb{N}$. Classically, this was studied by viewing it as describing a locus of points in $\mathbb{C}^3$ as a topological space. (We say that we are looking at solutions "over $\mathbb{C}$".) But for instance, we may be interested in solutions over $\mathbb{Q}$. How do we understand our solutions now?

In fact, we can look for solutions over any ring R. We can think of picking out three elements $(a, b, c) \in R^3$ as the same thing as describing a ring homomorphism $\mathbb{Z}[x, y, z] \to R$ (where $x \mapsto a$, $y \mapsto b$, and $z \mapsto c$.) So we obtain a bijection $\{(a, b, c) \in R^3 : a^n + b^n + c^n = 0\} \cong \mathrm{Hom}_{\mathbf{Rings}}(\mathbb{Z}[x, y, z]/(x^n + y^n + z^n), R)$. Thus, we might say that the ring $\mathbb{Z}[x, y, z]/(x^n + y^n + z^n)$ contains information about solutions to the equation in the ring R, for *all* rings R.

More formally, we can consider the functor $S : \mathbf{Ring} \to \mathbf{Set}$ given by $S(R) = \{(a, b, c) \in R^3 : a^n + b^n + c^n = 0\}$. To say that S is a *functor* means that:

1. For any $R \xrightarrow{f} R'$, we have a map $\theta_f : S(R) \to S(R')$ (given by $(a, b, c) \mapsto (f(a), f(b), f(c))$).

2. If $R \xrightarrow{f} R' \xrightarrow{g} R''$, then the diagram

$$\begin{tikzcd}
S(R) \arrow[r, "\theta_f"] \arrow[dr, "\theta_{g\circ f}"'] & S(R') \arrow[d, "\theta_g"] \\
 & S(R'')
\end{tikzcd}$$

commutes.

In fact, if for any ring A we denote by $h^A : \mathbf{Ring} \to \mathbf{Set}$ the functor $h^A(R) = \mathrm{Hom}_{\mathbf{Ring}}(A, R)$, then $S \simeq h^{\mathbb{Z}[x,y,z]/(x^n+y^n+z^n)}$; these functors are *isomorphic* (a term we will properly define later). We say that S is a *(co)representable functor.*

Let us turn to geometry. We will study *schemes.* These are global geometric objects, the local study of which should be exactly commutative ring theory. This is in analogy to the way that manifolds are locally isomorphic to $\mathbb{R}^n$. Thus, a scheme will be a functor $F : \mathbf{Ring} \to \mathbf{Set}$ which "locally is representable". In general, the functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}, \mathbf{Set})$ given by $X \mapsto h^X$ is *contravariant* (i.e. it reverses the direction of arrows).

Lemma 1 (Yoneda)**.** *The functor $X \mapsto h^X$ is fully faithful, i.e. for all $X, X' \in \mathcal{C}$,*

$$\mathrm{Hom}_{\mathcal{C}}(X, X') \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathbf{Set})}(h^{X'}, h^X)$$

is a bijection.

Example 1. We can use Yoneda's lemma to prove potentially messy ring-theoretic statements in clean ways. For example, let us prove that for any $A, B, C \in \mathbf{Rings}$, $P = (A \otimes_{\mathbb{Z}} B) \otimes_{\mathbb{Z}} C$ and $Q = A \otimes_{\mathbb{Z}} (B \otimes_Z C)$ are isomorphic. We consider the functor $h^P : \mathbf{Rings} \to \mathbf{Set}$ defined by $h^P(R) = \mathrm{Hom}_{\mathbf{Rings}}(P, R)$ and similarly the functor $h^Q : \mathbf{Rings} \to \mathbf{Set}$. Now since the Yoneda embedding is fully faithful, then the map $\mathrm{Hom}_{\mathbf{Rings}}(P, Q) \to \mathrm{Hom}_{\mathrm{Fun}(\mathbf{Rings},\mathbf{Set})}(h^Q, h^P)$ is a bijection. In general, the tensor product $D \otimes E$ is defined to be the initial ring admitting maps $D \to D \otimes E \leftarrow E$ from D and E, i.e. if we have maps $D \to S$ and $E \to S$ then we obtain a unique map $D \otimes E \to S$ making the diagram

$$\begin{tikzcd}
D \arrow[d] \arrow[dr] & \\
D \otimes E \arrow[r, dashed, "\exists!"] & S \\
E \arrow[u] \arrow[ur] &
\end{tikzcd}$$

commute. Thus, $D \otimes E$ represents the functor $S \mapsto \mathrm{Hom}_{\mathbf{Rings}}(D, S) \times \mathrm{Hom}_{\mathbf{Rings}}(E, S)$ (in a way which is induced by the diagram $D \to D \otimes E \leftarrow E$). Now in our particular case, h^P is the functor

$$S \mapsto \mathrm{Hom}((A \otimes B), S) \times \mathrm{Hom}(C, S) \simeq \mathrm{Hom}(A, S) \times \mathrm{Hom}(B, S) \times \mathrm{Hom}(C, S).$$

Of course we easily obtain that h^Q is isomorphic to that same functor. So there is an isomorphism of functors $h^P \simeq h^Q$, i.e. natural transformations $h^P \to h^Q$ and $h^Q \to h^P$ that are two-sided inverses. Going back through the bijection guaranteed by Yoneda's lemma (and using the fact that the Yoneda embedding is a functor), we obtain ring homomorphisms $P \to Q$ and $Q \to P$ that are two-sided inverses. Thus $P \cong Q$.

2 Category theory

Definition 1. A *category* consists of a class of *objects* $\mathrm{Ob}(\mathcal{C})$ and for any two objects $A, B \in \mathrm{Ob}(\mathcal{C})$ a set of *morphisms* $\mathrm{Hom}_{\mathcal{C}}(A, B)$ such that for all $A, B, C \in \mathrm{Ob}(\mathcal{C})$ we have a *composition law*

$$\begin{array}{ccc} \mathrm{Hom}(A,B) \times \mathrm{Hom}(B,C) & \overset{c_{A,B,C}}{\to} & \mathrm{Hom}(A,C) \\ (f,g) & \mapsto & g \circ f \end{array}$$

satisfying the following axioms:

1. (Associativity axiom) Given $A \overset{f}{\to} B \overset{g}{\to} C \overset{h}{\to} D$, $(hg)f = h(gf)$. That is, the diagram

$$\begin{tikzcd} \mathrm{Hom}(A,B) \times \mathrm{Hom}(B,C) \times \mathrm{Hom}(C,D) \arrow[r, "\mathrm{id} \times c_{B,C,D}"] \arrow[d, "c_{A,B,C} \times \mathrm{id}"'] & \mathrm{Hom}(A,B) \times \mathrm{Hom}(B,D) \arrow[d, "c_{A,B,D}"] \\ \mathrm{Hom}(A,C) \times \mathrm{Hom}(C,D) \arrow[r, "c_{A,C,D}"'] & \mathrm{Hom}(A,D) \end{tikzcd}$$

   commutes.

2. (Unit axiom) For every object $A \in \mathrm{Ob}(\mathcal{C})$ there exists a morphism $\mathrm{id}_A \in \mathrm{Hom}(A, A)$ such that for every $f \in \mathrm{Hom}(A, B)$, $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$.

Remark 1. We will ignore set-theoretic issues throughout this class. We will always pretend that our categories are *small*, i.e. that $\mathrm{Ob}(\mathcal{C})$ is a set.

Remark 2. Very often it's helpful to think of a category as a graph: objects are vertices, and morphisms are directed arrows. In this setup, we generally don't draw identity morphisms.

Example 2. We have the following examples of categories:

- sets;
- groups, Abelian groups;
- Rings (which for us will always be commutative with unit);
- $\mathbf{Mod}_R$, the category of modules over a fixed ring R;
- the opposite category $\mathcal{C}^{op}$ of a category $\mathcal{C}$, which has $\mathrm{Ob}(\mathcal{C}^{op}) = \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}^{op}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(B, A)$;
- the category $Op(X)$ of open subsets of a topological space X, with morphisms

$$\mathrm{Hom}_{Op(X)}(U,V) = \begin{cases} \{*\}, & U \subset V \\ \emptyset, & U \not\subset V. \end{cases}$$

Definition 2. Given categories $\mathcal{C}$ and $\mathcal{D}$, a *functor* $F : \mathcal{C} \to \mathcal{D}$ is a rule $A \in \mathcal{C} \rightsquigarrow F(A) \in \mathcal{D}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B) \rightsquigarrow F(f) \in \mathrm{Hom}_{\mathcal{C}}(F(A), F(B))$ such that:

1. For all $A \in \mathcal{C}$, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.
2. $F(g \circ f) = F(g) \circ F(f)$ whenever $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$; i.e., the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(A,B) \times \mathrm{Hom}_{\mathcal{C}}(B,C) \arrow[r, "c_{A,B,C}"] \arrow[d, "F"'] & \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "F"] \\ \mathrm{Hom}_{\mathcal{D}}(F(A),F(B)) \times \mathrm{Hom}_{\mathcal{D}}(F(B),F(C)) \arrow[r, "d_{F(A),F(B),F(C)}"'] & \mathrm{Hom}_{\mathcal{D}}(F(A),F(C)) \end{tikzcd}$$

   commutes.

Example 3. We have the following examples of functors.

- Forget : **Rings** → **Ab**.
- Given a topological space X, we have the functor

$$\begin{array}{ccc} Op(X)^{op} & \overset{\text{Fun}}{\to} & \mathbf{Set} \\ U & \mapsto & C^0(U \to \mathbb{R}) \end{array}$$

of continuous real-valued functions. Given $U \hookrightarrow V$, we have $\text{Fun}(V) \to \text{Fun}(U)$ given by restriction.

Definition 3. A functor $F : \mathcal{C} \to \mathcal{D}$ is called *fully faithful* if the maps $\text{Hom}_{\mathcal{C}}(A, B) \to \text{Hom}_{\mathcal{D}}(F(A), F(B))$ are bijections. In this case, we can think of $\mathcal{C}$ as a *subcategory* of $\mathcal{D}$. F is called *essentially surjective* if every object in $\mathcal{D}$ is isomorphic to $F(A)$ for some $A \in \mathcal{C}$. ($X, Y \in \mathcal{D}$ are *isomorphic* if there exist $f \in \text{Hom}_{\mathcal{D}}(X, Y)$ and $g \in \text{Hom}_{\mathcal{D}}(Y, X)$ such that $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$.)

Example 4. Given the 4-object category $\mathcal{C} = \bullet \leftrightarrows \bullet \leftrightarrows \bullet \leftrightarrows \bullet$, the unique functor to the trivial category $\mathcal{D} = \bullet$ is fully faithful: there's only the identity morphism in each endomorphism set of $\mathcal{C}$. So it doesn't really matter which objects there are in each category; what matters are the morphisms. This is also essentially surjective.

We can define a functor from $\mathcal{D}$ back to $\mathcal{C}$ by specifying the image of the unique object. There are no choices to be made about morphisms. This is also fully faithful and essentially surjective.

Definition 4. Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\eta : F \to G$ is a rule that for each $A \in \mathcal{C}$ assigns $\eta_A \in \text{Hom}_{\mathcal{D}}(F(A), G(A))$, such that for all $f \in \text{Hom}_{\mathcal{C}}(A, B)$ the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"'] & F(B) \arrow[d, "\eta_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

commutes. If $\mathcal{C}$ and $\mathcal{D}$ are categories, the functors $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ for a category: the objects are functors and the morphisms are natural transformations. An *isomorphism of functors* is an isomorphism in this category; this is the case exactly when $\eta_A : F(A) \to G(A)$ is an isomorphism for all $A \in \mathcal{C}$. (This agrees with the condition that there exist natural transformations back and forth between F and G such that the compositions are the identity natural transformations.)

Example 5. Let us return to the previous example. Write $\text{Ob}(\mathcal{C}) = \{1, 2, 3, 4\}$ for convenience. Let us call our functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, say $G(\bullet) = 3 \in \mathcal{C}$. Then $(G \circ F)(i) = 3$ for all $i \in \text{Ob}(\mathcal{C})$. This is not the identity functor. However, $G \circ F$ is nevertheless *isomorphic* to the identity functor, via the natural transformation $\eta : \text{id}_{\mathcal{C}} \to G \circ F$ defined by taking η_i to be the unique element of $\text{Hom}_{\mathcal{C}}(i, 3)$ for all i. On the other hand, $F \circ G$ is *equal* to the identity functor of $\mathcal{D}$.

Definition 5. A functor $\mathcal{C} \to \mathcal{D}$ is an *equivalence of categories* if there is a functor $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F \simeq \text{id}_{\mathcal{C}}$ and $F \circ G \simeq \text{id}_{\mathcal{D}}$.

Proposition 1. *A functor is an equivalence if and only if it is fully faithful and essentially surjective.*

Question 1. *Suppose that $\mathcal{C}$ is a category and $X, Y \in \mathcal{C}$. What does it mean to say that the product $X \times Y$ exists?*

Whenever one encounters such a question, it is useful to consider it as a question of whether a particular functor is representable. Consider $F : \mathcal{C}^{op} \to \mathbf{Set}$ given by $F(Z) = \text{Hom}_{\mathcal{C}}(Z, X) \times \text{Hom}_{\mathcal{C}}(Z, Y)$. (A morphism $Z' \to Z$ induces a function $F(Z) \to F(Z')$ by precomposition.)

Definition 6. A *product* $X \times Y$ is an object $P \in \mathcal{C}$ and an isomorphism $\iota : h_P \xrightarrow{\sim} F$, where we define $h_P(Z) = \text{Hom}_{\mathcal{C}}(Z, P)$.

Proposition 2. *This is equivalent to the usual definition that Q (with maps $b : Q \to X$ and $a : Q \to Y$) is the product $X \times Y$ if the diagram*

$$\begin{tikzcd}
Q \arrow[r, "a"] \arrow[d, "b"'] & Y \\
X &
\end{tikzcd}$$

is terminal among objects with pairs of maps to X and Y.

Proof. Take $\mathrm{id}_P \in h_P(P)$, and write $\iota(\mathrm{id}_P) = (b, a) \in \mathrm{Hom}_{\mathcal{C}}(P, X) \times \mathrm{Hom}_{\mathcal{C}}(P, Y)$. Now given $Z \in \mathcal{C}$, we can associate to each $\rho \in \mathrm{Hom}_{\mathcal{C}}(Z, P) = h_P(Z)$ the postcompositions $(f, g) = (b \circ \rho, a \circ \rho) \in F(Z)$, or alternatively we can say $\mathrm{Hom}_{\mathcal{C}}(Z, P) \simeq h_P(Z) \overset{\iota}{\to} F(Z)$. In fact, these are the same thing. (There's something to check here!) ☐

In particular, this makes clear the difference between direct sums and direct products: for finite diagrams of groups (for example) these happen to coincide, but they are totally different *as functors*.

3 Presheaves and sheaves

Let X be a topological space. Recall that we have the category $Op(X)$ of open sets and inclusions.

Definition 7. A *presheaf* (of sets) (on X) is a functor $F : Op(X)^{op} \to \mathbf{Set}$. (We obtain a presheaf of groups, abelian groups, spaces, etc. by replacing $\mathbf{Set}$ with a different category.) More concretely, a presheaf F on X consists of the following data:

- for each open $U \subset X$, a set $F(U)$, and
- for each inclusion $V \subset U$, a *restriction* function $\rho_{U,V} : F(U) \to F(V)$

such that $\rho_{U,U} = \mathrm{id}_{F(U)}$ and that whenever $W \subset V \subset U$, the diagram

$$\begin{tikzcd}
F(U) \arrow[r, "\rho_{V,U}"] \arrow[dr, "\rho_{U,W}"'] & F(V) \arrow[d, "\rho_{V,W}"] \\
& F(W)
\end{tikzcd}$$

commutes.

Example 6. Let Y be a topological space. For $U \subset X$ open, define $F(U) = \mathrm{Hom}_{\mathbf{Top}}(U, Y)$. This is actually a functor $h_Y : \mathbf{Top}^{op} \to \mathbf{Set}$, which when restricted to $Op(X)^{op}$ gives us the presheaf F.

Definition 8. A presheaf F on X is a *sheaf* if for any open subset $U \subset X$ and covering $U = \cup_i U_i$, the sequence $F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{(i,j)} F(U_i \cap U_j)$ is exact. The first map α is $\alpha(x) = (\rho_{U,U_i}(x))_i$, and the other arrows p_1 and p_2 are given by

$$p_1((y_i)_i) = \left(\rho_{U_i, U_i \cap U_j}(y_i)\right)_{i,j}$$
$$p_2((y_i)_i) = \left(\rho_{U_j, U_i \cap U_j}(y_i)\right)_{i,j},$$

where $y_i \in F(U_i)$. Here, *exactness* means that α is injective and its image in $\prod_i F(U_i)$ is exactly the subset of those elements $(y_i)_i \in \prod_i F(U_i)$ for which $p_1((y_i)_i) = p_2((y_i)_i)$.

We can rephrase the *sheaf condition* as follows: A presheaf F is a sheaf if for every open $U \subset X$ and covering $U = \cup_i U_i$:

- if $s, s' \in F(U)$ and $\rho_{U,U_i}(s) = \rho_{U,U_i}(s')$ for all i, then $s = s'$, and

- if $s_i \in F(U_i)$ and for all (i,j) we have $\rho_{U_i, U_i \cap U_j}(s_i) = \rho_{U_j, U_i \cap U_j}(s_j)$ then there is a unique $s \in F(U)$ such that $\rho_{U,U_i}(s) = s_i$ for all i.

(A presheaf that satisfies only the second of these conditions is called a *separated presheaf.*)

An advantage of working with abelian groups is that these form an abelian category. Thus we can take the difference $p_1 - p_2$, and we can simply demand that $F(U) \overset{\alpha}{\to} \prod_i F(U_i) \overset{p_1 - p_2}{\to} \prod_{(i,j)} F(U_i \cap U_j)$ is a short exact sequence in the usual sense.

Definition 9. If F and G are presheaves, a *morphism of presheaves* $f : F \to G$ consists of maps $f_U : F(U) \to G(U)$ for each $U \in Op(X)$ such that whenever $V \subset U$, the diagram

$$\begin{tikzcd}
F(U) \arrow[r, "f_U"] \arrow[d, "\rho^F_{U,V}"] & G(U) \arrow[d, "\rho^G_{U,V}"'] \\
F(V) \arrow[r, "f_V"'] & G(V).
\end{tikzcd}$$

If F and G are sheaves, then a *morphism of sheaves* is just a morphism of presheaves (that happens to be between sheaves); that is, we take sheaves to be a full subcategory of presheaves.

Example 7. Let $X = \mathbb{R}$ with the usual topology. To any open $U \subset X$ we can associate $\mathcal{O}^{cts}(U) = C^0(U, \mathbb{R})$, or $\mathcal{O}^{diff}(U) = C^\infty(U, \mathbb{C})$, or $\mathcal{O}^{dis}(U) = \mathrm{Hom}_{\mathbf{Set}}(U, \mathbb{R})$, etc. These are all sheaves, which is often proved by invoking the *localness* of continuity or of differentiability or of being a function. That is, one can check any of these properties globally by checking it on any open cover, so when one patches together such things on an open cover one necessarily obtains the same such thing globally.

Example 8. Let X be a topological space and let S be a set. Then the *constant presheaf* $U \mapsto S$ is generally not a sheaf. We usually remedy this by defining the *constant sheaf* to be $U \mapsto C^0(U, S)$ where S has the discrete topology.

Definition 10. Given a continuous map $f : X \to Y$ of topological spaces and a presheaf F on X, we define the *pushforward* presheaf f_*F on Y by setting $(f_*F)(U) = F(f^{-1}(U))$. The restriction maps are determined by the diagram of categories

$$\begin{tikzcd}
Op(X)^{op} \arrow[r, "F"] & \mathbf{Set} \\
Op(Y)^{op}. \arrow[u, "f^{-1}"] \arrow[ur, "f_*F"']
\end{tikzcd}$$

Proposition 3. *If F is a sheaf, then so is f_*F.*

Proof. Let $U \subset Y$ be open, and suppose that $U = \cup_i U_i$. We must check that the diagram

$$(f_*F)(U) \to \prod_i (f_*F)(U_i) \rightrightarrows \prod_{i,j} (f_*F)(U_i \cap U_j)$$

is exact. By definition, this is

$$F(f^{-1}(U)) \to \prod_i F(f^{-1}(U_i)) \rightrightarrows \prod_{i,j} F(f^{-1}(U_i \cap U_j)).$$

But the intersection of a preimage is the preimage of the intersection, so this last term is $\prod_{i,j} F(f^{-1}(U_i) \cap f^{-1}(U_j))$. Now this is exact by the sheaf property for F applied to the cover $f^{-1}(U) = \cup_i f^{-1}(U_i)$. $\square$

We now take a brief detour into the world of adjoint functors.

Definition 11. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then an *adjoint pair* is a triple (F, G, ι) of two functors $F : \mathcal{D} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{D}$ along with a natural transformation $\iota : \mathrm{id}_{\mathcal{D}} \to GF$, such that for all $X \in \mathcal{D}$ and $Y \in \mathcal{C}$, the composition

$$\mathrm{Hom}_{\mathcal{C}}(F(X), Y) \xrightarrow{G} \mathrm{Hom}_{\mathcal{D}}(GF(X), G(Y)) \xrightarrow{\iota_X^*} \mathrm{Hom}_{\mathcal{D}}(X, G(Y))$$

is an isomorphism (i.e. a bijection of sets). We call F a *left adjoint* to G, and we call G a *right adjoint* to F.

Remark 3. Given F (resp. G), the pair (G, ι) (resp. (F, ι)) is unique up to unique isomorphism assuming it exists.

Example 9 (Example/Proposition/Definition)**.** Let X be a topological space and let $x \in X$ be a point. We consider x as a morphism $j : \{*\} \to X$. Now, a sheaf on $\{*\}$ is just an abelian group (or a set, etc.), and we can define the pushforward $j_* : \mathbf{Ab} = Sh_{\mathbf{Ab}}(\{*\}) \to Sh_{\mathbf{Ab}}(X) =: Sh(X)$. Then j_* has a left adjoint $F \mapsto F_x$, called the *stalk* at x. This must satisfy the property that

$$\mathrm{Hom}_{\mathbf{Ab}}(F_x, A) \simeq \mathrm{Hom}_{Sh(X)}(F, j_*X),$$

i.e. we should have an isomorphism between the functors

$$\mathrm{Hom}_{\mathbf{Ab}}(F_x, -), \mathrm{Hom}_{Sh(X)}(F, j_*(-)) : \mathbf{Ab} \to \mathbf{Set}.$$

(This will also have to be functorial in F.) Now, observe that

$$(j_*A)(U) = \begin{cases} A, & x \in U \\ 0, & x \notin U. \end{cases}$$

(For presheaves taking values in $\mathcal{C}$, the evaluation of the pushforward on those open sets not containing x will have to be the terminal object of $\mathcal{C}$.) Thus, a morphism of sheaves $f : F \to j_*A$ is a collection of morphisms $f_U : F(U) \to (j_*A)(U)$ such that if $x \in V \subset U \cap U'$ then the diagram

$$\begin{tikzcd}
F(V) \arrow[r] \arrow[d] & f(U) \arrow[d] \\
F(U') \arrow[r] & A
\end{tikzcd}$$

commutes. That is, for all the open sets containing x we need maps off their sections. The initial object under a diagram is called the *colimit* (or *inverse limit*): we define

$$F_* = \varinjlim_{x \in U \subset X} F(U).$$

By its universal property, this admits maps from all $F(U)$ where $x \in U$. We call its elements "germs of sections" of F: they are represented by pairs (U, s) for $s \in F(U)$, under the equivalence relation that $(U, s) \sim (U', s')$ if there is some $V \subset U \cap U'$ such that $s|_V = s'|_V \in F(V)$.

Proposition 4. *Let X be a topological space, and let $\varphi : F \to G$ be a morphism of sheaves on X. Then φ is an isomorphism iff the induced morphism on stalks $\varphi_x : F_x \to G_x$ is an isomorphism for every point $x \in X$.*

Proposition 5. *Let X be a topological space. Then the inclusion $Sh(X) \hookrightarrow PSh(X)$ has a left adjoint, called* sheafification[1][2]*. We denote this by $F \mapsto F^a$, taking a presheaf F to its* associated sheaf *F^a.*

That is, if F is a presheaf and G is a sheaf then $\mathrm{Hom}_{PSh(X)}(F, G) \simeq \mathrm{Hom}_{Sh(X)}(F^a, G)$. Taking $G = F^a$, we see that a sheafification comes with a morphism $F \to F^a$ which is associated to the identity morphism $\mathrm{id}_{F^a} \in \mathrm{Hom}_{Sh(X)}(F^a, F^a)$. Thus we have

$$\begin{tikzcd}
F \arrow[r] \arrow[dr] & F^a \arrow[d, dashed, "\exists!"] \\
 & G.
\end{tikzcd}$$

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[1] Christ, you know it ain't easy / You know how hard it can be / The way things are going / They're gonna sheafify me
- John Lennon, *The Ballad of John and Yoko (Sheafy Remix)*

[2] I gotta testify, come up in the spot looking extra fly / For the day I die, I'mma sheafify
- Kanye West, *Touch the Sky (Shizzeafy Rizzizzlemizzle)*

Proof/construction. The idea is as follows. The sheafification F^a satisfies two properties:

- $(F^a)_x = F_x$;
- if G is a sheaf then for all $U \subset X$, the morphism $G(U) \to \prod_{x \in U} G_x$ is injective.

This inspires us to make the definition

$$F^a(U) = \{(f^x)_{x \in U} : f^x \in F_x \text{ and } \forall x \in U \exists \text{ open } V \subset U \text{ with } x \in V \text{ and } g \in F(V) \text{ s.t. } g_y = f^y \forall y \in V\}.$$

We might say that the sections of F^a over U are precisely the coherent elements in $\prod_{x \in U} F_x$.

We should check the following things.

1. *F^a is a presheaf.*
   Given $V \subset U$, we define the restriction map $F^a(U) \to F^a(V)$ by $(f^x)_{x \in U} \mapsto (f^x)_{x \in V}$.
2. *F^a is a sheaf.*
   This comes directly from the definition of F^a.
3. *We have a map $F \to F^a$ of presheaves.*
   We define the presheaf morphism $F \to F^a$ by sending $f \in F(U)$ to $(f_x)_{x \in U}$.
4. *For all $x \in X$, $(F^a)_x \cong F_x$.*
   The map $(F^a)_x \to F_x$ is given by $(f^x)_{x \in U} \mapsto f^x$. We can check that this is an isomorphism by examining the composition $F_x \to (F^a)_x \to F_x$.
5. *If F is already a sheaf then $F \to F^a$ is an isomorphism.*
   This is true by the previous thing, since (as we didn't prove) a morphism of sheaves is an isomorphism iff it is an isomorphism on stalks.
6. *The association $F \to F^a$ is a functor.*
   This is clear from the definition.

Because of the last thing, a morphism $F \to G$ of presheaves induces

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & F^a \arrow[d] \\
G \arrow[r] & G^a.
\end{tikzcd}$$

So if G is a sheaf, then by the previous thing we get our map $F^a \to G$. This is unique since for every $x \in X$ the map $(F^a)_x \to G_x$ is uniquely determined via the diagram

$$\begin{tikzcd}
F_x \arrow[r, "\sim"] \arrow[dr] & (F^a)_x \arrow[d] \\
 & G_x,
\end{tikzcd}$$

and a map of sheaves is uniquely determined by the induced maps on stalks. $\square$

This is very convenient, because even though sheaves are better behaved, often it's easier to make explicit constructions on the level of presheaves.

Example 10. We can use sheafification to show that cokernels exist in the category of abelian sheaves on X. What this means is that given a morphism $F \to G$, whenever we have a morphism $G \to H$ such that the composition $F \to G \to H$ is zero, then we have a *cokernel*, denoted G/F, which satisfies the universal property

$$\begin{tikzcd}
F \arrow[r] \arrow[dr, "0"'] & G \arrow[r] \arrow[d] & G/F \arrow[dl, dashed, "\exists!"] \\
 & H. &
\end{tikzcd}$$

We first obtain this result in the category of abelian presheaves. It turns out that we can simply define $G/^{ps}F$ by $(G/^{ps}F)(U) = \mathrm{Coker}(F(U) \to G(U))$, with restriction maps given by the universal property of cokernels, and then this has exactly the same universal property in this larger category of presheaves.

However, this construction ruins the sheaf property: even if F and G are sheaves, $G/^{ps}F$ need not be a sheaf. Note however that we can simply define $G/F = (G/^{ps}F)^a$, since

$$\begin{aligned}&\mathrm{Hom}_{Sh(X)}((G/^{ps}F)^a, H) = \mathrm{Hom}_{PSh(X)}(G/^{ps}F, H)\\&= \mathrm{Ker}(\mathrm{Hom}_{PSh(X)}(G,H) \to \mathrm{Hom}_{PSh(X)}(F,H))\\&= \mathrm{Ker}(\mathrm{Hom}_{Sh(X)}(G,H) \to \mathrm{Hom}_{Sh(X)}(F,H))\end{aligned}$$

(because $Sh(X) \subset PSh(X)$ is a full subcategory).

Example 11. Let us consider an actual example. Consider the 2-to-1 cover $\pi : X = S^1 \to Y = S^1$ (defined on $S^1 \subset \mathbb{C}$ by $z \mapsto z^2$). Define the sheaf of sections on Y via $F(U) = \{s : U \to X \text{ s.t. } \pi \circ s = \mathrm{id}_U\}$. Such a map s is precisely a choice of $\sqrt{z}$ over U. Of course, if there's one choice then there are two choices: there's a $\mathbb{Z}/2$-action on this sheaf, and every set of sections has either 0 or 2 elements. We define the presheaf $G(U) = F(U)/\mathbb{Z}/2$ is

$$U \mapsto \begin{cases} *, & \exists\sqrt{z} \text{ over } U \\ \emptyset, & \text{otherwise.}\end{cases}$$

Now, G^a is the constant sheaf associated to the singleton set $*$. Then, $F(S^1) \to G^a(S^1)$ is *not* surjective, despite being an epimorphism in the category of sheaves on X.

Proposition 6. *Let $f : X \to Y$ be a continuous map of topological spaces. Then:*

1. $f_* : PSh(X) \to PSh(Y)$ *has a left adjoint* $\hat{f}^{-1} : PSh(Y) \to PSh(X)$*, called* (presheaf) pullback.
2. $f_* : Sh(X) \to Sh(Y)$ *has a left adjoint* $f^{-1} : Sh_{\mathbf{Ab}}(Y) \to Sh_{\mathbf{Ab}}(X)$*, called* pullback.

Taking the stalk is the special case where $X = \{*\}$.

Proof. First observe that the first claim implies the second. This is thanks to the sheafification functor, the left adjoint to the forgetful functor from sheaves to presheaves. Explicitly,

$$\mathrm{Hom}_{Sh(X)}((\hat{f}^{-1}F)^a, G) = \mathrm{Hom}_{PSh(X)}(\hat{f}^{-1}F, G) = \mathrm{Hom}_{PSh(Y)}(F, f_*G) = \mathrm{Hom}_{Sh(Y)}(F, f_*G).$$

So it only remains to prove the first claim. Suppose that $F \in PSh(Y)$. We then define the presheaf pullback by

$$(\hat{f}^{-1}F)(U) = \varinjlim_{\{V \subset Y : f(U) \subset V\}} F(V).$$

If $U' \subset U$, then we get the map

$$\varinjlim_{\{V \subset Y : f(U) \subset V\}} F(V) \longrightarrow \varinjlim_{\{V \subset Y : f(U') \subset V\}} F(V)$$

since $f(U') \subset f(U)$ so if $f(U) \subset V$ then $f(U') \subset V$. Note that there is a distinguished map

$$F \to f_*\hat{f}^{-1}F$$

selected by $\mathrm{id}_{\hat{f}^{-1}F}\mathrm{Hom}_{PSh(X)}(\hat{f}^{-1}F, \hat{f}^{-1}F) = \mathrm{Hom}_{PSh(X)}(F, f_*\hat{f}^{-1}F)$. This map is determined by

$$F(V) \to (\hat{f}^{-1}F)(f^{-1}(V)) = \varinjlim_{\{W : f(f^{-1}(V)) \subset W\}} F(W).$$

Therefore, given $F \to f_*G$ for some G, we get a map

$$\hat{f}^{-1}F(U) = \varinjlim_{\{V \subset Y : f(U) \subset V\}} F(V) \longrightarrow \varinjlim_{\{V \subset Y : f(U) \subset V\}} (f_*G)(V)$$

via the map

$$G(U) \overset{\theta_{f^{-1}(V),U}}{\longleftarrow} \varinjlim_{\{V \subset Y : f(U) \subset V\}} G(f^{-1}(V)).$$

Conversely, given $\hat{f}^{-1}F \to G$ we apply f_* to get $F \to f_*\hat{f}^{-1}F \to f_*G$. These define inverse bijections $\mathrm{Hom}_{PSh(Y)}(F, f_*G) \leftrightarrows \mathrm{Hom}_{PSh(X)}(\hat{f}^{-1}F, G)$. ☐

There are two special cases to think about:

1. When $X = *$, then $(\hat{f}^{-1}F)$ is the stalk at $y = f(*)$.
2. When $Y = *$, then $(f_*G) = G(X)$, the global sections.

To summarize, we have f_*, f^{-1}, $\hat{f}^{-1}$, and the sheaf associated to a presheaf. For a morphism $f : F \to G$ of abelian sheaves on a topological space X, we can also define the following operations. (Indeed, the category of abelian sheaves on X is an *abelian category.*)

- The *sheaf kernel* is simply the presheaf kernel $U \mapsto \operatorname{Ker}(F(U) \to G(U))$.
- The *sheaf cokernel* is the sheafification of the presheaf cokernel $U \mapsto \operatorname{Coker}(F(U) \to G(U))$, i.e. $(U \mapsto \operatorname{Coker}(F(U) \to G(U)))^a$.
- There are a few possibilities for the *image sheaf.* We could take $(U \mapsto \operatorname{Im}(F(U) \to G(U)))^a$ or we could take $\operatorname{Ker}(G \to \operatorname{Coker}(f))$ (the sheaf cokernel). As it turns out, these are the same thing.

4 Schemes

4.1 The underlying set of a scheme

Definition 12. Let A be a ring. Then *spectrum* of A, denoted Spec A, is a topological space whose underlying space is the set of prime ideals $\mathfrak{p} \subset A$. (We will define the topology later.)

The motivation here is that taking the prime spectrum defines a functor Spec : $\mathbf{Rings}^{op} \to \mathbf{LocallyRingedSpaces}$. This will end up being fully faithful, but there are a lot of things we need to check first.

Example 12. Let k be a field, let V be a finite-dimensional k-vector space, and let $A : V \to V$ be a linear map. Then we get $k[T] \to \operatorname{End}_k(V)$ via $T \mapsto A$. (Once we choose a basis, the $\operatorname{End}_k(V) \simeq M_n(k)$.) Note that $k[T]$ is a PID, so the kernel is of the form (P) for some $P \in k[T]$. Thus $R = k[T]/(P)$. This P is the *minimal polynomial* for A (which we'll take by definition to be monic). Then, Spec R is precisely the spectrum of A in the sense of linear algebra (namely, its set of eigenvalues).

Let us unwind this. Suppose $P \in k[T]$ is any polynomial. The prime ideals in $k[T]/(P)$ are determined exactly by prime ideals in $k[T]$ which contain P. Even more explicitly, suppose that we have the decomposition $P = P_1^{e_1} \cdots P_r^{e_r}$ into irreducible factors. By the Chinese remainder theorem,

$$k[T]/(P) \simeq (k[T]/p_1^{e_1}) \times \cdots \times (k[T]/p_r^{e_r}).$$

Then, prime ideals are exactly kernels of maps $k[T]/(P) \to k[T]/(P)$. For example, if $k = \overline{k}$ and $P = (T - \lambda_1) \cdots (T - \lambda_r)$ (where the λ_i are the eigenvalues of A) then the set Spec A is in bijection with $\{\lambda_1, \ldots, \lambda_r\}$ (not counting multiplicities).

Example 13. Let $A = \mathbb{C}[T]$. Then Spec A has the subset of maximal ideals, which since $\mathbb{C} = \overline{\mathbb{C}}$ all take the form $(T - a)$ for $a \in \mathbb{C}$. Thus, the maximal ideals correspond to the affine line $\mathbb{A}^1_{\mathbb{C}}$ (over $\mathbb{C}$). But there's one more point in Spec A: the one associated to the prime ideal (0). This is called the *generic point.*

Example 14. Let $A = \mathbb{Z}$. (This is very similar to the previous example, since both are Dedekind domains.) Here, maximal ideals correspond to prime numbers (via $p \mapsto (p)$). The other prime ideal is (0).

4.2 The topology

The *Zariski topology* (which has yet to be defined) will reduce for the examples $A = \mathbb{C}[T]$ and $A = \mathbb{Z}$ to the "complements of finite sets of maximal ideals are open" topology. It is easy to see in this examples that this does indeed define a topology. Note for example that every open set is *quasi-compact* (i.e. every open cover has a finite subcover, absent of a Hausdorffity). Note that on Spec $\mathbb{C}[T] \cong \mathbb{A}^1_{\mathbb{C}}$, this is very different from the *analytic topology* on $\mathbb{C}[T]$. Note also that in both examples, all nonempty open subsets contain the generic point.

Example 15. Suppose A is a discrete valuation ring (e.g. the p-adics $\mathbb{Z}_p$, the power series ring $k[[T]]$). Then Spec A consists of two points, $m = (0)$ and $s = \eta$. Here, s is a closed point ant η is the generic point. So the open sets are $\{m, \eta\} =$ Spec A, $\{m\}$, and $\emptyset$.

Remark 4 (Functoriality; or, "Why primes and not just maximal ideals")**.** If $f : A \to B$ is a ring homomorphism and $\mathfrak{p} \subset B$ is prime, then $f^{-1}(\mathfrak{p}) \subset A$ is also prime. Thus we get a map Spec $B \to$ Spec A. This is false when we only consider maximal ideals.

If we think of a prime ideal $\mathfrak{p} \subset B$ as a surjective homomorphism $B \to B/\mathfrak{p}$ to an integral domain, the case where $\mathfrak{p}$ is exactly when $B/\mathfrak{p}$ is a field. But there's no reason that the composition $A \to B \to B/\mathfrak{p}$ should have the image of A be a field. (For example, take $A = \mathbb{Z}$, $B = \mathbb{Q}$, and $\mathfrak{p} = (0) \subset B$.)

(Incidentally, a subring of an integral domain is an integral domain, which proves that the map from A to its image in $B/\mathfrak{p}$ is a surjective homomorphism to an integral domain.)

Definition 13. Let A be a ring, and suppose that $\mathfrak{a} \subset A$ is an ideal. Define $V(\mathfrak{a}) = \{\mathfrak{p} : \mathfrak{a} \subset \mathfrak{p}\} \subset$ Spec A. The *Zariski topology* is defined by declaring these sets to be closed.

Example 16. Let us return to $A = \mathbb{C}[T]$. Fix some $a \in A$. It is very important for intuition to think carefully about the map $ev_a : \mathbb{C}[T] \to \mathbb{C}[T]/(T-a) \cong \mathbb{C}$. This is called the *evaluation* map: it takes $F(T)$ to $F(a)$. (Concretely, if $F(T) = \sum_{i=0}^{N} \alpha_i T^i$, then $F(a) = \sum_{i=0}^{N} \alpha_i \cdot a^i$.) In fact, the isomorphism $\mathbb{C}[T]/(T-a) \simeq \mathbb{C}$ is characterized by the requirement that

$$\begin{tikzcd}
\mathbb{C} \arrow[d] \arrow[dr] & \\
\mathbb{C}[T] \arrow[r, "ev_a"'] & \mathbb{C}[T]/(T-a)
\end{tikzcd}$$

be a diagram of $\mathbb{C}$-algebras.

What does this have to do with the Zariski topology? Well, suppose that we have a locus $\{z : F(z) = 0\} \subset \mathbb{C}$ for some $F \in \mathbb{C}[T]$. This locus corresponds to the set of prime ideals $\mathfrak{p} \subset \mathbb{C}[T]$ such that $F \in \mathfrak{p}$, i.e. those such that $ev_a(F) = 0$.

Example 17. Let $A = k[X_1, \ldots, X_n] = k[\underline{X}]$. Given a collection $f_1, \ldots, f_r \in k[\underline{X}]$, we define $V(f_1, \ldots, f_r)$ to be the subset of exactly those prime ideals $\mathfrak{p} \subset k[\underline{X}]$ containing $(f_1, \ldots, f_r)$. This corresponds exactly to the (joint) zero locus of the polynomials.

Lemma 2. *The Zariski topology is a topology. Explicitly:*

1. *If $\mathfrak{a}, \mathfrak{b} \subset A$ are ideals, then $V(\mathfrak{a} \cdot \mathfrak{b}) = V(a) \cup V(b)$.*
2. *If $\{\mathfrak{a}_i\}$ is an arbitrary set of ideals of A, then $V(\sum \mathfrak{a}_i) = \cap V(\mathfrak{a}_i)$.*
3. *If $\mathfrak{a}, \mathfrak{b} \subset A$ are ideals, then $V(a) \subset V(b)$ iff $\sqrt{\mathfrak{a}} \supset \sqrt{\mathfrak{b}}$.*

(Note that the unit ideal is not considered prime.)

4.3 A convenient open cover

Definition 14. We have *basic open sets* which will sometimes be convenient to work with. Given $f \in A$, define $D(f) = \{\mathfrak{p} \subset A : f \notin \mathfrak{p}\} = V((f))^c \subset$ Spec A.

Example 18. When is it true that $D(f_1) \cup \cdots \cup D(f_r) =$ Spec A? Well, this union is exactly the set of prime ideals $\mathfrak{p}$ such that at least one of the f_i lies outside $\mathfrak{p}$. So, its complement is the set of prime ideals $\mathfrak{p}$ such that $f_1, \ldots, f_r \in \mathfrak{p}$. So its complement is empty when the ideal $(f_1, \ldots, f_r)$ is not contained in any prime ideal, which by basic algebra means that this ideal is the entire ring, a/k/a the unit ideal.

Example 19. The same argument shows that $\cup_{i \in I} D(f_1) =$ Spec A iff $(f_i)_{i \in I} = A$ (even if the indexing set I is infinite). But this means that the ideal (f_i) is the unit ideal, meaning that we have a finite expression $1 = \alpha_1 f_{i_1} + \cdots + \alpha_s f_{i_s}$. Thus, already $D(f_{i_1}) \cup \cdots \cup D(f_{i_s}) =$ Spec A. That is, every open cover of Spec A (by basic opens) has a finite subcover.

Remark 5. We think of $D(f) = \{\mathfrak{p} \subset A : f \notin \mathfrak{p}\}$, the set of prime ideals that **don't** contain f. By the homework, this is in bijection with Spec A_f, and this is compatible with the topologies on the latter two sets.

Definition 15. Let X be a topological space. A *base* for the topology on X is a collection of open sets $\mathcal{B} = \{B_i\}_{i \in I}$ such that every open set $U \subset X$ can be covered by elements of $\mathcal{B}$ (i.e. $U = \cup_{j \in J} B_j$ for some $J \subset I$).

Example 20. In metric space theory, one generally takes open balls as a base for the topology.

Proposition 7. *The basic open sets form a base for the Zariski topology on* Spec A.

Proof. Consider the open set $V(\mathfrak{a})^c \subset \operatorname{Spec} A$. The left side is equal to the union of basic opens on any generating set for $\mathfrak{a}$ (e.g. the entire ideal). ☐

This is a very nice base, because we're super stoked about localization of rings and modules. From here on out, we'll start defining sheaves only on the base of basic opens for the topology.

Intuition 1. If M is an A-module, we will want to say that $(f) \mapsto M_f$ defines a sheaf on Spec A. In particular, taking $M = A$ we should get a sheaf of rings, usually denoted $\mathcal{O}$.

Let X be a topological space and let $\mathcal{B}$ be a base for its topology. Then $\mathcal{B} \subset Op(X)$. Thus a sheaf $\mathcal{F}$ on X, which is just a special functor, defines a functor

$$\begin{tikzcd}
\mathcal{B}^{op} \arrow[r, "\mathcal{F}_B"] \arrow[d, hook] & \textbf{Set}. \\
Op(X)^{op} \arrow[ur, "\mathcal{F}"'] &
\end{tikzcd}$$

Now, suppose that $U \subset X$ is open. Note that we can always write $B_i \cap B_j = \cup_k B_{ijk}$ for some $B_{ijk} \in \mathcal{B}$. We obtain an injection

$$\prod_{i,j} \mathcal{F}(B_i \cap B_j) \hookrightarrow \prod_{i,j} \prod_k \mathcal{F}_{\mathcal{B}}(B_{ijk}),$$

and thus

$$\text{Equalizer}\left(\prod_i \mathcal{F}_{\mathcal{B}}(B_i) \rightrightarrows \prod_{i,j} \mathcal{F}(B_i \cap B_j)\right) \cong \text{Equalizer}\left(\prod_i \mathcal{F}_{\mathcal{B}}(B_i) \rightrightarrows \prod_{i,j} \mathcal{F}(B_i \cap B_j) \rightrightarrows \prod_{i,j} \prod_k \mathcal{F}_{\mathcal{B}}(B_{ijk})\right),$$

and so we can happily define $\mathcal{F}(U)$ as their shared value. So, we can recover a sheaf from its restriction to a base.

Definition 16. *A sheaf on the base* $\mathcal{B}$ is a functor $G : \mathcal{B}^{op} \to \textbf{Set}$ such that whenever $B \in \mathcal{B}$, $B = \cup_i B_i$ for some $B_i \in \mathcal{B}$, and $B_i \cap B_j = \cup_k B_{ijk}$ for some $B_{ijk} \in \mathcal{B}$, then

$$G(B) \to \prod_i G(B_i) \rightrightarrows \prod_{i,j} \prod_j G(B_{ijk})$$

is an equalizer diagram. A *morphism* of sheaves on the base $\mathcal{B}$ is just a morphism of functors (i.e. a natural transformation).

Theorem 1. *The functor*

$$(\text{sheaves on } X) \to (\text{sheaves on } \mathcal{B})$$

given by $\mathcal{F} \mapsto \mathcal{F}_{\mathcal{B}}$ *is an equivalence of categories.*

The easiest way to prove this is to redefine $\mathcal{F}(U)$ be the limit over *all* possible coverings $U = \cup_i B_i$ of the corner $\prod_i \mathcal{F}_{\mathcal{B}}(B_i) \to \prod_{i,j} \prod_k \mathcal{F}_{\mathcal{B}}((B_{ijk})) \leftarrow \prod_i \mathcal{F}_{\mathcal{B}}(B_i)$.

4.4 The structure sheaf

Definition 17. Let $\mathcal{B}$ be the base of basic opens on Spec A. We then define the sheaf $\mathcal{O}$ on $\mathcal{B}$ by $\mathcal{O}(D(f)) = A_f$.

We must check that this even defines a functor. Suppose that $D(g) \subset D(f)$. We then have the diagram of localizations

$$\begin{tikzcd} A \arrow[d] \arrow[dr] & \\ A_f \arrow[r, dashed] & A_g. \end{tikzcd}$$

To obtain a unique such dashed arrow, we need that f maps to a unit in A_g. This is true because if not, then there would be a map from A_g to an integral domain with f in the kernel. But this is the same as saying that

$$\{\mathfrak{p} \subset A : g \notin \mathfrak{p},\ f \in \mathfrak{p}\} = D(g) \backslash D(g) \cap D(f).$$

Moreover, we must check the sheaf axiom. We write this as a proposition.

Proposition 8. *Let A be a ring and let M be an A-module. Suppose that $f \in A$, and $\{h_i\}_{i \in I}$ are elements are A such that $D(f) = \cup_i D(h_i)$. Then the sequence*

$$M_f \to \prod_i M_{h_i} \rightrightarrows \prod_{i,j} M_{h_i h_j}$$

is exact.

(Note that $D(a) \cap D(b) = D(ab)$.) This construction is nice, because it illustrates that the sheaf $\widetilde{M}$ that we obtain has $\widetilde{M}(D(f)) = M_f$; otherwise, if we simply appealed to sheafification, we wouldn't necessarily have a good handle what the sheaf looks like locally.

Proof. We make the following reductions:

1. We can assume that $f = 1$, so that $D(f) = \text{Spec } A$. Namely, we can replace A by A_f, M by M_f, and the h_i by their images in A_f. (By what we have just seen, since $D(h_i) \subset D(f)$ then we get a unique map $A_f \to A_{h_i}$ of A-algebras. If we let $\overline{h_i} \in A_f$ be the image of h_i, then $A_{h_i} \cong (A_f)_{\overline{h_i}}$ and $M_{h_i} \cong (M_f)_{\overline{h_i}}$, and these isomorphisms are compatible in such a way that our diagram that we must show is an equalizer just becomes $M_f \to \prod_i M_{h_i} \rightrightarrows \prod_{i,j} M_{h_i h_j}$.)

2. We can assume that $\{h_i\}_{i \in I}$ is finite. Indeed, suppose the result holds for a finite indexing set, and consider any collection $\{h_i\}_{i \in I}$. Then there exists a finite subset $J \subset I$ such that $\text{Spec } A = \cup_{j \in J} D(h_j)$ (as Spec A is quasi-compact). Then we have the map of diagrams

$$\begin{tikzcd} M \arrow[r] \arrow[dr, hook] & \prod_i M_{h_i} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d] & \prod_{i,s} M_{h_i h_s} \arrow[d] \\ & \prod_{j \in J} M_{h_j} \arrow[r, shift left] \arrow[r, shift right] & \prod_{j,j' \in J} M_{h_j h_{j'}}. \end{tikzcd}$$

   First, this implies that the first horizontal map is injective. Then given $(m_i)_{i \in I}$ with $m_i \in M_{h_i}$ such that $p_1((m_i)) = p_2((m_i))$, then there is some $m \in M$ such that the image of m in M_{h_j} is equal to m_j for all $j \in J$. In fact, we claim that for all i, the image of m in M_{h_i} is equal to m_i. Indeed, the map $M_{h_i} \to \prod_{j \in J} M_{h_i h_j}$ is injective (by the first observation applied to A_{h_i} and the fact that $D(h_i) = \text{Spec } R_{h_i} = \cup_{j \in J}(D(h_i) \cap D(h_j)) = \cup_{j \in J} D(h_i h_j)$). So if $m, m_i \in M_{h_i}$ then we can map these forward to the product, $m \mapsto m$ and $m_i \mapsto (m_i)_j = (m_j)_j$ (since these agree under p_1 and p_2).

Thus, we can and will assume that $A = A_f$ and $\{h_i\}_{i \in I} = \{h_1, \ldots, h_r\}$. Note that since $(h_1, \ldots, h_r) = (1)$, then also $(h_1^{n_1}, \ldots, h_r^{n_r}) = (1)$ for any choices $n_i \geq 1$ as well; geometrically, this is because $D(h_i) = D(h_i^n)$ for all $n \geq 1$. Note also that $M_{h_i} \cong M_{{h_i}^{n_i}}$, because declaring h_i to be a unit is equivalent to declaring $h_i^{n_i}$ to be a unit.

We first check injectivity of $M \to \prod_i M_{h_i}$. So suppose that $m \in M$ and $m \mapsto 0 \in M_{h_i}$ for all i. Observe that

$$\ker(M \to M_{h_i}) = \{m \in M : \exists n_i \geq 1 \text{ s.t. } h_i^{n_i} m = 0\}.$$

So we can choose such n_i, and set $N = \max\{n_i\}$ (using the fact that I is finite). Then write $1 = \sum_i \alpha_i h_i^N$; then $m = \sum_i \alpha_i h_i^N \cdot m = 0$. Thus the map is indeed injective.

Finally, we check exactness in the middle. Consider elements $m_i \in M_{h_i}$ such that for all pairs (i, j), m_i and m_j have the same image in $M_{h_i h_j}$. Write $m_i = a_i/h_i^{s_i}$ for $a_i \in M_{h_i}$ and $s_i \geq 1$. (If we started with an element m_i with no denominator, we can just multiply by h_i/h_i.) We replace h_i by $h_i^{s_i}$, and then we can assume that $s_i = 1$ and hence $m_i = a_i/h_i$. Now, saying that m_i and m_j have the same image in $M_{h_i h_j}$ is saying that there is some N_{ij} such that $(h_i h_j)^{N_{ij}}(h_j a_i - h_i a_j) = 0$. Then again since we have a finite number of indices we can take $N = \max\{N_{ij}\}$, and this value will work for all pairs (i, j). Thus, for all pairs (i, j),

$$h_j^{N+1} h_i^N a_i = h_i^{N+1} h_j^N a_j.$$

Now for all $i, j \in I$ we replace h_i by h_i^{N+1} and a_i by $h_i^N a_i$. This simplifies the equation to $h_j a_i = h_i a_j$. We now write $1 = \sum_i \alpha_i h_i$, and let $m = \sum_i \alpha_i a_i \in M$. Then $h_j m = \sum_i \alpha_i h_j a_i = \sum_i \alpha_i h_i a_j = (\sum_i \alpha_i h_i) \cdot a_j = 1 \cdot a_j = a_j$. So, m maps to $m_j = a_j/h_j$ under the localization $M \to M_{h_j}$. ☐

More generally, if B is an A-module, then the sequence

$$B_f \to \prod_i B_{h_i} \rightrightarrows \prod_{i,j} B_{h_i h_k}$$

is exact. In particular, taking $A = B$, we get a sheaf of rings $\mathcal{O}$ on on Spec A, called the *structure sheaf.*

Definition 18. A *ringed space* is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X. A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that for all $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring. (Recall that a ring is called *local* if it has a unique maximal ideal, e.g. a field k, a power series ring $k[[X_1, \ldots, X_d]]$, et al.)

Lemma 3. (Spec $A, \mathcal{O}$) = (Spec $A, \mathcal{O}_{\text{Spec } A}$) *is a locally ringed space.*

Proof. Let $\mathfrak{p} \subset A$ be a prime ideal, i.e. a point of Spec A. The stalk of the structure sheaf at this point is

$$\mathcal{O}_{\text{Spec } A, \mathfrak{p}} = \varinjlim_{\mathfrak{p} \in U \subset \text{Spec } A} \mathfrak{O}_{\text{Spec } A}(U) = \varinjlim_{\mathfrak{p} \in D(f) \subset \text{Spec } A} \mathfrak{O}_{\text{Spec } A}(U) = \varinjlim_{f \notin \mathfrak{p}} \mathcal{O}_{\text{Spec } A}(D(f)) = \varinjlim_{f \notin \mathfrak{p}} A_f = A_{\mathfrak{p}}.$$

This is a local ring, the localization "away from $\mathfrak{p}$"; its maximal ideal is $\mathfrak{p}A_{\mathfrak{p}}$. ☐

Definition 19. A *morphism of locally ringed spaces* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$, where $f : X \to Y$ is a continuous map of topological spaces and a morphism $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves on Y such that for all $x \in X$ the induced map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism of rings.

Remark 6. Given an open set $U \subset Y$, we should think of $\mathcal{O}_Y(U) \to f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U))$ as given by precomposing functions (sections) that are defined over U.

Remark 7. We can also think of $f^\#$ as a map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ since f^{-1} and f_* are adjoint functors. We will generally not use different notation for this map.

Remark 8. The map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is described as follows. We consider the point x as a map $j : * \to X$, and then the point $f(x)$ is picked out by the composition $fj : * \to Y$. Then we have

$$\mathcal{O}_{Y,f(x)} = (fj)^{-1}\mathcal{O}_Y = j^{-1}f^{-1}\mathcal{O}_Y \overset{f^\#}{\to} j^{-1}\mathcal{O}_X = \mathcal{O}_{X,x}.$$

More directly, we can say that for every $U \subset Y$ containing $f(x)$, we get a map $\mathcal{O}_Y(U) \overset{f^\#}{\to} \mathcal{O}_X(f^{-1}(U)) \to \mathcal{O}_{X,x}$, and taking the colimit over all such U gives us the map on stalks.

Remark 9. A morphism $\varphi : A \to B$ is called *local* if $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$. (In general, the preimage of a maximal ideal only must be prime.)

Remark 10. A morphism of ringed spaces is exactly the same except there's of course no locality condition on the morphisms on stalks. However, locally ringed spaces is *not* a full subcategory of ringed spaces, because we have fewer morphisms.

This rather artificial-looking definition will be justified by the following fact.

Theorem 2. Spec *defines a fully faithful functor* $\mathbf{Rings}^{op} \to \mathbf{LocallyRingedSpaces}$.

Example 21. Consider the standard inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$. Applying Spec gives a morphism Spec $\mathbb{Q} \to$ Spec $\mathbb{Z}$. The source Spec $\mathbb{Q}$ has one point, associated to (0), and the target Spec $\mathbb{Z}$ has the point associated to (0) and all the points associated to (p). We compute the stalks

$$\begin{aligned}\mathcal{O}_{\text{Spec } \mathbb{Q},(0)} &= \mathbb{Q}\\ \mathcal{O}_{\text{Spec } \mathbb{Z},(0)} &= \mathbb{Q}\\ \mathcal{O}_{\text{Spec } \mathbb{Z},(p)} &= \mathbb{Z}_{(p)}\end{aligned}$$

which illustrates that the image of the unique point of Spec $\mathbb{Q}$ must be $(0) \in$ Spec $\mathbb{Z}$. (We can also compute it explicitly.)

Example 22. Let us look at maps Spec $\mathbb{Q} \to$ Spec $\mathbb{Z}$ in the category of ringed spaces instead of locally ringed spaces (which, as we'll see, is the wrong thing to do!). Recall that Spec $\mathbb{Q} = (\{*\}, \mathbb{Q})$ and that Spec $\mathbb{Z} = (\{(0),(p)\}, \mathcal{O}_{\text{Spec } \mathbb{Z}})$. Let's attempt to define the map on topological spaces by sending $*$ to $(59) \in$ Spec $\mathbb{Z}$. The map on sheaves $f^{\#} : f^{-1}\mathcal{O}_{\text{Spec } \mathbb{Z}}$ must be a ring homomorphism $\mathbb{Z}_{(59)} \to \mathbb{Q}$, and there is a unique choice. So we've done it! Thus, in the category of ringed spaces, there is one morphism Spec $\mathbb{Q} \to$ Spec $\mathbb{Z}$ for each point of Spec $\mathbb{Z}$. But this is *bad*, because we want a unique map since there's exactly one ring homomorphism $\mathbb{Q} \leftarrow \mathbb{Z}$. The problem is that our map on stalks $\mathbb{Z}_{(59)} \to \mathbb{Q}$ is *not* a local homomorphism; the only point of Spec $\mathbb{Z}$ that doesn't have the same issue is $(0) \in$ Spec $\mathbb{Z}$. This illustrates why we don't want **LocallyRingedSpaces** to form a full subcategory of **RingedSpaces**.

Proof of theorem. We have the functors

$$\text{Spec} : \text{Hom}_{\mathbf{Rings}}(A, B) \leftrightarrows \text{Hom}_{\mathbf{LRS}}(\text{Spec } B, \text{Spec } A) : \Gamma.$$

Here, Γ is the *global sections* functor: $(f, f^{\#}) : \text{Spec } B, \text{Spec } A$ gets sent to

$$f^{\#} : A = \mathcal{O}_{\text{Spec } A}(\text{Spec } A) \to f_*\mathcal{O}_{\text{Spec } B}(\text{Spec } A) = \mathcal{O}_{\text{Spec } B}(\text{Spec } B) = B.$$

We first observe that $\Gamma \circ \text{Spec} = \text{id}_{\mathbf{Rings}}$. Indeed, given $\varphi : A \to B$ we get the map on topological spaces $\text{Spec}(\varphi) : \text{Spec } B \to \text{Spec } A$ by sending $\mathfrak{p} \subset B$ to $\varphi^{-1}(\mathfrak{p})$. We can check that this is continuous on a base, and indeed

$$\text{Spec}(\varphi)^{-1}(D(f)) = \text{Spec}(\varphi)^{-1}(D(f)) = \{\mathfrak{p} \subset B : f \notin \varphi^{-1}(\mathfrak{p})\} = \{\mathfrak{p} \subset B : \varphi(f) \notin \mathfrak{p} = D(\varphi(f)).$$

We define the map $\mathcal{O}_{\text{Spec } A} \to \text{Spec}(\varphi)_*\mathcal{O}_{\text{Spec } B}$ on basic opens as follows. Note that

$$\begin{aligned}&\mathcal{O}_{\text{Spec } A}(D(f)) = A_f\\ &(\text{Spec}(\varphi)_*\mathcal{O}_{\text{Spec } B})(D(f)) = \mathcal{O}_{\text{Spec } B}(D(\varphi(f))) = B_{\varphi(f)},\end{aligned}$$

so we take the map to be the unique map filling in the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ A_f \arrow[r, dashed, "\exists!"'] & B_{\varphi(f)}. \end{tikzcd}$$

(On fractions, this is $a/f^s \mapsto \varphi(a)/\varphi(f)^s$.)

Now suppose that $(f, f^{\#}) : \text{Spec } B \to \text{Spec } A$ is a morphism of locally ringed spaces. Let $\gamma : A \to B$ denote $\Gamma(f, f^{\#})$, and let $\mathfrak{p} \subset B$ be a prime. Then

$$\begin{tikzcd} A \arrow[r, "\gamma"] \arrow[d, "l_A"'] & B \arrow[d, "l_B"] \\ A_{f(\mathfrak{p})} \arrow[r, "f_{\mathfrak{p}}"'] & B_{\mathfrak{p}} \end{tikzcd}$$

commutes. Thus

$$\begin{aligned} f(\mathfrak{p}) &= l_A^{-1}(\mathfrak{m}_{A_{f(\mathfrak{p})}}) \\ &= l_A^{-1} f_{\mathfrak{p}}^{-1}(\mathfrak{m}_{B_{\mathfrak{p}}}) \\ &= \gamma^{-1} l_B^{-1}(\mathfrak{m}_{B_{\mathfrak{p}}}) \\ &= \gamma^{-1}(\mathfrak{p}). \end{aligned}$$

So it only remains to see that the maps of sheaves of rings coincide. But both maps are characterized by the property that they uniquely fill in the diagram of sheaves

$$\begin{tikzcd} \underline{A} \arrow[r, "\gamma"] \arrow[d] & \underline{B} \arrow[d] \\ \mathcal{O}_{\mathrm{Spec}\ A} \arrow[r, dashed] & f_*\mathcal{O}_{\mathrm{Spec}\ B} \end{tikzcd}$$

(where the vertical morphisms are determined on basic opens by the localization maps), so they must be equal. ☐

Example 23. Let us look at Spec $\mathbb{Z}$. For any $n \in \mathbb{Z}$, we have $\mathcal{O}(D(n)) = \mathbb{Z}[1/n]$, the rational numbers that only have powers of n in their denominator. Taking the limit as n gets more divisible, we end up with $\mathbb{Q} = \varinjlim_n \mathbb{Z}[1/n]$.

Example 24. Let us consider Spec $\mathbb{Z}/(60)$. Since $60 = 2^2 \cdot 3 \cdot 5$, there are the three points (2), (3), and (5). The stalks are $(\mathbb{Z}/(60))_{(2)} = \mathbb{Z}/(4)$, $(\mathbb{Z}/(60))_{(3)} = \mathbb{Z}/(3)$, and $(\mathbb{Z}/(60))_{(5)} = \mathbb{Z}/(5)$ respectively. This subset has the discrete topology, and so the sheaf is entirely determined by these stalks.

Now if we look at the ring $\mathbb{Z}/(4) \times \mathbb{Z}/(3) \times \mathbb{Z}/(5)$, we get the same locally ringed space, and so there is an isomorphism $\mathbb{Z}/(60) \cong \mathbb{Z}/(4) \times \mathbb{Z}/(3) \times \mathbb{Z}/(5)$. This is of course just the Chinese remainder theorem, but in fact this is quite a useful technique in general. And even in this example, if we write Spec $A = \cup D(f_i)$, then $A \to \prod_i A_{f_i} \rightrightarrows \prod_{i,j} A_{f_i f_j}$ is exact. This is fancier version of the Chinese remainder theorem: if things agree when we pass to localization, then they can be glued back together.

Example 25. Let k be a field, and consider Spec $k[x,y]/(y^2 = x^3)$. If we draw this over $\mathbb{R}$, it's got a cusp at $(x,y) = (0,0)$. We might imagine that we could straighten this out into a line. In the world of schemes, the line is Spec $k[z]$. Now, there's a map of rings

$$\begin{aligned} k[x,y]/(y^2 = x^3) &\to k[z] \\ x &\mapsto z^2 \\ y &\mapsto z^3. \end{aligned}$$

This induces a homeomorphism on topological spaces, but the map on rings is *not* an isomorphism. So the structure sheaf really does encode extra information beyond the topological space itself.

In fact, $k[z]$ is a regular ring and $k[x,y]/(y^2 = x^3)$ is not, so these cannot be made isomorphic. But we can see this geometrically too. The "tangent spaces" are going screwy at the origin (x,y) in the first scheme, which is the image of (z) (or some other $(z-a)$, but it's all the same). We think of the tangent space to $\mathfrak{p}$ as $\mathfrak{p}/\mathfrak{p}^2$, so then we're comparing $(x,y)/(x,y)^2$ and $(z)/(z)^2$; these have different dimensions as k-vector spaces, so they cannot be equal.

Example 26. Consider Spec A where A is an artinian local ring (e.g. $k[\varepsilon]/(\varepsilon^2)$). Then $(\mathrm{Spec}\ A, \mathcal{O}_{\mathrm{Spec}\ A}) = (*, A)$.

The amazing thing about schemes is that we can really do differential geometry with schemes, relying heavily on schemes such as the one given above.

Example 27. Define $\mathbb{A}^2 = \mathrm{Spec}\ k[x,y]$. This has the point $\underline{0} = (x,y)$ ("the origin"), and the sheaf over this closed point evaluates to k; indeed, this is the isomorphic image of Spec k. Then we obtain a a locally ringed space $\mathbb{A}^2 - \underline{0}$, which we claim is not isomorphic to Spec R for any ring R (i.e., it is not affine). Nevertheless, it equals $D(x) \cup D(y)$. This motivates that we will need to *glue* schemes (just as one glues open sets in $\mathbb{R}^n$ to obtain manifolds).

Example 28. Consider the map $\mathbb{A}^1_{\mathbb{Z}} = \mathrm{Spec}\ \mathbb{Z}[x] \to \mathrm{Spec}\ \mathbb{Z}$ which is opposite to the inclusion $\mathbb{Z} \to \mathbb{Z}[x]$. The fiber over $[(p)]$ is supposed to look like $\mathbb{A}^1_{\mathbb{F}_p}$, and the fiber over $[(0)]$ is supposed to look like $\mathbb{A}^1_{\mathbb{Q}}$. How does this work?

Take some $[\mathfrak{q}] \in \mathbb{A}^1_{\mathbb{Z}}$. This gives us some prime ideal $\mathfrak{p} = \mathfrak{q} \cap \mathbb{Z}$. In a general topological setting, we consider the *fiber* of $B \to A$ over a point $\text{pt} \to A$ to be the pullback $\text{pt} \times_A B$. Of course, we want this to be a *closed* point (so that as a space it really is a singleton), and so for our purposes the *fiber* over $[\mathfrak{p}] \in \text{Spec } \mathbb{Z}$ will be the pullback

$$\begin{tikzcd}
F \arrow[rr] \arrow[d] & & \text{Spec } \mathbb{Z}[x] \arrow[d] \\
\text{Spec}(\text{Frac}(\mathbb{Z}/\mathfrak{p})) \arrow[r] & \text{Spec } \mathbb{Z}/\mathfrak{p} \arrow[r] & \text{Spec } \mathbb{Z}
\end{tikzcd}$$

(as $\text{Spec}(\text{Frac}(\mathbb{Z}/\mathfrak{p}))$ is terminal among one-point schemes over $\text{Spec } \mathbb{Z}/\mathfrak{p}$). Luckily for us, given a diagram D of rings we have an isomorphism $\text{Spec}(\text{colim}(D)) \cong \lim(\text{Spec}(D))$, so we obtain that $F = \text{Spec}(\text{Frac}(\mathbb{Z}/\mathfrak{p}) \otimes_{\mathbb{Z}} \mathbb{Z}[x])$. Thus when $\mathfrak{p} = (p)$ (for prime p) the fiber is $\text{Spec}(\mathbb{Z}/(p) \otimes \mathbb{Z}[x]) = \text{Spec } \mathbb{F}_p[x] = \mathbb{A}^1_{\mathbb{F}_p}$, and when $\mathfrak{p} = (0)$ the fiber is $\text{Spec}(\mathbb{Q} \otimes \mathbb{Z}[x]) = \text{Spec}\mathbb{Q}[x] = \mathbb{A}^1_{\mathbb{Q}}$.

4.5 Non-affine schemes

We now finally come to the definition of a general scheme.

Definition 20. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ such that there exists an open cover $X = \cup_i U_i$ and rings R_i such that $(U_i, \mathcal{O}_X|_{U_i}) \cong \text{Spec } R_i$ for every i. A morphism is a morphism of locally ringed spaces.

Thus, we have the functor

$$\text{Spec} \colon \textbf{Rings}^{op} \to \textbf{Schemes} \subset \textbf{LocallyRingedSpaces}.$$

This is fully faithful, but it is *not* essentially surjective: there will be schemes that are not affine (i.e. isomorphic to some $\text{Spec } R$).

Remark 11. By now, we're at the point that $\text{Spec } R$ means the whole package of the topological space and its sheaf of local rings. In the future, we will often just write X for a scheme and later write $\mathcal{O}_X$ for the implied structure sheaf.

Proposition 9. *If $(X, \mathcal{O}_X)$ is a scheme and $U \subset X$ is an open subset, then $(U, \mathcal{O}_U = \mathcal{O}_X|_U)$ is a scheme.*

Proof. Let $X = \cup_i U_i$ with $(U_i, \mathcal{O}_{U_i}) \cong \text{Spec } R_i$. Define $V_i = U \cap U_i$; then $U = \cup_i V_i$. So it suffices to show that $(V_i, \mathcal{O}_{V_i})$ is a scheme. Thus, it is enough to consider the case when $(X, \mathcal{O}_X) = \text{Spec } R$ since $V_i \subset U_i$ is an open subset. (Explicitly, if $V_i \subset U_i \subset X$ then to show that $(V_i, \mathcal{O}_{V_i})$ is a scheme we are already assuming $(U_i, \mathcal{O}_{U_i}) \cong \text{Spec } R_i$ and now V_i is an open subset of an affine scheme.) In this case, we can write $U = \cup_i D(f_i)$, and then $D(f_i) = \text{Spec } R_{f_i}$. ☐

Example 29. Let k be a field. Define $\mathbb{A}^2_k = \text{Spec } k[X, Y]$. We have the closed point $0 = [(X, Y)]$ (corresponding to the maximal ideal (X, Y)). Define the open set $U = \mathbb{A}^2_k - 0$. We claim that U (considered as a scheme) is not affine. We can prove this using a quite general method. If it were affine, we'd be able to compute the resulting ring by taking global sections of the structure sheaf, i.e. by computing the ring of global functions. Even more explicitly, if U were affine, then we would have an isomorphism $U \to \text{Spec } (\Gamma(U, \mathcal{O}_U))$. (Recall that Spec and Γ are adjoints, and they define an equivalence on the full subcategory of affine schemes.) We can make the cover $U = D(X) \cup D(Y)$ by the locus where X is nonzero and the locus where Y is nonzero. We have $D(X) \cap D(Y) = D(XY)$. Thus, by the sheaf condition we can compute that

$$\begin{aligned}
\Gamma(U, \mathcal{O}_U) &= \text{Equalizer}(\mathcal{O}_{\mathbb{A}^2}(D(X)) \times \mathcal{O}_{\mathbb{A}^2}(D(Y)) \rightrightarrows \mathcal{O}_{\mathbb{A}^2}(D(XY))) \\
&= \text{Equalizer}(k[X^{\pm}, Y] \times k[X, Y^{\pm}] \rightrightarrows k[X^{\pm}, Y^{\pm}]) \\
&= k[X, Y].
\end{aligned}$$

(Here, $k[X^{\pm}, Y] = k[X, Y]_{(X)}$ consists of finite sums of monomials $X^a Y^b$ where $a \in \mathbb{Z}$ and $b \in \mathbb{N}$.) So in fact we've shown that the restriction map $\Gamma(\mathbb{A}^2, \mathcal{O}_{\mathbb{A}^2}) \to \Gamma(U, \mathcal{O}_U)$ is an isomorphism. Thus U cannot be affine, because the inclusion $U \hookrightarrow \mathbb{A}^2$ would be an isomorphism. But this is false because there's a point of $\mathbb{A}^2$ missing from U.

One of the most unpleasant (but occasionally necessary) methods of constructing schemes is by *gluing*.

Proposition 10. *Let $\{X_i\}_{i\in I}$ be a collection of schemes. Suppose that for all i, j we have open subsets $U_{ij} \in X_i$ and $U_{ji} \subset X_j$ and isomorphisms $\varphi_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$ of schemes, such that:*

- $\varphi_{ji} = \varphi_{ij}^{-1}$*;*
- $U_{ii} = X_i$ *and* $\varphi_{ii} = \mathrm{id}$*;*
- $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ *so that the diagram*

$$\begin{tikzcd} U_{ij} \cap U_{ik} \arrow[r, "\varphi_{ij}"] \arrow[d, "\varphi_{ik}"'] & U_{ji} \cap U_{jk} \arrow[d, equal] \\ U_{jk} \cap U_{ki} & U_{jk} \cap U_{ji} \arrow[l, "\varphi_{jk}"] \end{tikzcd}$$

  commutes.

Then we can glue our schemes X_i along the identifications $U_{ij} \sim U_{ji}$ into a single scheme

$$X = \operatorname{colim}\left(\coprod_{(i,j)} U_{ij} \overset{\iota_{U_{ij},X_i}, \coprod_i X_i}{\rightrightarrows} \coprod_i X_i\right).$$

In the interest of not making our heads explode, we will skip the proof of this important fact. We prefer to use its universal property. As a topological space, we just obtain X as the space $\coprod X_i/\sim$, where the relation is that if there is some (i, j) so that $x \in U_{ij}$ and $y \in U_{ji}$ and $\varphi_{ij}(x) = y$, then $x \sim y$. The inclusions $X_i \hookrightarrow X$ will be *open* inclusions.

4.6 Examples of schemes

Example 30. Of course, the first examples of schemes we have are Spec R for various commutative rings R.

Example 31. Given any ring R, we have *affine n-space* $\mathbb{A}^n_R = \text{Spec } R[x_1, \ldots, x_n]$. This has maximal ideals in canonical bijection with the set R^n.

Example 32. Given any ring R, we have the *multiplicative group* $\mathbb{G}_{m,R} = \text{Spec } R[x^{\pm}]$. Observe that the Yoneda functor of this scheme sends an affine scheme over Spec R to the multiplicative group of its global functions, i.e.

$$\begin{aligned} \mathbb{G}_{m,R} \colon \textbf{AffSch}/\text{Spec } R \quad &\longrightarrow \quad \textbf{Sets} \\ (\text{Spec } A \to \text{Spec } R) \quad &\mapsto \quad \{R[x^{\pm}] \to A \text{ extending the algebra map } R \to A\} \cong A^{\times}. \end{aligned}$$

This is our first example of a *group scheme*, which is the algebro-geometric analog of a Lie group.

Example 33. As we discussed, $\mathbb{A}^2 - \{0\}$ is a scheme which is not affine.

Example 34. We can obtain new schemes by gluing. The primordial example of this is $\mathbb{P}^d_R$, projective d-space over the ring R. At $d = 2$ we have

$$\begin{tikzcd} \mathbb{A}^1_R - V(x) = \text{Spec } R[x^{\pm}] \arrow[r] \arrow[d, hook] & \text{Spec } R[y^{\pm}] = \mathbb{A}^1_R - V(y) \arrow[d, hook] \\ \mathbb{A}^1_R = \text{Spec } R[x] & \text{Spec } R[y] = \mathbb{A}^1_R. \end{tikzcd}$$

Classically, we know we should have a map $\mathbb{A}^2_R - \{0\} \to \mathbb{P}^1_R$ given by taking the point p to the line that it spans. For us, this corresponds to the map

$$\begin{aligned} D(t) = \text{Spec } R[s, t^{\pm}] \quad &\to \quad \text{Spec } R[x] \\ R[s, t^{\pm}] \ni \frac{s}{t} \quad &\leftarrow \quad x \in R[x]. \end{aligned}$$

In the gluing diagram, this gives us

$$\begin{tikzcd}
D(s) = \text{Spec } R[s^{\pm}, t] \arrow[r] & \text{Spec } R[y] \arrow[d, hook] \\
\mathbb{A}^1 - \{0\} \arrow[r] & \mathbb{P}^1 \\
D(t) = \text{Spec } R[s, t^{\pm}] \arrow[r] & \text{Spec } R[x]. \arrow[u, hook]
\end{tikzcd}$$

Note that really what we want is to define $\mathbb{P}^1_R = (\mathbb{A}^2_R - \{0\})/\mathbb{G}_{m,R}$, where $\mathbb{G}_{m,R}$ acts by scaling. But for now, we need to work with this actual construction.

Example 35 (The point of exercise 2 on homework 5)**.** Often we think of a scheme not as a locally ringed space, but in terms of the functor it represents. For example, given a ring R, how should we think about maps from Spec R to $\mathbb{P}^n_{\mathbb{Z}}$? Certainly we don't want to have to take an open cover of $\mathbb{P}^n$ and mess around like that. Rather, we'd like a functorial interpretation of $\mathbb{P}^n$, i.e. we'd like to understand the functor $h_{\mathbb{P}^n}$. The classical definition would suggest that we should be thinking about lines in affine $(n+1)$-space, i.e. the functor F : **Rings** $\to$ **Set** given by setting $F(R)$ to be the set of free direct summands $L \subset R^{n+1}$ of rank 1. However, we found in our homework that this functor is *not* representable by any scheme. Briefly, we can say that F is not a sheaf in the Zariski topology – we will explain this.

In general, if Y and X are schemes and $Y = \cup U_i$ is an open cover, then

$$h_X(Y) \to \prod_i h_x(U_i) \rightrightarrows \prod_{i,j} h_x(U_i \cap U_j)$$

is exact: a morphism of schemes is equivalent to compatible morphisms on the open sets of a cover. So, we find a ring R and an open cover Spec $R = \cup$Spec R_{f_i} such that (abusing notation)

$$F(R) \to \prod_i F(R_{f_i}) \rightrightarrows \prod_{i,j} F(R_{f_i f_j})$$

is not exact. The point here is that there are rings R with a R-module P such that P is not a free module but P_{f_i} is free of rank 1 for every i and $(f_1, \ldots, f_r) = R$, and P is a direct summand of R^{n+1} for some n.

We will need a ring which is not a PID, of which the typical example is $R = \mathbb{Z}[\sqrt{-5}]$: the ideal $(2, 1+\sqrt{-5})$ determines a submodule of the R-module R^2 (given by surjecting $R^2 \to P$ and then choosing a splitting guaranteed by projectiveness). In this ring, $(1+\sqrt{-5})(1-\sqrt{-5}) = 6 = 2 \cdot 3$ so we have the localizations $P_3 = (1+\sqrt{-5})$ and $P_2 = R_2$. So, why does this contradict the sheaf property? Well, $P \subset R^{n+1}$ does not define an element of $F(R)$, but it wants to because it defines compatible elements of $F(R_3)$ and $F(R_2)$.

Ultimately, the correct interpretation will be that $h_{\mathbb{P}^n}|_{\textbf{Rings}}$ is isomorphic to the functor

$$R \mapsto \{P \subset R^{n+1} \colon P \text{ a direct summand and projective of rank } 1\}.$$

(Rank here means the rank of any localization that becomes free.) This is the *sheafification* (in the appropriate sense – with respect to the Zariski topology on **Schemes**) of the functor F.

Example 36. Sometimes schemes are *nonseparated* – this is the algebro-geometric analog of *Hausdorffness* with usual topological spaces (although schemes are not Hausdorff in general, so really it's something else going on). Our first example of this is given by gluing along the morphism

$$\begin{tikzcd}
\text{Spec } R[x^{\pm}] \arrow[r, "x \mapsto y"] \arrow[d, hook] & \text{Spec } R[y^{\pm}] \arrow[d, hook] \\
\mathbb{A}^1_{R,x} & \mathbb{A}^1_{R,y}.
\end{tikzcd}$$

This is called "the affine line with two origins".

4.7 Proj

Since explicit gluing is difficult, there are few global constructions in algebraic geometry. However, one of the few things we do have is Proj, which takes a graded ring and returns a projective scheme, just as Spec takes an ungraded ring and returns an affine scheme.

Definition 21. An $\mathbb{N}$-*graded ring* is a ring R and a decomposition $R = \bigoplus_{d\geq 0} R_d$ as an abelian group such that $R_d \cdot R_e \to R_{d+e}$. In this situation, we define $R_+ = \bigoplus_{d>0} R_d$, and we call R_d the *degree-d homogeneous elements of* R.

Example 37. The typical example of this is $R = S[x_1, \dots, x_n]$, where $R_d = \{\sum c_\alpha x_1^{i_{\alpha,1}} \cdots x_n^{i_{\alpha,n}} : i_{\alpha,1} + \cdots + i_{\alpha,n} = d \text{ for all } \alpha\}$.

Remark 12. A $\mathbb{Z}$-grading on a ring R is the same thing as an action of $\mathbb{G}_m$. Let us unpack this a bit. Suppose that R is a $\mathbb{Z}$-graded ring. Let us describe a map $\alpha : \mathbb{G}_m \times \operatorname{Spec} R \to \operatorname{Spec} R$. Given a map $\operatorname{Spec} S \to \mathbb{G}_m \times \operatorname{Spec} R$, which is the same as a pair (u, f) of a unit $u \in S^\times$ and a ring map $f : R \to S$, we define a new map $\alpha(u, f) : R \to S$ by $r \mapsto u^{\deg(r)} \cdot f(r)$. Conversely, given an action of $\mathbb{G}_m$ on Spec R, we can pick out the grading on R. (Alternatively, we will eventually learn that the product of affine schemes corresponds to the tensor product of the rings. So then $\mathbb{G}_m \times \operatorname{Spec} R = \operatorname{Spec} R[x^\pm]$, and our map $\mathbb{G}_m \times \operatorname{Spec} R \to \operatorname{Spec} R$ is the same as a map $R[x^\pm] \leftarrow R$, and the preimage of x^d is exactly R_d.)

We want to construct a scheme Proj R, which will be a generalization of $\mathbb{P}^1_R$. This will be the union over all homogenous elements $f \in R$ of something we will call $D_+(f)$. Consider $R_f = \bigoplus_{d\in\mathbb{Z}} (R_f)_d$. Then if $x \in R_e$, we set $\deg(x/f^n) = e - n \cdot \deg(f)$, and we set $D_+(f) = \operatorname{Spec}(R_f)_0$.

To see why this is reasonable, consider $R[s, t^\pm]$, a graded ring with $\deg(s) = \deg(t) = 1$. Then $R[s/t]$ consists of degree-0 elements, which is what we want. To see why this construction makes sense, we need to do some serious commutative algebra...

Definition 22. In an $\mathbb{N}$-graded ring $R = \bigoplus_{d\geq 0} R_d$, an ideal $I \subset R$ is called *homogeneous* if $I = \bigoplus_{d\geq 0}(I \cap R_d)$. This is equivalent to saying that I is generated by homogeneous elements.

This implies that $R/I = \bigoplus_{d\geq 0} R_d/(I \cap R_d)$ is graded. For example, suppose $R = k[x_1, \dots, x_n]$ graded by degree. Let $I = (x_1^4 + 17x_2^4 + \cdots + 3x_{n-1}^4)$. Then the quotient R/I is graded.

The basic construction is as follows. Given $R = \bigoplus_{d\geq 0} R_d$ as above and $f \in R$ homogeneous of degree 1, we define $R_f = \bigoplus_{d\in\mathbb{Z}} (R_f)_d$ (here the subscript d stands for "degree-d part"). Note that for every $d \in \mathbb{Z}$, multiplication by f^d defines an isomorphism $(R_f)_0 \xrightarrow{\sim} (R_f)_d$. Thus $(R_f)_0[z^\pm] \cong R_f$ given including $(R_f)_0 \hookrightarrow R_f$ and mapping $z \mapsto f$. (Thus $\mathbb{G}_{m,(R_f)_0} \cong \operatorname{Spec} R_f$, so the quotient by the multiplicative group action should indeed be Spec $(R_f)_0$.)

Proposition 11. *The map*

$$\begin{array}{rcl} \varphi : \{\text{homog. primes } \mathfrak{p} \subset R \text{ not containing } f\} & \longrightarrow & \{\text{prime ideals in } (R_f)_0\} \\ \mathfrak{p} & \mapsto & (\mathfrak{p}R_f)_0 \end{array}$$

is a bijection.

Proof. First, observe that $(\mathfrak{p}R_f)_0$ is the preimage of $\mathfrak{p}R_f$ under $(R_f)_0 \hookrightarrow R_f$, whence we see that $(\mathfrak{p}R_f)_0$ is prime.

We first show that φ is surjective. Suppose that $\mathfrak{q} \subset (R_f)_0$ is prime. Then let $\tilde{\mathfrak{q}} = \mathfrak{q} \cdot R_f$; as modules over $(R_f)_0$, we have the isomorphism $\tilde{\mathfrak{q}} \cong \bigoplus_{d\geq\mathbb{Z}} \mathfrak{q} \cdot z^d$. Then $\tilde{\mathfrak{q}} \subset R_f$ is prime since we can explicitly write $R_f/\tilde{\mathfrak{q}} \cong ((R_f)_0/\mathfrak{q})[z^\pm]$ which is an integral domain, and moreover $\tilde{\mathfrak{q}} \cap (R_f)_0 = \mathfrak{q}$. Now, let $\mathfrak{p}$ be the preimage of $\tilde{\mathfrak{q}}$ under the localization map $R \to R_f$. Then $\mathfrak{p}$ is a prime ideal in R such that $\mathfrak{p} \cdot R_f = \tilde{\mathfrak{q}}$, and $\mathfrak{p} = \operatorname{Ker}(R \to R_f \to R_f/\tilde{\mathfrak{q}})$, a composition of morphisms of graded rings (it's just $\bigoplus R_d \to \bigoplus (R_f)_d \to \bigoplus (R_f/\tilde{\mathfrak{q}})_d$), which means that $\mathfrak{p}$ is graded. In fact, $\mathfrak{p} = \bigoplus_{d\geq 0} \operatorname{Ker}(R_d \to (R_f/\tilde{\mathfrak{q}})_d)$.

We now show that φ is injective. We have the injection

$$\{\text{homog. primes } \mathfrak{p} \subset R \text{ not containing } f\} \hookrightarrow \{\text{homog. primes } \tilde{\mathfrak{q}} \subset R_f\}$$

since this is already an inclusion without the word "homogeneous" everywhere. Now, we take such a $\tilde{\mathfrak{q}}$ and associate it to the prime ideal $\tilde{\mathfrak{q}} \cap (R_f)_0$ in $(R_f)_0$. We would like to show that this association is bijective. The map backwards is $\mathfrak{q} \mapsto \mathfrak{q} \cdot R_f$, so it suffices to show that any homogeneous prime $\tilde{\mathfrak{q}} \subset R_f$ is generated by degree-0 elements. If $(g_i)_{i\in I}$ is a set of homogeneous generators for $\tilde{\mathfrak{q}}$, then so is $(g_i \cdot f^{-\deg(g_i)})_{i\in I}$. □

Definition 23. Assume that R is generated as an R_0-algebra by R_1. As a set, we define

$$\text{Proj } R = \{\text{homog. primes } \mathfrak{p} \text{ not containing } R_+\}.$$

Here, $R_+ = \bigoplus_{d>0} R_d$, so this is just Proj $R = \bigcup_{f \in R_1} D_+(f)$, where $D_+(f) = \{\text{homog. primes } \mathfrak{p} \subset R \colon f \notin \mathfrak{p}\} =$ Spec $(R_f)_0$. Thus, we are simply avoiding the *irrelevant ideal* $R_+ \subset R$.

Example 38. Let $R = k[x, y]$ with $|x| = |y| = 1$. Then homogeneous primes that aren't irrelevant take the form $(ax + by)$, and these correspond to lines in the plane.

Theorem 3. *The scheme structures on* $D_+(f)$ *are compatible and define a scheme structure on* Proj R.

Let us unpack this. The topology is given by declaring the sets $V(\mathfrak{a})$ to be closed, where $V(\mathfrak{a}) = \{\text{homog. primes } \mathfrak{p} \supset \mathfrak{a}\}$ for any homogenous ideal $\mathfrak{a} \subset R$. We should check that this defines a topology, and that for any (homogeneous) $f \in R_1$, $D_+(f) \subset$ Proj R is an open set and has the induced topology. These $D_+(f)$ define a base for the topology on Proj R, and the structure sheaf is determined on this base by $\mathcal{O}|_{D_+(f)} = \mathcal{O}_{\mathrm{Spec}(R_f)_0}$.

Example 39. A classical example is to look at the vanishing locus V of the ideal $I = (x^4 + y^4 + z^4)$ in $\mathbb{P}^2 =$ Proj $k[x, y, z]$. This ambient scheme is covered by $D_+(x) \cup D_+(y) \cup D_+(z)$. Then, for example, $D_+(x) =$ Spec $(k[x, y, z]_x)_0 \cong$ Spec $k[y/x, z/x] \cong \mathbb{A}^2$. Then, we can think about our solutions in several ways. For starters, we will want $V =$ Proj $k[x, y, z]/I \subset \mathbb{P}^2$. Then we should have

$$V \cap D_+(x) = \text{Spec } (k[x, y, z]/I)_{x,0} = \text{Spec } k[y/x, z/x]/(1 + (y/x)^4 + (z/x)^4) \cong \text{Spec } k[Y, Z]/(1 + Y^4 + Z^4).$$

Example 40 (one of Prof. Olsson's faves!)**.** Let k be a field. We consider the ring $k[\ldots, x_n, x_{n+1}, \ldots]_{n \in \mathbb{Z}}$, where $|x_n| = 1$. We would like to quotient by the relation that $x_n \cdot x_m = 0$ whenever $|n - m| \geq 2$. What do we get when we take Proj of this?

We visualize this as follows. Take the line $x = 1$ in the xy-plane, and take all integer points $(1, n)$ on this line. These generate a monoid of integer points in the plane, namely those integer points (x, y) with $x \geq 1$ (along with the origin $(0, 0)$). Whenever we have a monoid M we can take the monoid-algebra $k[M]$ (of which a group-algebra is a special case), and the region $nx \leq y \leq (n+1)x$ for various $n \in \mathbb{Z}$ we get a free monoid isomorphic to $\mathbb{N}^2$. Thus, we should expect a bunch of lines (copies of $\mathbb{P}^1$) glued "head to tail".

Now we do it for real. Observe that the $D_+(x_i) \subset$ Proj R form a cover for $i \in \mathbb{Z}$. Explicitly, $D_+(x_i) =$ Spec R_{x_i}, and when we make x_i a unit, then anybody who products with it to 0 has to get killed. So we get $R_{x_i} = k[x_{i-1}, x_i^{\pm}, x_{i+1}]/(x_{i-1}x_{i+1})$. The degree-0 part of this is

$$(R_{x_i})_0 = k\left[\frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}\right] / \left(\frac{x_{i-1}}{x_i} \cdot \frac{x_{i+1}}{x_i}\right) \simeq k[y_{i-\frac{1}{2}}, y_{i+\frac{1}{2}}]/(y_{i-\frac{1}{2}} y_{i+\frac{1}{2}}).$$

The prime spectrum of this is just the union of the two axes in the $y_{i-\frac{1}{2}} y_{i+\frac{1}{2}}$-plane. More explicitly, when we kill $y_{i-\frac{1}{2}}$ we just get the affine line $k[y_{i+\frac{1}{2}}]$. Abstractly, this is just two intersecting lines.

Now to glue these all together, we have for example the isomorphism

$$D_+(x_{i+1}) \supset \text{Spec } k[y^{\pm}_{(i+1)-\frac{1}{2}}] \xrightarrow{\sim} \text{Spec } k[y^{\pm}_{i+\frac{1}{2}}] \subset D_+(x_i),$$

and when you work it out you get $y_{(i+1)-\frac{1}{2}} \mapsto y^{-1}_{i+\frac{1}{2}}$ since $y_{i+\frac{1}{2}} = x_{i+1}/x_i$ while $y_{(i+1)-\frac{1}{2}} = x_i/x_{i+1}$. (These get projectivized exactly through this gluing.)

Example 41. Here is a natural example of a graded ring. Let's start with the expression $xy = z^2$. Going back to the technique from the previous example, we can obtain a graded ring out of the monoid M of integer points in $\{(x, y) \in \mathbb{R}^2 : x \geq 0, |y| \leq x\}$ by defining $R = \bigoplus_{m \in M} k \bullet x^m$; each x^m is just a basis element as a module, and we define $x^m \cdot x^{m'} = x^{m+m'}$ (this is where we are using the module structure). We obtain an $\mathbb{N}$-grading by setting $\deg(x^m)$ to be the horizontal coordinate of m. We have the relation $x^{(1,1)} \cdot x^{(1,-1)} = x^{(2,0)} = (x^{(1,0)})^2$. One can check that in fact these are all the relations: $R \simeq k[x, y, z]/(xy - z^2)$.

So, what is Proj $k[x, y, z]/(xy - z^2)$? In fact, this is again $\mathbb{P}^1$! Consider first $R = k[s, t]$. Taking Proj of this gives us $\mathbb{P}^1$ for real; the graded subring of $k[s, t]$ of elements of only even degree, i.e. $\bigoplus_{i \geq 0} R_{2i}$. This is actually isomorphic to $k[s^2, st, t^2]/(s^2 \cdot t^2 = (st)^2)$. We calculate

$$\mathcal{O}(D_+(z)) = (k[x, y, z^{\pm}]/xy - z^2)_0 = (k[x^{\pm}, y^{\pm}, z^{\pm}]/xy - z^2)_0.$$

That is, if we invert z then we've also inverted x and y. So $D_+(z) \subset D_+(x) \cap D_+(y)$. So, Proj $k[x,y,z]/(xy - z^2) = D_+(x) \cup D_+(y)$. Now we compute that

$$\begin{aligned}\mathcal{O}(D_+(x)) &= (k[x^{\pm}, y, z]/xy - z^2)_0 = (k[x^{\pm}, z])_0 = k[z/x] \\ \mathcal{O}(D_+(y)) &= k[z, y].\end{aligned}$$

These are glued along the locus where z is invertible, i.e. we associate $k[(z/y)^{\pm}] \to k[(z/x)^{\pm}]$ via

$$\frac{z}{x} = \frac{yz}{z^2} = \frac{y}{z},$$

and so we get exactly $\mathbb{P}^1_k$.

4.8 Basic properties of schemes

4.8.1 Global properties

We begin with some topological definitions. These tell us about general schemes. Later we will see what these mean for affine schemes, which will bring us back to commutative algebra.

Definition 24. A scheme $X = (|X|, \mathcal{O}_X)$ is called:

- *connected* if $|X|$ is not a disjoint union of two open subsets;
- *irreducible* if $|X|$ is not a union of two proper closed subsets;
- *reducible* if X is not irreducible;
- *reduced* if for every open $U \subset X$, the ring $\mathcal{O}_X(U)$ has no nilpotent elements;
- *integral* if for every open $U \subset X$, the ring $\mathcal{O}_X(U)$ is an integral domain.

Example 42. The union of the two axes in $\mathbb{A}^2$ is reducible – this is given by Spec $k[x,y]/(xy)$. On the other hand, Spec $k[x,y]/(y^2 - x^2(x+1))$ is irreducible.

Example 43. Let $X = \text{Spec } A$. Then:

- X is reduced iff the nilradical of A, denoted $\mathfrak{N}(A)$, is 0.
- X is integral iff A is an integral domain.
- X is irreducible iff $\mathfrak{N}(A)$ is prime. Recall that $\mathfrak{N}(A)$ is the ideal of nilpotent elements of A (i.e. the set of $f \in A$ for which there exist $n > 0$ such that $f^n = 0$). This is indeed an ideal, and it may also be written as $\sqrt{0}$ (where $\sqrt{I}$ denotes the radical of the ideal I). Moreover, $\sqrt{0}$ is the intersection of all prime ideals of A. Geometrically, we are looking for global functions whose power is 0. To check that $\sqrt{0} \subset \cap\mathfrak{p}$, we check that if $f^n = 0$ then $f^n \in \mathfrak{p}$ for all $\mathfrak{p}$, so $f \in \mathfrak{p}$ by primeness. On the other hand, to check that $\sqrt{0} \supset \cap\mathfrak{p}$, supposed that $f \in \mathfrak{p}$ for all prime ideals $\mathfrak{p}$. This is saying that $V(f) = |\text{Spec } A|$, so $D(f) = (V(f))^c = \emptyset$, so $A_f = 0$: if we localize a ring at a non-nilpotent element, we get a nontrivial ring. So $f^n = 0$ for some $n > 0$.

Example 44. Returning to our example $k[x,y]/(xy)$, we see that (0) is not a prime ideal. This is why Spec $k[x,y]/(xy)$ is not reduced.

Note that integral implies reduced.

Example 45. Let $X = \text{Spec } k[x,y]/(xy, x^2)$ for some field k. This is:

- not reduced, since $x^2 = 0$ but $x \neq 0$;
- not integral, because integrality implies reduced;
- irreducible, as we will explain.

In general, $|\text{Spec } R| \cong |\text{Spec } R/\sqrt{0}|$ is a natural homeomorphism. Thus, $|\text{Spec } k[x,y]/(xy, x^2)| \cong |\text{Spec } k[x,y]/(xy, x)| \cong |\text{Spec } k[y]|$. So, people generally draw this as a line with a fattened origin. This is meant to represent (x, y), whose complement $D(y)$ is actually isomorphic as a scheme to Spec $k[y^{\pm}]$.

Lemma 4. *If X is irreducible, then any two nonempty open sets $U, V \subset X$ intersect nontrivially.*

Proof. If $U \cap V = \emptyset$ then $U^c \cup V^c = X$. $\square$

Example 46. In $\mathbb{A}^1_{\mathbb{C}}$, the Zariski topology has that the nonempty open sets are all complements of finite sets. In this case it's clear that any two open sets intersect.

Lemma 5. *Let X be an irreducible scheme. Then there is a unique point $\eta \in X$ with $\overline{\eta} = X$. (This point is called the* generic point *of X.)*

Proof. Recall that if $x = [\mathfrak{p}_x], y = [\mathfrak{p}_y] \in \text{Spec } A$ and $y \in \overline{x}$, then $\mathfrak{p}_x \subset \mathfrak{p}_y$. So in the affine case $X = \text{Spec } A$, clearly $\eta = \sqrt{0} = \mathfrak{N}(A)$ is the unique such point. For the general case, we need to check that if we have Spec B, Spec $A \subset X$ with generic points $\eta_A \in \text{Spec } A$ and $\eta_B \in \text{Spec } B$, then these map to the same point of X. Of course since any nontrivial open set in X is dense, $\overline{\eta_A} = \overline{\text{Spec } A} = X$. We must check uniqueness, though. First, note that if in fact Spec $B \subset$ Spec A, then $\eta_A = \eta_B$ (by what we have already seen in the affine case). More generally, if Spec $C \subset$ Spec $A \cap$ Spec $B \subset X$, then $\eta_A = \eta_C = \eta_B$. $\square$

In practice, we just take η to be the generic point of *any* open affine Spec $A \subset X$. If X is integral, then $\eta \in \text{Spec } A \subset X$ is the point corresponding to $(0) \subset A$. The local ring $\mathcal{O}_{X,\eta}$ is usually denoted $k(X)$, and is called the *function field* of X. (In the case of a Riemann surface, this is the field of meromorphic functions.)

Example 47. Let $X = \text{Spec } \mathbb{Z}$. This is an integral domain, so $\eta = [(0)]$ and $k(X) = \mathbb{Q}$.

Example 48. Let $X = \text{Spec } k[t]$ where k is a field. This is again an integral domain, so $\eta = [(0)]$ and (terribly) $k(X) = k(t)$.

Example 49. Let $X = \mathbb{P}^n_k$. We choose an affine $\mathbb{A}^n_k \subset \mathbb{P}^n_k$, and then $k(\mathbb{P}^n_k) = k(\mathbb{A}^n_k) = k(x_1, \dots, x_n)$.

4.8.2 Local properties

Now we will discuss the local versions of these properties.

Definition 25. A scheme X is called:

- *locally noetherian* if $X = \cup \text{Spec } A_i$ is an open cover by noetherian rings and each A_i is a noetherian ring;
- *noetherian* if X is locally noetherian and quasi-compact.

Remark 13. Any time you say a definition like this, you should know that it's not actually exactly what you want. These shouldn't depend on the choices of covers. The definitions are generally made in the weaker form, and then it's a fact that this is equivalent to the stronger form.

Lemma 6. *If X is locally noetherian, then for every affine open* Spec $A \subset X$*, the ring A is noetherian.*

Proof. Note first that if B is a noetherian ring, then so is B_f for any $f \in B$ (since the ideals of B_f are just some of the ideals of B). Now, let $X = \cup \text{Spec } A_i$ where A_i is noetherian. Then we have

$$\text{Spec } A_i \supset \text{Spec } A_i \cap \text{Spec } A \subset \text{Spec } A.$$

Write $\text{Spec } A_i \cap \text{Spec } A = \bigcup_j D^{\text{Spec } A_i}(g_{ij})$ (i.e. we are taking basic opens inside of Spec A_i determined by the $g_{ij} \in A_i$). Thus this intersection is covered by $\bigcup_j \text{Spec} A_{i,g_{ij}}$. This means we can cover Spec A by affine opens Spec B where B is a noetherian ring; that is, Spec A is locally noetherian. So it remains to show that a locally noetherian affine scheme is actually noetherian.

Note that if $f \in A$ satisfies $D^{\text{Spec } A}(f) \subset \text{Spec } B \subset \text{Spec } A$, then $A_f \simeq B_f$, i.e. $\text{Spec } B \cap D^{\text{Spec } A}(f) = \text{Spec } B_f$. (We are identifying $f \in A$ with its image in B.) (In **Rings**, if we have a map $B \to R$ so that $f \in B$ gets sent to a unit, then this is the same as asking that the composite $A \to B \to R$ takes f to a unit; we're intersecting with $D^{\text{Spec } A}(f)$ so that f isn't accidentally zero to start with. So both sides of this equation represent the same functor on affine schemes, so they are isomophic.) So if Spec A is locally noetherian then there exist $f_1, \dots, f_r \in A$ such that $(f_1, \dots, f_r) = A$ and A_{f_i} is noetherian for every i by our opening observation.

Let $\varphi_i : A \to A_{f_i}$ be the localization map. We claim that if $\mathfrak{a} \subset A$ is any ideal, then $\mathfrak{a} = \bigcap_{i=1}^r \varphi_i^{-1}(\varphi_i(\mathfrak{a}) \cdot A_{f_i})$. The containment $\subset$ is clear. For the other containment $\supset$, suppose that $b \in \bigcap_{i=1}^r \varphi_i^{-1}(\varphi_i(\mathfrak{a}) \cdot A_{f_i})$. Then $\varphi_i(b) = \varphi_i(c_i)/f_i^N$ with $c_i \in \mathfrak{a}$; by finiteness we can choose the same N for all i. This is equivalent to saying that $\varphi_i(f_i^N b - c_i) = 0$ in A_{f_i}, which implies that there is some M such that $f_i^M(f_i^N b - c_i) = 0$ in A. Since this is true for all i, we can take M to work for all i. Thus $f_i^{M+N}b = f_i^M c_i \in \mathfrak{a}$. Now, write $1 = \sum_{i=1}^r \alpha_i f_i^{M+N}$. (Recall that if $(f_1, \ldots, f_r) = A$ then $(f_1^{M+N}, \ldots, f_r^{M+N}) = A$.) So now $D(f_i) = D(f_i^{M+N})$, so $b = \sum_{i=1}^r \alpha_i f_i^{M+N} b = \sum_{i=1}^r (\alpha_i f_i^M) c_i \in \mathfrak{a}$.

This is good because we need to check an ascending chain condition on ideals of A, and we know this is true in each of this finite number of localizations, so it must be true in A too. More explicitly, suppose that $\mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \cdots$ is an ascending chain of ideals in A. Then for each i, there is some N_i such that $\varphi_i(\mathfrak{a}_k) \cdot A_{f_i} = \varphi_i(\mathfrak{a}_{k+1}) \cdot A_{f_i}$ for all $k \geq N_i$. Let $N = \max\{N_i\}$. Then for any $k \geq N$, we have

$$\mathfrak{a}_k = \bigcap_{i=1}^r \varphi_i^{-1}(\varphi_i(\mathfrak{a}_k) \cdot A_{f_i}) = \bigcap_{i=1}^r \varphi_i^{-1}(\varphi_i(\mathfrak{a}_{k+1} \cdot A_{f_i})) = \mathfrak{a}_{k+1},$$

so this sequence stabilizes. ☐

4.9 Basic properties of morphisms

Definition 26. A morphism of schemes $f : X \to Y$ is called *locally of finite type* if there are coverings $Y = \bigcup_i \text{Spec } A_i$ and $f^{-1}(\text{Spec } A_i) = \bigcup_{j \in I_i} \text{Spec } B_{ij}$ such that the corresponding maps $A_i \to B_{ij}$ are of finite type (i.e. B_{ij} becomes a finitely generated A_i-algebra). The morphism f is said to have *finite type* if it is locally of finite type and for every i we can choose the indexing set I_i to be finite.

As before, we have the following fact which strengthens the definition.

Proposition 12. *If $f : X \to Y$ is locally of finite type, then for every affine open* Spec $A \subset Y$ *with* Spec $B \subset f^{-1}(\text{Spec } A)$*, B becomes a finitely generated A-algebra. Moreover, if f is of finite type then for any affine* Spec $A \subset Y$*, $f^{-1}(\text{Spec } A)$ is quasi-compact.*

Proof. For the first statement, choose coverings $Y = \bigcup_i \text{Spec } A_i$ and $f^{-1}(\text{Spec } A_i) = \bigcup_j \text{Spec } B_{ij}$ as in the definition. For any $x \in \text{Spec}\mathbb{A}$, there is some i and f_i such that $x \in \text{Spec } A_{i,f_i} \subset \text{Spec } A_i \subset \text{Spec } A$. We do the same thing upstairs: given a point $\sigma \in \text{Spec } B$ we can find some i, j and g such that $\sigma \in \text{Spec } B_{ij,g} \subset \text{Spec } B$ and

$$\begin{tikzcd}
\text{Spec } B_{ij,g} \arrow[r, hook] \arrow[d, "h'"] & \text{Spec } B \arrow[d, "h"] \\
\text{Spec } A_{i,f_i} \arrow[r, hook] & \text{Spec } A.
\end{tikzcd}$$

That is,

$$\begin{tikzcd}
L \arrow[r, hook] \arrow[d, hook] & \text{Spec } B \arrow[d] \\
f^{-1}(\text{Spec } A_{i,f_i}) \arrow[r, hook] \arrow[d, hook] & f^{-1}(\text{Spec } A) \arrow[d] \\
\text{Spec } A_{i,f_i} \arrow[r, hook] & \text{Spec} A
\end{tikzcd}$$

(where L is the intersection). Then $B_{ij,g}$ is a finitely generated A_{i,f_i}-algebra by construction. (The localization of a finitely generated algebra is again finitely generated.) Now we go even smaller to get to a neighborhood that's a localization of B, $\sigma \in \text{Spec } B_\kappa \hookrightarrow \text{Spec } B_{ij,g} \hookrightarrow \text{Spec } B$. Like last time, B_κ is also a localization of $B_{ij,g}$ and similarly downstairs. So we conclude that there exists a covering $\text{Spec } B = \bigcup_i \text{Spec } B_{\kappa_i}$ with each B_{κ_i} a finitely generated A-algebra and $(\kappa_1, \ldots, \kappa_r) = B$. Thus, we are reduced to showing that in this case, B is a finitely generated A-algebra. So, let $B' \subset B$ be a finitely generated subalgebra such that $\kappa_i \in B'$ for all i and $(\kappa_1, \ldots, \kappa_r) = B'$. Then $B'_{\kappa_i} = B_{\kappa_i}$ for all i. We claim that this implies that $B' = B$. This is because the quotient $Q = B/B'$ is a B'-module for which $Q_{\kappa_i} = 0$ for every i. But then $Q = 0$ because $Q \hookrightarrow \prod_i Q_{\kappa_i}$.

We will leave the proof of the second claim to the reader. ☐

Remark 14. In the case of affines (where finite type and locally finite type mean the same thing), a morphism Spec $B \to$ Spec A is of finite type iff there exists a factorization

$$\begin{tikzcd}
\text{Spec } B \arrow[r, hook, "\text{closed immersion}"] \arrow[rd] & \mathbb{A}^n_A \arrow[d] \\
& \text{Spec } A.
\end{tikzcd}$$

A *closed immersion* is opposite to a surjection. We study these presently.

Definition 27. A morphism $f : X \to Y$ is called *finite* if there is a covering $Y = \bigcup \text{Spec } A_i$ such that:

- $f^{-1}(\text{Spec } A_i)$ is affine for all i,
- if $f^{-1}(\text{Spec } A_i) = \text{Spec } B_i$, then B_i is a finite A_i-algebra (i.e., it is finitely-generated as a module).

Example 50. Consider Spec $k[s,t] \to$ Spec $k[x,y,z]/(xy-z^2)$, corresponding to the map $k[x,y,z]/(xy-z^2) \to k[s,t]$ given by $x \mapsto s^2$, $y \mapsto t^2$, $z \mapsto st$. This is finite: as a module, $k[s,t]$ is generated by $\{1, s, t\}$.

Example 51. Consider Spec $k[t^{\pm}] \hookrightarrow$ Spec $k[t]$. This is *not* finite, since $k[t^{\pm}]$ is not finitely generated as a $k[t]$-module.

Example 52. Let us draw two pictures. In picture (a), the graphs of $y = \sin x$ and $y = -\sin x$ project down to $y = -2$. This is finite. In picture (b), we take picture (a) and puncture one of the waves at one point. This is not finite.

Definition 28. A morphism $f : X \to Y$ is an *open immersion* if f induces an isomorphism $X \overset{\sim}{\to} (U, \mathcal{O}_U)$, where $U \subset Y$ is an open subset and $\mathcal{O}_U = \mathcal{O}_Y|_U$.

Definition 29. A morphism $f : X \to Y$ is a *closed immersion* if f induces a homeomorphism $|X| \overset{\cong}{\to} Z$ where $Z \subset |Y|$ is a closed subset, and $\mathcal{O}_Y \overset{f^\#}{\to} f_*\mathcal{O}_X$ is surjective.

Remark 15. If $Y = $ Spec A, then we have a bijection

$$\begin{array}{rcl} \{\text{ideal } I \subset A\} & \leftrightarrow & \{\text{closed immersions } X \to Y\} \\ I & \mapsto & (\text{Spec } A/I \to \text{Spec } A). \end{array}$$

More precisely, it will turn out that any such X is of this form. We then have that $|\text{Spec } A/I| \simeq V(I)$.

Definition 30. A *closed subscheme* of a scheme Y is an equivalence class of closed immersions, where $(f : X \to Y) \sim (f' : X' \to Y)$ iff there is a diagram

$$\begin{tikzcd}
& X' \arrow[d, "f'"] \\
X \arrow[ur, "\sim"] \arrow[r, "f"'] & Y.
\end{tikzcd}$$

Remark 16. A closed subset $Z \subset |Y|$ usually has many different structures as a closed subscheme. For example, take Z to be the horizontal axis in $\mathbb{A}^2_k$. This could be associated to any number of ideals, e.g. Spec $k[x,y]/y =$ Spec $k[x]$, or Spec $k[x,y]/y^2$, or more generally Spec $k[x,y]/y^n$. We think of this as an "n^{th}-order tubular neighborhood of of the axis, which we draw as a somewhat thickened line. But also we could take Spec $k[x,y]/(y^2, xy)$, which we think of as being the thin everywhere but fatter at the origin.

Proposition 13. *Suppose Y is a scheme and $Z \subset |Y|$ is a closed subset. Then there is a closed subscheme $X \to Y$ such that $|X| = Z$ and moreover X is minimal with this property: if $X' \to Y$ is a closed subscheme with $|X'| = Z$, then we have a unique factorization*

$$\begin{tikzcd}
& X' \arrow[d] \\
X \arrow[ur, dashed] \arrow[r] & Y.
\end{tikzcd}$$

If $Y = \text{Spec } A$, we take $I = \bigcap_{\mathfrak{p} \subset Z} \mathfrak{p}$. Then $\text{Spec } A/I \to \text{Spec } A$ has the required properties. (If $Z = V(J)$, then $I = \sqrt{J}$.)

4.10 Fiber products

We would like to define the pullback $X \times_Y Z$ of a corner $X \to Y \leftarrow Z$. This is just a limit.

Theorem 4. *Fiber products exist in the category of schemes.*

It is equivalent to say that the functor $h_X \times_{h_Y} h_Z : \mathbf{Schemes}^{op} \to \mathbf{Set}$ is representable (by a scheme). This is because $(h_X \times_{h_Y} h_Z)(T) = h_X(T) \times_{h_Y(T)} h_Z(T)$; an element of the latter is exactly a pair $(T \to X, T \to Z)$ such that the postcompositions to Y agree. This is what we will prove.

Remark 17. It will not be true that $|X \times_Y Z| = |X| \times_{|Y|} |Z|$. The point here is that this would certainly be true if we forgot down to **Top**, but schemes will have to carry more structure that will make there exist a scheme which as a topological space is *before* this one which is nevertheless the limit in the category of schemes.

As a warm-up, we begin with the case that $X = \text{Spec } B$, $Y = \text{Spec } C$, $Z = \text{Spec } A$. For a scheme T, recall that $\text{Hom}(T, \text{Spec } R) \simeq \text{Hom}(R, \Gamma(T, \mathcal{O}_T))$. So in this case, $h_X \times_{h_Y} h_Z$ is isomorphic to the functor

$$T \mapsto \text{Hom}(B, \Gamma(T, \mathcal{O}_T)) \times_{\text{Hom}(C, \Gamma(T, \mathcal{O}_T))} \text{Hom}(C, \Gamma(T, \mathcal{O}_T)) \cong \text{Hom}(B \otimes_C A, \Gamma(T, \mathcal{O}_T)).$$

So in this case, we obtain that $\text{Spec } B \times_{\text{Spec } C} \text{Spec } A \cong \text{Spec}(B \otimes_C A)$. (The canonical morphisms to Spec B and Spec A are given by the maps $B \overset{b \mapsto b \otimes 1}{\longrightarrow} B \otimes_C A$ and $A \overset{a \mapsto 1 \otimes a}{\longrightarrow} B \otimes_C A$.)

In general, we will give some conditions on a functor $F : \mathbf{Schemes}^{op} \to \mathbf{Set}$ for it to be representable. Roughly, these will be that it is "locally representable" and that it behaves "well" with respect to coverings of T. First, we need the following terminology.

Definition 31. A commutative square

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

is called *cartesian* if the induced map $W \to X \times_Z Y$ is an isomorphism. (This is equivalent to saying that W satisfies the universal property of fibered products.)

Now, our next definition will be motivated by the following fact. Given a functor $F : \mathbf{Schemes}^{op} \to \mathbf{Set}$, we can write $F_T : Op(T)^{op} \to \mathbf{Schemes}^{op} \to \mathbf{Set}$ for the presheaf given by considering each open $U \subset T$ as a scheme.

Definition 32. A functor $F : \mathbf{Schemes}^{op} \to \mathbf{Set}$ is called a *big Zariski sheaf* if F_T is a sheaf on T for all schemes T.

Example 53. If X is a scheme, then h_X is a big Zariski sheaf. Explicitly, for any open subset $U \subset T$ and covering $U = \bigcup U_i$, the sequence $h_X(U) \to \prod_i h_X(U_i) \rightrightarrows \prod_{i,j} h_X(U_i \cap U_j)$ is exact by definition of morphisms of schemes.

Definition 33. Fix a functor $F : \mathbf{Schemes}^{op} \to \mathbf{Set}$. A *subfunctor* $G \subset F$ is a functor $G : \mathbf{Schemes}^{op} \to \mathbf{Set}$ along with a natural transformation $\eta : G \to F$ such that for all schemes T, the function $\eta_T : G(T) \to F(T)$ is an injection of sets. G is called an *open subfunctor* if for any scheme T and any element $t \in F(T)$, the functor

$$\begin{array}{rcl} P_t : (\mathbf{Schemes}/T)^{op} & \to & \mathbf{Set} \\ (S \xrightarrow{f} T) & \mapsto & \begin{cases} \{*\}, & f^*(t) \in G(S) \\ \emptyset, & \text{otherwise.} \end{cases} \end{array}$$

is represented by some object $U \xrightarrow{\iota} T$, the inclusion of an open subscheme of T. (The category $\mathbf{Schemes}/T$ is the *overcategory* of T: its objects are maps $S \xrightarrow{f} T$ and a morphism from $S \xrightarrow{f} T$ to $S' \xrightarrow{f'} T$ is given by a morphism

$S \xrightarrow{g} S'$ such that the triangle

$$\begin{tikzcd}
& S' \arrow[d, "f'"] \\
S \arrow[ru, "g"] \arrow[r, "f"'] & T
\end{tikzcd}$$

commutes.) By the Yoneda lemma, we may consider $t \in F(T) \in \mathrm{Hom}_{\mathrm{Fun}(\mathbf{Schemes}^{op},\mathbf{Set})}(h_T, F)$, and then we can consider

$$\begin{tikzcd}
P_t = h_T \times_F G \arrow[r] \arrow[d] & G \arrow[d, "\sigma"] \\
h_T \arrow[r, "t"'] & F.
\end{tikzcd}$$

This gives us $P_t : \mathbf{Schemes}^{op} \to \mathbf{Set}$ as

$$S \mapsto h_T(S) \times_{F(S)} G(S) = \{(g : S \to T, h \in G(S)) \colon g^*t = \sigma(h)\} = \{g : S \to T \colon g^*t \in G(S) \subset F(S)\}.$$

Example 54. Suppose $U \subset X$ is an open subscheme. Then $h_U \hookrightarrow h_X$ is an open subfunctor. Indeed, suppose $t \in h_X(T)$. Then the functor $P_t : (\mathbf{Schemes}/T)^{op} \to \mathbf{Set}$ takes an object $(S \to T)$ and gives us the set of factorizations

$$\begin{tikzcd}
T' \arrow[r] \arrow[rd, dashed] & T \arrow[r, "t"] & X \\
& t^{-1}(U) \arrow[u, hook] \arrow[r, "t|_{t^{-1}(U)}"'] & U \arrow[u, hook]
\end{tikzcd}$$

(which since $U \subset X$ is of size either 0 or 1). So $(t^{-1}(U) \hookrightarrow T) \in (\mathbf{Schemes}/T)^{op}$ does indeed represent P_t.

Definition 34. A map $H \to F$ of big Zariski sheaves is called *surjective* if for all T, the map $H_T \to F_T$ is a sheaf epimorphism.

The following theorem will be the key ingredient in proving that **Schemes** has fibered products.

Theorem 5. *A functor $F : \mathbf{Schemes}^{op} \to \mathbf{Set}$ is representable (i.e. $F \simeq h_X$ for some $X \in \mathbf{Schemes}^{op}$) iff the following two conditions hold:*

1. *F is a big Zariski sheaf.*
2. *There is a collection of open subfunctors $H_i \subset F$ with each H_i representable and the map $\coprod H_i \to F$ a surjection of big Zariski sheaves.*

Before proving the theorem, we investigate its implications for the particular case we care about.

Example 55. Suppose we have a corner $X \xrightarrow{a} Y \xleftarrow{b} Z$ in **Schemes**. Define the functor $F : \mathbf{Schemes}^{op} \to \mathbf{Set}$ by $F = h_X \times_{h_Y} h_Z$, i.e. $F(T) = \{(T \xrightarrow{f} X, T \xrightarrow{g} Y) \colon a \circ f = b \circ g\}$. The theorem then implies that this is representable.

Condition 1 is obvious: we just check the sheaf property. For condition 2, suppose that Spec $A \subset Y$ is affine and that Spec $B \subset a^{-1}(\text{Spec } A)$, Spec $C \subset b^{-1}(\text{Spec } A)$. Then we have the diagram

$$\begin{tikzcd}
 & \text{Spec } B \arrow[r, hook] \arrow[d] & X \arrow[dd, "a"] \\
\text{Spec } C \arrow[r] \arrow[d, hook] & \text{Spec } A \arrow[dr, hook] & \\
Z \arrow[rr, "b"'] & & Y.
\end{tikzcd}$$

Let $G = h_{\text{Spec } C} \times_{h_{\text{Spec } A}} h_{\text{Spec } B}$. To see that this is an open subfunctor, suppose that $t \in F(T)$. Then $t = (T \xrightarrow{f} X, T \xrightarrow{g} Z)$ such that $a \circ f = b \circ g$. This gives an element of $G(T)$ exactly when $f^{-1}(\text{Spec } B) = g^{-1}(\text{Spec } C) = T$. Thus, P_t is represented by $f^{-1}(\text{Spec } B) \cap g^{-1}(\text{Spec } C)$. Moreover, G is representable: $G \simeq h_{\text{Spec } B \otimes_A C}$.

Now, choose a covering $Y = \bigcup_i \text{Spec } A_i$, and then choose coverings $a^{-1}(\text{Spec } A_i) = \bigcup_j \text{Spec } B_{ij}$ and $b^{-1}(\text{Spec } A_i) = \bigcup_k \text{Spec } C_{ik}$. Certainly $H_{ijk} = h_{\text{Spec } C_{ik}} \times_{h_{\text{Spec } A_i}} h_{\text{Spec } B_{ij}}$ is a representable open subfunctor of F. We need only check that $\coprod_{ijk} H_{ijk} \to F$ is a surjection of big Zariski sheaves. For this, suppose we have a scheme T and an element $t \in F(T)$, i.e. $t = (T \xrightarrow{f} X, T \xrightarrow{g} Y)$ such that $a \circ f = b \circ g$. Then $T = \bigcup_{ijk}(f^{-1}(\text{Spec } B_{ij}) \cap g^{-1}(\text{Spec } C_{ik}))$. Let $T_{ijk} = f^{-1}(\text{Spec } B_{ij}) \cap g^{-1}(\text{Spec } C_{ik})$. Then $t|_{T_{ijk}} = (f, g)|_{T_{ijk}} \in H_{ijk}(T_{ijk})$. So the map is a surjection.

Proof. We first give a sketch of the proof, then we fill in the gluing-related details, and then we finish the proof carefully.

Suppose that the two conditions are satisfied, i.e. F is a big Zariski sheaf onto which surjects the disjoint union $\coprod H_i$ of some collection of representable open subfunctors $H_i \subset F$. Write $H_i = h_{U_i}$. Then define the pullback diagram

$$\begin{tikzcd}
P_{ij} \arrow[r] \arrow[d] & h_{U_i} \arrow[d] \\
h_{U_j} \arrow[r] & F,
\end{tikzcd}$$

i.e. glue together the U_i to get a scheme X such that $h_X \simeq F$. The idea is that P_{ij} should be something like $h_{U_i \cap U_j}$.

On the other hand, suppose that $F \simeq h_X$. Clearly F is a big Zariski sheaf. By the Yoneda lemma, $h_{U_j} \to F$ corresponds to an element $u_j \in F(U_j)$. If we set $P_{ij} = P_{U_j}$ from before (i.e. P_t applied to the data $(h_{U_j}, u_j \in F(U_j))$), then $P_{ij} = h_{V_{ij}}$ with $V_{ij} \hookrightarrow U_j$ open, and also $P_{ij} = h_{V_{ji}}$ with $V_{ji} \hookrightarrow U_i$ open. Thus we have the diagrams

$$\begin{tikzcd}
V_{ij} \arrow[r, "\varphi"] \arrow[d, hook] & V_{ji} \arrow[d, hook] \\
U_j & U_i,
\end{tikzcd}$$

which look suspiciously like the gluing conditions. In fact this does work out, of course, and so we obtain the desired open subfunctors H_i.

Now, as for gluing, the key diagram to keep in mind is

$$\begin{tikzcd}
h_{U_{ij}} = h_{U_i} \times_F h_{U_j} \arrow[r] \arrow[d] & h_{U_i} \simeq H_i \arrow[d, "u_i \in F(U_i)"] \\
h_{U_j} \simeq H_j \arrow[r, "u_j \in F(U_j)"'] & F
\end{tikzcd}$$

of functors. Recall how we glue: we need a family $\{U_i\}$ of schemes such that

- for all $i \neq j$ we have open subsets $\lambda_i : U_{ij} \hookrightarrow U_i$ and isomorphisms $\varphi_{ij} : U_{ij} \overset{\sim}{\to} U_{ji}$;
- $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ and $\varphi_{ji} = \varphi_{ij}^{-1}$;
- the diagram

$$\begin{tikzcd}
U_{ij} \cap U_{ik} \arrow[r, "\varphi_{ij}"] \arrow[dr, "\varphi_{ik}"'] & U_{ji} \cap U_{jk} = U_{jk} \cap U_{ji} \arrow[d, "\varphi_{jk}"] \\
 & U_{kj} \cap U_{ki}
\end{tikzcd}$$

commutes.

The upper map $h_{U_{ij}} \to h_{U_i}$ in the diagram of functors above will just be λ_i, and the isomorphisms

$$h_{U_{ji}} \simeq H_i \times_F H_j \simeq H_j \times_F H_i \simeq h_{U_{ij}}$$

determine $\varphi_{ij} : h_{U_{ij}} \to h_{U_{ji}}$, which does indeed make the diagram

$$\begin{tikzcd}
h_{U_{ji}} \arrow[d, hook, "\lambda_j"'] & h_{U_{ij}} \arrow[l, "\varphi_{ij}"'] \arrow[d, hook, "\lambda_i"] \\
h_{U_j} \arrow[dr, hook] & h_{U_i} \arrow[d, hook] \\
 & F.
\end{tikzcd}$$

commute.

We claim that the hypotheses necessary for gluing are implied by the commutativity of these diagrams. Indeed, the most recent diagram implies that φ_{ij} is the unique isomorphism which is compatible with the inclusions as subfunctors of F. Now, note that

$$h_{U_{ij} \cap U_{ik}} \simeq h_{U_{ij}} \times_{h_{U_i}} h_{U_{ik}} = (H_j \times_F H_i) \times_{H_i} (H_k \times_F H_i) = H_j \times_F H_k \times_F H_i.$$

More precisely, the inclusion $h_{U_{ij} \cap U_{ik}} \hookrightarrow h_{U_{ij}} \hookrightarrow h_{U_i} \hookrightarrow F$ identifies $h_{U_{ij} \cap U_{ik}}$ with $H_i \times_F H_j \times_F H_k \subset F$. So to check the second hypothesis for gluing, we observe that in the diagram

$$\begin{tikzcd}
h_{U_{ij} \cap U_{ik}} \arrow[r, dashed] \arrow[d, hook] & h_{U_{ji} \cap U_{jk}} \arrow[d, hook] \\
h_{U_{ij}} \arrow[r, "\varphi_{ij}"] \arrow[dr, hook] & h_{U_{ji}} \arrow[d, hook] \\
 & F,
\end{tikzcd}$$

the dotted arrow exists and is an isomorphism since both compositions down to F map to $H_i \times_F H_j \times_F H_k \subset F$. (Of course, this must be unique.) For the last hypothesis for gluing, we must check that the composition

$$h_{U_{ij} \cap U_{ik}} \xrightarrow{\varphi_{ij}} h_{U_{ji} \cap U_{ki}} = h_{U_{jk} \cap U_{ji}} \xrightarrow{\varphi_{jk}} h_{U_{kj} \cap U_{ki}}$$

is equal to φ_{ik}. To check this, we map every object of this diagram to F via its specified inclusion, and every triangle commutes. Since both the displayed composition and φ_{ik} both the triangle

$$\begin{tikzcd}
h_{U_{ij}\cap U_{ik}} \arrow[r, dashed] \arrow[dr, hook] & h_{U_{kj}\cap U_{ki}} \arrow[d, hook] \\
 & F
\end{tikzcd}$$

commute, they must be equal.

So, we get a scheme X by gluing the U_i. We need to show that $h_X \simeq F$. To do this, we will show that both are equalizers in the category of big Zariski sheaves of the diagram $\coprod H_i \times_F H_j \rightrightarrows \coprod H_i$.

First of all, $X = \bigcup_i U_i$. For each i we have the identity element $u_i \in h_{U_i}(U_i) \xrightarrow{\sim} H(U_i) \subset F_X(U_i)$ (recall that $F_X = F|_{Op(X)}$). So, we have the sequence

$$F_(X) \to \prod_i F_X(U_i) \rightrightarrows \prod_{i,j} F_X(U_i \cap U_j).$$

For all i, j, $u_i|_{U_i \cap U_j} = u_j|_{U_i \cap U_j} \in F_X(U_i \cap U_j)$. Indeed, recall that we defined the U_{ij} via the commutative diagram

$$\begin{tikzcd}
h_{U_{ij}} \arrow[r, equal] & H_j \times_F h_{U_i} \arrow[r, "\sim"] \arrow[d, hook] & H_j \times_F H_i & h_{U_j} \times_F H_i \arrow[l, "\sim"'] \arrow[d, hook] \arrow[r, equal] & h_{U_{ji}} \\
 & h_{U_i} \arrow[dr, hook, "u_i"'] & & h_{U_j} \arrow[dl, hook', "u_j"] & \\
 & & F. & &
\end{tikzcd}$$

Since F is a big Zariski sheaf, the F_X is a sheaf and the above sequence is exact, so $(u_i) \in \prod_i F_X(U_i)$ is induced by a unique element $x \in F_X(X)$. We can think of this as a morphism $x : h_X \to F$ of big Zariski sheaves, which we must show is an isomorphism to complete the proof (again by the Yoneda lemma). For this, it is enough to show that for any scheme T, $h_X|_{Op(T)} \to F_T$ is an isomorphism. To see that it is an epimorphism, we refer to the commutative diagram

$$\begin{tikzcd}
\coprod (h_{U_i}|_{Op(T)}) \arrow[r, "\sim"] \arrow[d] & \coprod (H_i|_{Op(T)}) \arrow[d] \\
h_X|_{Op(T)} \arrow[r, dashed] & F_T.
\end{tikzcd}$$

To see that it is a monomorphism, given two maps $f, g \in h_X(T)$ such that $f^*x = g^*x \in F(T)$, we must show that $f = g$. Note that

$$\begin{tikzcd}
H_i \times_F h_X \arrow[r] \arrow[d] & h_X \arrow[d, "x"] \\
H_i \arrow[r] & F,
\end{tikzcd}$$

identifies $H_i \times_F h_X$ with h_{U_i}. This means that $f^{-1}(U_i) = f^{-1}(U_i)$ for all i, since we have the diagram of fibered products

$$\begin{tikzcd}
h_{f^{-1}(U_i)} \arrow[r] \arrow[d] & h_T \arrow[d, "f"] \\
H_i \times_F h_X \arrow[r] \arrow[d] & h_X \arrow[d, "x"] \\
H_i \arrow[r] & F,
\end{tikzcd}$$

but $h_{g^{-1}(U_i)}$ could equally well sit as the pullback in the big rectangle. (In general, to show that two open subsets are the same, it suffices to show that they are the same when we intersect them with a covering.) So we are reduced to the case that we can factor $f, g : T \to U_i \hookrightarrow X$. In this case, the diagram

$$\begin{tikzcd}
T \arrow[r, "f", "g"'] \arrow[d, equal] & U \arrow[r, hook] \arrow[d] & X \\
H_i \arrow[r, hook] & F &
\end{tikzcd}$$

proves that we must have $f = g$. ☐

Let us return to some examples of fibered products of schemes.

Example 56. Suppose we have a corner $U \overset{j}{\hookrightarrow} Y \overset{f}{\leftarrow} X$ where j is an open immersion. Then we have a fibered product $X \times_Y U = f^{-1}(U)$. Note that this simple example actually requires no appeal to the above construction.

Example 57 (the "fiber")**.** Given a morphism $f : X \to Y$ of schemes, we would like to talk about the "fiber" over a point $y \in Y$. The stalk $\mathcal{O}_{Y,y}$ is a local ring with some unique maximal ideal $\mathfrak{m}$, and so we have the field $k(y) = \mathcal{O}_{Y,y}/\mathfrak{m}$. This gives us a morphism Spec $k(y) \to Y$ selecting the point y. More precisely, if $y \in \text{Spec } A \subset Y$, then this point corresponds to some $\mathfrak{p} \subset A$, and the map on rings is just given by $k(y) = Frac(A/\mathfrak{p}) \leftarrow A$. The *(scheme-theoretic) fiber*, often denoted X_y, is by definition the fibered product

$$\begin{tikzcd}
\text{Spec } k(y) \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\
\text{Spec } k(y) \arrow[r] & Y.
\end{tikzcd}$$

For example, we have a map Spec $k[z] = \mathbb{A}^1_k \to \mathbb{A}^1_k =$ Spec $k[x]$ given by $z^2 \leftarrow x$, the "branched double cover". Let us compute the fiber over $1 = (x-1) \in \text{Spec } k[x]$. The stalk here is k, so we get

$$\begin{tikzcd}
k[z]/(z^2-1) & k[z] \arrow[l] \\
k \arrow[u] & k[x]. \arrow[l] \arrow[u]
\end{tikzcd}$$

Thus, if char $k \neq 2$, then we get fiber Spec $k[z]/(z-1)(z+1)$, i.e. the two points ± 1. If char $k = 2$, then we get Spec $k[z]/(z-1)^2 \cong$ Spec $k[\varepsilon]/\varepsilon^2$. Abusing notation, we might denote this as $f^{-1}(1)$.

Similarly, we could calculate that in any characteristic, $f^{-1}(0) =$ Spec $k[z]/z^2$.

If we compute the fiber over the generic point, we have the cartesian diagram

$$\begin{tikzcd}
\operatorname{Spec} k(z) \arrow[r, equal] & \operatorname{Spec} k(x) \otimes_{k[x]} k[z] \arrow[r] \arrow[d] & \operatorname{Spec} k[z] \arrow[d] \\
 & \operatorname{Spec} k(x) \arrow[r] & \operatorname{Spec} k[x].
\end{tikzcd}$$

(If we invert even-degree monomials in z, we invert z itself.)

Example 58 (Prof. Olsson's favorite statement of Galois theory)**.** Suppose we have a finite separable extension $L \to K$ of fields. This gives us Spec $K \to$ Spec L, a map of one-point spaces which on the face of it doesn't look that interesting. But let's take an embedding $L \hookrightarrow L^{sep}$ into a separable closure. This gives us the diagram

$$\begin{tikzcd}
\prod_{\sigma: K \hookrightarrow L^{sep}} L^{sep} \arrow[r, equal, "\sim", "\text{(over L)}"'] & L^{sep} \otimes_L K \arrow[r] & K \\
 & L^{sep} \arrow[u] & L. \arrow[l] \arrow[u]
\end{tikzcd}$$

Thus the fiber of the map Spec $K \to$ Spec L is $\coprod_{\sigma: K \hookrightarrow L^{sep}}$ Spec L^{sep}. The map $L^{sep} \otimes_L K \to \prod_{\sigma: K \hookrightarrow L^{sep}} L^{sep}$ is given by $a \otimes b \mapsto (a \cdot \sigma(b))_\sigma$, and Galois theory tells us things about the injectivity and surjectivity of this map.

Example 59 (the blowup)**.** Consider the scheme-theoretic product $\mathbb{A}^2 \times \mathbb{P}^1$ (over a field, or even over a base ring). Take coordinates (x, y) and $[u, v]$ on the factors. We define the *blowup* of the plane at the origin to be cut out globally as $X = V(xv = yu)$. More precisely, we have the decomposition $\mathbb{A}^2 \times \mathbb{P}^1 = (\mathbb{A}^2_{(x,y)} \times \mathbb{A}^1_{u/v}) \cup (\mathbb{A}^2_{(x,y)} \times \mathbb{A}^1_{v/u})$, where the intersection is $\mathbb{A}^2_{(x,y)} \times \mathbb{G}_m$. It is with respect to this decomposition that we can actually locally define X. To be completely clear, we have the diagram

$$\begin{tikzcd}
\operatorname{Spec} k[x, y, (u/v)^{\pm}] \arrow[r, "(u/v) \mapsto (v/u)^{-1}"] \arrow[d, hook] & \operatorname{Spec} k[x, y, (v/u)^{\pm}] \arrow[d, hook] \\
\operatorname{Spec} k[x, y, u/v] & \operatorname{Spec} k[x, y, v/u].
\end{tikzcd}$$

Locally, X is cut out in Spec $k[x, y, u/v]$ by $x = y \cdot (u/v)$ and in Spec $k[x, y, v/u]$ by $x \cdot (v/u) = y$. These do indeed agree along the gluing:

$$\begin{tikzcd}
\operatorname{Spec} \frac{k[x,y,(u/v)^{\pm}]}{x = y \cdot (u/v)} \arrow[r] \arrow[d, hook] & \operatorname{Spec} \frac{k[x,y,(v/u)^{\pm}]}{x \cdot (v/u) = y} \arrow[d, hook] \\
\operatorname{Spec} \frac{k[x,y,u/v]}{x = y \cdot (u/v)} & \operatorname{Spec} \frac{k[x,y,v/u]}{x \cdot (v/u) = y}.
\end{tikzcd}$$

Observe that the projection $\mathbb{A}^2 \times \mathbb{P}^1 \to \mathbb{P}^1$ is locally given by killing x and y, so the restriction $X \to \mathbb{P}^1$ is given locally by e.g.

$$\begin{array}{ccc}
\operatorname{Spec} \dfrac{k[x, y, u/v]}{x = y \cdot (u/v)} & \to & \operatorname{Spec} k[u/v] \\
\dfrac{k[x, y, u/v]}{x = y \cdot (u/v)} & \hookleftarrow & k[u, v].
\end{array}$$

The first scheme is isomorphic to Spec $k[y, u/v] \simeq \mathbb{A}^1_{k[u/v]}$. So this is "the affine line over the affine line" – this will end up being the *tautological line bundle* $\mathcal{O}(-1)$ over $\mathbb{P}^1$.

Now consider the restriction of the projection $\mathbb{A}^2 \times \mathbb{P}^1 \to \mathbb{A}^2$ to X. This is given locally by e.g.

$$\begin{tikzcd}
\operatorname{Spec} \dfrac{k[x,y,u/v]}{x = y \cdot (u/v)} \arrow[r] & \operatorname{Spec} k[x,y] \\
\dfrac{k[x,y,u/v]}{x = y \cdot (u,v)} & k[x,y], \arrow[l, hook']
\end{tikzcd}$$

Over $D(x) \cap D(y)$, this looks like

$$\begin{tikzcd}
\operatorname{Spec} \dfrac{k[x^{\pm}, y^{\pm}, (u/v)^{\pm}]}{x = y \cdot (u/v)} \arrow[r] & \operatorname{Spec} k[x^{\pm}, y] \\
k[x^{\pm}, y^{\pm}] & k[x^{\pm}, y]. \arrow[l, hook']
\end{tikzcd}$$

Over just $D(x)$, this looks like

$$\begin{tikzcd}
\operatorname{Spec} \dfrac{k[x,y,v/u]}{x \cdot (v/u) = y} \arrow[r] & \operatorname{Spec} k[x,y].
\end{tikzcd}$$

Finally, over $0 \in \mathbb{A}^2$ we have $x = y = 0$, so the map is locally $\operatorname{Spec} k[(u/v)^{\pm}] \xrightarrow{\sim} \operatorname{Spec} k[(v/u)^{\pm}]$, and so these glue to $\mathbb{P}^1$.

Remark 18. In the above example, we should clarify the difference between the scheme-theoretic fiber and the set-theoretic fiber. The scheme-theoretic fiber is defined over a point of $\mathbb{A}^2_k$ by mapping Spec k onto that point and taking the pullback $X \times_{\mathbb{A}^2_k} \operatorname{Spec} k$. When we select the origin $0 \in \mathbb{A}^2_k$, this gives us $\mathbb{P}^1$. So we need that the set-theoretic fiber is just the points of the fibered product. More generally, if $\mathfrak{p}$ is associated to a point $x \in \operatorname{Spec} A$, then we would look at the cartesian diagram

$$\begin{tikzcd}
\operatorname{Spec} S \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\
\operatorname{Spec} k(x) \arrow[r] & \operatorname{Spec} A,
\end{tikzcd}$$

so that when we reverse into **Rings** we get

$$\begin{tikzcd}
S \arrow[d] & B/\mathfrak{p}B \arrow[l] & B \arrow[l] \\
Frac(A/\mathfrak{p}) & A/\mathfrak{p} \arrow[l] \arrow[u] & A. \arrow[l] \arrow[u]
\end{tikzcd}$$

That is, the prime ideals in S are in bijection with the prime ideals of B which contain the image of $\mathfrak{p}$ in B.

We formalize this as follows. Suppose that we have a morphism $f : X \to Y$ of schemes and $y \in Y$ is a point. If we write $X_y = \operatorname{Spec} k(y) \times_Y X$ (here $k(y) = \mathcal{O}_{Y,y}/\mathfrak{m}_y$, the fraction field of the stalk at y), then we claim that $f^{-1}(y)$ is in bijection with $|X_y|$. More precisely, in the diagram

$$\begin{tikzcd}
X_y \arrow[r, "g"] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec} k(y) \arrow[r, "y"'] & Y
\end{tikzcd}$$

the map g induces the bijection $|X_y| \to f^{-1}(y)$.

First, we can assume $Y = \operatorname{Spec} A$, since any point of Y is contained in an affine neighborhood. If g induces a bijection $|V_y| \to f^{-1}(y) \cap V$ for every affine $V \subset X$, then g induces the desired bijection $|X_y| \to f^{-1}(y)$. (We can check bijectivity affine-locally in the target; by construction, $V_y = g^{-1}(V)$.) So it suffices to consider $X = \operatorname{Spec} B$, too. Now, the proof of this statement for $\operatorname{Spec} B \to \operatorname{Spec} A$ is exactly given by the argument above.

5 Quasicoherent sheaves

5.1 Definitions and basic facts

Recall that if R is a ring and $(f_1, \ldots, f_r) = R$, then for any R-module M, the sequence

$$M \to \prod_i M_{f_i} \rightrightarrows \prod_{i,j} M_{f_i f_j}$$

is exact. So, we get a presheaf on the basic opens in Spec R by the association $D(f) \mapsto M_f$. In fact, this gives a *sheaf* on Spec R, which we denote $\widetilde{M}$. In fact, this is not just a sheaf of abelian groups, but comes with the structure of an $\mathcal{O}_{\text{Spec } R}$*-module*, which we now define precisely.

Definition 35. If $(X, \mathcal{O}_X)$ is a ringed space, an $\mathcal{O}_X$*-module* is a sheaf of abelian groups M and a map $\varphi : \mathcal{O}_X \times M \to M$ of sheaves (of sets) such that for all open $U \subset X$, φ makes $M(U)$ into an $\mathcal{O}_X(U)$-module.

Continuing from above, for every $f \in R$, we have that $M_f = \widetilde{M}(D(f))$ is an R_f-module, and if we restrict further to $D(g) \subset D(f)$ then the restriction maps are compatible with the module structure, i.e. the diagram

$$\begin{tikzcd}
R_f \times M_f \arrow[r, "\varphi_{D(f)}"] \arrow[d, "\text{localize}"'] & M_f \arrow[d, "\text{localize}"] \\
R_{fg} \times M_{fg} \arrow[r, "\varphi_{D(fg)}"'] & M_{fg}
\end{tikzcd}$$

commutes.

Thus we have a category $\mathbf{Mod}_{\mathcal{O}_X}$ of $\mathcal{O}_X$-modules. This has the following extra structure.

- Recall that a homework assignment defined an *internal hom*: for any $\mathcal{F}, \mathcal{G} \in \mathbf{Mod}_{\mathcal{O}_X}$, we have $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ given by $U \mapsto \operatorname{Hom}_{\mathbf{Mod}_{\mathcal{O}_X}}(\mathcal{F}|_U, \mathcal{G}|_U)$.
- Recall that *tensor product* is always meant to be left-adjoint to internal hom. This category has a tensor product, which we will denote $\otimes_{\mathcal{O}_X}$, which is given explicitly by declaring that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheafification of the presheaf tensor product,

  $$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U) = (\mathcal{F}(U) \otimes^{PSh}_{\mathcal{O}_X(U)} \mathcal{G}(U))^a.$$

  In the special case that $X =$ Spec R and M is an R-module, it turns out that $\widetilde{M} = \underline{M} \otimes_{\underline{R}} \mathcal{O}_X$, where underlines denote the constant sheaf. (This is because for any $f \in R$, $M_f = M \otimes_R R_f$. For this construction we are using the ringed space $(|\text{Spec } R|, \underline{R})$.)

Proposition 14. *Let $(X, \mathcal{O}_X)$ be a ringed space and let $R = \Gamma(X, \mathcal{O}_X)$. Then the functor*

$$\begin{array}{ccc} \mathbf{Mod}_{\mathcal{O}_X} & \longrightarrow & \mathbf{Mod}_R \\ \mathcal{F} & \mapsto & \Gamma(X, \mathcal{F}) \end{array}$$

has a left adjoint, namely $M \mapsto \underline{M} \otimes_{\underline{R}} \mathcal{O}_X$.

Proof. Recall that if $\varphi : A \to B$ is a ring homomorphism with $M \in \mathbf{Mod}_A$ and $N \in \mathbf{Mod}_B$, then

$$\begin{array}{ccc} \operatorname{Hom}_{\mathbf{Mod}_A}(M, N) & \xrightarrow{\sim} & \operatorname{Hom}_{\mathbf{Mod}_B}(M \otimes_A B, N) \\ (f : M \to N) & \mapsto & (m \otimes b \mapsto b \cdot f(m)). \end{array}$$

(The inverse is $(g : M \otimes_A B \to N) \mapsto (m \mapsto g(m \otimes 1))$.) The analogous statement in categories of sheaves is that if $\varphi : \mathcal{A} \to \mathcal{B}$ is a map of sheaves of rings on a topological space X with $\mathcal{F} \in \mathbf{Mod}_{\mathcal{A}}$ and $\mathcal{G} \in \mathbf{Mod}_{\mathcal{B}}$, then

$$\operatorname{Hom}_{\mathbf{Mod}_{\mathcal{A}}}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{Mod}_{\mathcal{B}}}(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{B}, \mathcal{G})$$

via the same formula. To see this, we recall that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = (\mathcal{F} \otimes^{PSh}_{\mathcal{O}_X} \mathcal{G})^a$, the sheafification of the presheaf tensor product, and so we calculate

$$\mathrm{Hom}_{\mathbf{Mod}_{\mathcal{B}}}(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{B}, \mathcal{G}) \cong \mathrm{Hom}_{\mathbf{Mod}^{PSh}_{\mathcal{B}}}(\mathcal{F} \otimes^{PSh}_{\mathcal{A}} \mathcal{B}, \mathcal{G}) \cong \mathrm{Hom}_{\mathbf{Mod}^{PSh}_{\mathcal{A}}}(\mathcal{F}, \mathcal{G}).$$

Indeed, we have a map $\mathcal{F} \to \mathcal{F} \otimes^{PSh}_{\mathcal{A}} \mathcal{B}$ which induces the map, which for all U gives the map

$$\mathrm{Hom}_{\mathbf{Mod}_{\mathcal{B}(U)}}(\mathcal{F}(U) \otimes_{\mathcal{A}(U)} \mathcal{B}(U), \mathcal{G}(U)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Mod}_{\mathcal{A}(U)}}(\mathcal{F}(U), \mathcal{G}(U)).$$

Now, we compute that

$$\mathrm{Hom}_{\mathbf{Mod}_{\mathcal{O}_X}}(\underline{M} \otimes_{\underline{R}} \mathcal{O}_X, \mathcal{F}) \cong \mathrm{Hom}_{\mathbf{Mod}_{\underline{R}}}(\underline{M}, \mathcal{F}) \cong \mathrm{Hom}_R(M, \Gamma(X, \mathcal{F})).$$

Here, the first isomorphism is associated to the canonical map $\underline{R} \to \mathcal{O}_X$ of sheaves of rings on X guaranteed by the universal property of a constant sheaf. The second isomorphism comes directly from that same universal property. ☐

Remark 19. Note that we can't define internal hom by $U \mapsto \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U), \mathcal{G}(U))$, because this has no canonical restriction map to $\mathrm{Hom}_{\mathcal{O}_X(V)}(\mathcal{F}(V), \mathcal{G}(V))$ for an open subset $V \subset U$.

Remark 20. If $X = \text{Spec } R$ and $M \in \mathbf{Mod}_R$ gives $\widetilde{M}$, then $\Gamma(X, \widetilde{M}) = M$. In fact, we can strengthen this.

Proposition 15. *The functor*

$$\begin{array}{ccc} \mathbf{Mod}_R & \longrightarrow & \mathbf{Mod}_{\mathcal{O}_{\text{Spec } R}} \\ M & \mapsto & \widetilde{M} \end{array}$$

is fully faithful.

Proof. Given $M, N \in \mathbf{Mod}_R$, we have that

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Mod}_{\mathcal{O}_X}}(\widetilde{M}, \widetilde{N}) &\cong \mathrm{Hom}_{\mathbf{Mod}_{\mathcal{O}_X}}(\underline{M} \otimes_{\underline{R}} \mathcal{O}_X, \widetilde{N}) \\ &\cong \mathrm{Hom}_{\mathbf{Mod}_{\underline{R}}}(\underline{M}, \widetilde{N}) \\ &\cong \mathrm{Hom}_{\mathbf{Mod}_R}(M, \Gamma(X, \widetilde{N})) \\ &\cong \mathrm{Hom}_{\mathbf{Mod}_R}(M, N). \end{aligned}$$

☐

Now, just as we glued affine schemes to get general schemes, we can glue sheaves on affine schemes to get general sheaves.

Definition 36. Let X be a scheme. Then an $\mathcal{O}_X$-module $\mathcal{F}$ is called *quasi-coherent* if there is an open covering $X = \bigcup U_i$ with $U_i \cong \text{Spec } R_i$ for each i, such that $\mathcal{F}_{U_i} \cong \widetilde{M_i}$ for some $M_i \in \mathbf{Mod}_{R_i}$.

Proposition 16. *If $\mathcal{F}$ is quasi-coherent, then for any open affine* $\text{Spec } A \subset X$*, in fact* $\mathcal{F}|_{\text{Spec } A} = \widetilde{\mathcal{F}(\text{Spec } A)}$.

Proof. Given $\mathcal{F} \in QCoh_{\text{Spec } R}$, write $M = \Gamma(\text{Spec } R, \mathcal{F}) = \mathcal{F}(\text{Spec } R)$. We then have a natural map $\widetilde{M} \to \mathcal{F}$ given by

$$\mathrm{Hom}_{\mathcal{O}_X}(\underline{M} \otimes_{\underline{R}} \mathcal{O}_X, \mathcal{F}) = \mathrm{Hom}_{\underline{R}}(\underline{M}, \mathcal{F}) = \mathrm{Hom}_R(M, \Gamma(\text{Spec } R, \mathcal{F})) = \mathrm{Hom}_R(M, M).$$

Given any $f \in R$, we obtain the commutative diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d] & \Gamma(\text{Spec } R, \mathcal{F}) \arrow[d] \\ M_f \arrow[r, dashed, "\exists!"] & \mathcal{F}(D(f)), \end{tikzcd}$$

where $\mathcal{F}(D(f))$ is an R_f-module. It will suffice to show two things:

- injectivity: if $m \in M$ maps to $0 \in \mathcal{F}(D(f))$, then there is some $r \geq 1$ such that $f^r m = 0$;
- surjectivity: if $x \in \mathcal{F}(D(f))$, there is some $r \geq 1$ such that $f^r x$ extends to a section in $\Gamma(\text{Spec } R, \mathcal{F})$.

So choose generators $(g_1, \ldots, g_s) = R$, and write $\mathcal{F}|_{D(g_i)} = \widetilde{M_i}$, where M_i is an R_{g_i}-module. To check injectivity, since

$$D(f) \cap D(g) = D(fg_i) \subset D(g_i) \subset \text{Spec } R,$$

we have that

$$\mathcal{F}(D(f)) \hookrightarrow \prod_i \mathcal{F}(D(fg_i)) = \prod_i M_{i,f}.$$

So if $m \in M$ maps to 0 in each $M_{i,f}$, then if we write $y_i \in M_i$ for the images of m under the above map, then there exist $n_i \geq 1$ such that $f^{n_i} \cdot y_i = 0$ in M_i. So, we can take $r = \max\{n_i\}$.

To check surjectivity, observe that we have the diagram

$$\begin{tikzcd}
 & \mathcal{F}(D(f)) \arrow[r, hook] & \prod_i M_{i,f} \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j} M_{i,g_j f} \\
M \arrow[r, equal] & \mathcal{F}(\text{Spec } R) \arrow[r] \arrow[u] & \prod_i M_i \arrow[r, shift left] \arrow[r, shift right] \arrow[u] & \prod_{i,j} M_{i,g_j} \arrow[u]
\end{tikzcd}$$

of equalizer sequences. Our element x lives in $\mathcal{F}(D(f))$, and pushes forward to some (x_i) in $\prod_i M_{i,f}$. So we begin by choosing r_1 such that there exist $z_i \in M_i$ such that $z_i \mapsto f^{r_1} x_i$ for all i. However, the vector (z_i) may not map to the same element of $\prod_{i,j} M_{i,g_j}$ under the two maps. But they map to the same element of $M_{i,g_j f}$, so there is some r_2 such that $f^{r_2}(z_i - z_j) = 0$ in M_{i,g_j}. So there exists some $w \in M$ mapping to $(f^{r_2} z_i) \in \prod_i M_i$, and w maps to $f^{r_1+r_2} m$ in $\mathcal{F}(D(f))$ since $(f^{r_2} z_i) \mapsto (f^{r_1+r_2} x_i)$.

(A quicker way to see surjectivity is just to localize the bottom row; since localization is an exact functor, this gives us that $M_f \cong \mathcal{F}(D(f))$. The above prove is essentially just explicitly checking this fact.) ☐

Corollary 1. *If R is a ring, then the tilde-construction gives an equivalence of categories $\mathbf{Mod}_R \simeq QCoh_{\text{Spec } R}$ between R-modules and quasi-coherent sheaves over* Spec R.

Corollary 2. *If X is a scheme and $\mathcal{F}$ is quasi-coherent, then for every affine $U \in X$, $\mathcal{F}|_U \cong \widetilde{\mathcal{F}(U)}$.*

Thus we might say that "quasi-coherent sheaf" is the global version of "module".

Definition 37. If X is locally noetherian, a quasi-coherent sheaf is called *coherent* if there exists a cover $X = \bigcup_i \text{Spec } R_i$ such that $\mathcal{F}(\text{Spec } R_i)$ is a finitely generated R_i-module.

Remark 21. One can make this definition without local noetherianity, but the category of finitely-generated modules over non-noetherian rings does not behave as one might expect. (An R-module is called *finitely presented* if there is an exact sequence $R^n \to R^m \to M \to 0$. When R is not noetherian, being finitely generated does not imply being finitely presented.)

Example 60. Let $R = k[x_1, x_2, \ldots]$, and take $M = k$ given by $x_i \mapsto 0$ for all i. Thus M is a finitely generated module, but it is not finitely presented. (To prove this, we'd have to consider *any* surjection $R^m \to M$.)

Now, let $f : X \to Y$ be a morphism of schemes. We have the functor $f_* : \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_Y}$, and $f_*\mathcal{F}$ is a module over $f_*\mathcal{O}_X$ which itself admits the structure map $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$, so in this way $f_*\mathcal{F}$ becomes a module over $\mathcal{O}_Y$.

This functor has a left adjoint $f^* : \mathbf{Mod}_{\mathcal{O}_Y} \to \mathbf{Mod}_{\mathcal{O}_X}$. The first thing we might do with some $\mathcal{G} \in \mathbf{Mod}_{\mathcal{O}_Y}$ is pull it back: this gives us $f^{-1}\mathcal{G}$, the pullback *as sheaves of sets*. This is an $f^{-1}\mathcal{O}_Y$-module, and we have an adjoint structure map $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. So, the only thing we can do (and luckily this works out!) is to define $f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. This is the sheafification of the presheaf tensor product.

Theorem 6.

1. *For any $f : X \to Y$, $f^* : Qcoh(Y) \to Qcoh(X)$, i.e. if $\mathcal{G} \in \mathbf{Mod}_{\mathcal{O}_Y}$ is quasi-coherent, then so is $f^*\mathcal{G}$.*
2. *If $f : X \to Y$ is quasicompact and quasiseparated, then $f_* : Qcoh(X) \to Qcoh(Y) \subset Sh(Y)$.*

Example 61 (A morphism where the pushforward of a quasicoherent sheaf is not quasicoherent)**.** Consider the morphism $f : X = \coprod_{z\in I} \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ which is the identity map on each component. If I is infinite, then this is not quasicompact. (Quasicompactness means that for every affine the preimage is quasicompact, but even the preimage of $\mathbb{A}^1_{\mathbb{C}}$ itself is not quasicompact.)

Now, $\mathcal{O}_X$ is certainly a quasicoherent sheaf over X. So, if f_* presevered quasicoherence, we would have that $(f_*\mathcal{O}_X)(\mathbb{A}^1_{\mathbb{C}}) = \prod_{z\in I} \mathbb{C}[t]$. This is supposed to be a module over $\mathbb{C}[t] = \Gamma(\mathbb{A}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}})$. Given any $g \in \mathbb{C}[t]$, we should then have $(f_*\mathcal{O}_X)(D(g)) = \prod_{z\in I} \mathbb{C}[t]_g$. So, the question becomes: is the morphism

$$\left(\prod_{z\in I} \mathbb{C}[t]\right)_g \longrightarrow \prod_{z\in I} (\mathbb{C}[t]_g)$$

is an isomorphism? The answer is yes is I is finite, but the answer is no if I is infinite. (For example, take a vector where there is no common bound for the power in the denominator.)

Proof of theorem. We begin with the special case that $X = \text{Spec } B$ and $Y = \text{Spec } A$. Then $f : X \to Y$ corresponds to some $\varphi : A \to B$. We will check that:

1. If $M \in \mathbf{Mod}_A$, then $f^*\widetilde{M} = \widetilde{M \otimes_A B}$.
2. If $N \in \mathbf{Mod}_B$, then $f_*\widetilde{N} = \widetilde{\varphi_* N}$.

For the first claim, recall that by definition,

$$\begin{aligned}
f^*\widetilde{M} &= (f^{-1}\widetilde{M}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \\
&= f^{-1}(\underline{M} \otimes_{\underline{A}} \mathcal{O}_Y) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \\
&= (f^{-1}\underline{M}) \otimes_{f^{-1}\underline{A}} f^{-1}\mathcal{O}_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_X \\
&= \underline{M} \otimes_A \mathcal{O}_X \quad \text{by the universal property of constant sheaves} \\
&= (\underline{M} \otimes_{\underline{A}} \underline{B}) \otimes_{\underline{B}} \mathcal{O}_X \\
&= \widetilde{M \otimes_A B}
\end{aligned}$$

(where the tensor product is in the category of sheaves, so we're not writing sheafification). To explain getting from the second line to the third, we are using the isomorphism $f^{-1}M \otimes_{f^{-1}A} f^{-1}\mathcal{O}_Y \overset{\sim}{\to} f^{-1}(M \otimes_A \mathcal{O}_Y)$ given via the adjunction

$$\text{Hom}_{f^{-1}\mathcal{O}_Y}(f^{-1}M \otimes_{f^{-1}A} f^{-1}\mathcal{O}_Y, f^{-1}(M \otimes_A \mathcal{O}_Y)) = \text{Hom}_{f^{-1}A}(f^{-1}M, f^{-1}(M \otimes_A \mathcal{O}_Y))$$

by $M \to M \otimes_A \mathcal{O}_Y$, $m \mapsto m \otimes 1$. We can check this on stalks, which is nice since it avoids sheafification; since pulling back sheaves commutes with taking stalks, we have that

$$(f^{-1}M \otimes_{f^{-1}A} f^{-1}\mathcal{O}_Y)_x = (f^{-1}M)_x \otimes_{(f^{-1}A)_x} (f^{-1}\mathcal{O}_Y)_x = M_{f(x)} \otimes_{A_{f(x)}} \mathcal{O}_{Y,f(y)} = M \otimes_A \mathcal{O}_{Y,f(x)}$$

and

$$(f^{-1}(M \otimes_A \mathcal{O}_Y))_x = (M \otimes_A \mathcal{O}_Y)_{f(y)} = M \otimes_A \mathcal{O}_{Y,f(x)}.$$

For the second claim, over a basic open $D(g) \subset \text{Spec } A$, we compute that

$$(f_*\widetilde{N})(D(g)) = \widetilde{N}(f^{-1}(D(g))) = \widetilde{N}(D(\varphi(g))) = N_{\varphi(g)}$$

and that

$$\widetilde{\varphi_* N}(D(g)) = (\varphi_* N)_g = N \otimes_A A_g = N \otimes_B (B \otimes_A A_g) = N \otimes_B B_{\varphi(g)}.$$

So these do match up on each basic open, and with a little more work we can check that this defines an isomorphism of sheaves on the distinguished base for the topology.

We now move to the general case. Let $f : X \to Y$ be an arbitrary morphism of schemes. If $\mathcal{F}$ is a quasicoherent sheaf on Y, to check that $f^*\mathcal{F}$ is a quasicoherent sheaf on X it suffices to find a neighborhood U of each $x \in X$ such that $(f^*\mathcal{F})|_U$ is quasicoherent. So given $x \in X$, choose Spec $A \subset Y$ containing $f(x)$ and Spec $B \subset f^{-1}(\text{Spec } A) \subset X$ containing x. We then have the diagram

$$\begin{tikzcd}
X \arrow[d, "f"'] & \text{Spec } B \arrow[l, hook', "j"'] \arrow[d, "g"] \\
Y & \text{Spec } A, \arrow[l, hook', "i"]
\end{tikzcd}$$

and now $(f^*\mathcal{F})|_{\text{Spec } B} = j^*f^*\mathcal{F} = g^*i^*\mathcal{F}$, and $i^*\mathcal{F}$ is quasicoherent so we've reduced to the affine case.

Now, suppose that f is quasicompact and quasiseparated, and suppose that $\mathcal{G}$ is a quasicoherent sheaf on X. Notice that if $U \subset Y$ is open then we get

$$\begin{tikzcd}
f^{-1}(U) \arrow[r, equal] & X_U \arrow[r, hook] \arrow[d, "f_U"] & X \arrow[d, "f"] \\
& U \arrow[r, hook] & Y,
\end{tikzcd}$$

and $(f_*\mathcal{G})|_U = (f_U)_*(\mathcal{G}|_{X_U})$. Thus it is enough to consider the case when Y is affine. Now, f being quasicompact means that for a cover of Y by affines then their preimages are affine, which now that Y is affine means that X itself is quasicompact: $X = \bigcup_{i=1}^r U_i$. Moreover, f being quasiseparated implies that each intersection $U_i \cap U_j$ can be covered by finitely many affines, $U_i \cap U_j = \bigcup_{k=1}^{m_{ij}} V_{ijk}$. Now, writing $f_i = f|_{U_i}$ and $f_{ij} = f|_{U_i \cap U_j}$, we have the equalizer diagram

$$f_{\mathcal{G}} \to \bigoplus_i (f_i)_*(\mathcal{G}|_{U_i}) \rightrightarrows \bigoplus_{i,j} (f_{ij})_*(\mathcal{G}|_{U_i \cap U_j}).$$

We know nothing about the last term here, but we do know that it injects into $\bigoplus_{i,j,k}(f_{ijk})_*(\mathcal{G}|_{V_{ijk}})$, which is a finite sum. So we can write the equalizer diagram

$$f_{\mathcal{G}} \to \bigoplus_i (f_i)_*(\mathcal{G}|_{U_i}) \rightrightarrows \bigoplus_{i,j,k} (f_{ijk})_*(\mathcal{G}|_{V_{ijk}}).$$

Now we see that $f_*\mathcal{G}$ is the kernel of a map between quasicoherent sheaves, and so for $f_*\mathcal{G}$ to be quasicoherent it suffices to check that the kernel of a map between quasicoherent sheaves is quasicoherent.

So let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of quasicoherent sheaves on Y, and let $K = \text{Ker}(\varphi)$. To check that K is quasicoherent we can work locally on Y, i.e. we can take $Y = \text{Spec } A$ so that $\mathcal{F} = \widetilde{M}$ and $\mathcal{G} = \widetilde{N}$. Then φ corresponds to a map $\gamma : M \to N$, and then since localization is an exact functor, $K = \widetilde{\text{Ker}(\varphi)}$. ☐

Remark 22. If X and Y are locally noetherian, then f^* of a coherent sheaf on Y is a coherent sheaf on X. (Recall that a sheaf is *coherent* if we can cover the scheme so that locally we have the tilde of a *finitely generated* module.) Indeed, it's enough to consider the affine case Spec $B \to$ Spec A. Then certainly $M \mapsto M \otimes_A B$ preserves finite generation.

Remark 23. Usually, f_* of a coherent sheaf is not coherent. For example, take $f : \mathbb{A}^1_k \to \text{Spec } k$ coming from $k \hookrightarrow k[x]$. Then $f_*\mathcal{O}_{\mathbb{A}^1_k} = k[x]$, viewed as a k-module. This is finitely generated as a $k[x]$-module, but not as a k-module.

Example 62 (unrelated/future example)**.** However, sometimes f_* of a coherent sheaf *is* coherent. If Σ is a compact Riemann surface, then there are no nonconstant holomorphic functions on Σ. (For example, $\mathbb{CP}^1 \cong S^2$ admits no nonconstant holomorphic functions.) Indeed, $H^0(\Sigma, \mathcal{O}_\Sigma) = \mathbb{C}$.

Suppose X is a scheme. If we have an exact sequence $0 \to F' \to F \to F'' \to 0$ of $\mathcal{O}_X$-modules, we obtain an exact sequence $0 \to \Gamma(X, F') \to \Gamma(X, F) \to \Gamma(X, F'')$. That is, the global sections functor is *left-exact* but not right-exact. However, we have the following theorem.

Theorem 7. *If X is affine and F' is quasicoherent, then the map $\Gamma(X,F) \to \Gamma(X,F'')$ is surjective.*

Remark 24. In general, the above sequence extends to $H^1(X, F')$. So the real theorem here is that $H^i(X, F') = 0$ for $i > 0$. By the way, it may seem weird that this depends only on F'; this is because of the *long exact sequence in cohomology* (of which we have just indicated the first little chunk).

Remark 25. We might hope to prove this (in a restricted case where all our sheaves are quasicoherent) by using the equivalence of categories $\mathbf{Mod}_A \simeq Qcoh(X)$, which preserves limits and colimits. However, our statement is about an exact sequence of arbitrary sheaves, not just of quasicoherent ones. It's not immediate that the functor $\mathbf{Mod}_A \xrightarrow{\sim} Qcoh(X) \hookrightarrow \mathbf{Mod}_{\mathcal{O}_X}$ preserves limits and colimits.

Proof. Let us write $s \in F''(X)$ and $X = \text{Spec } A$. Suppose that $f \in A$, and $t \in F(D(f))$ maps to $s|_{D(f)} \in F''(D(f))$. We claim that there is some $r \geq 0$ such that $f^r t$ extends to a section $\tilde{t} \in F(X)$ mapping to $f^r s \in F''(X)$. This will imply the theorem as follows. We can choose $(g_1, \ldots, g_N) = A$ and $t_i \in F(D(g_i))$ lifting $s|_{D(g_i)}$ (since $F \to F''$ is a surjective map of sheaves, so it is surjective when sufficiently restricted). For each of these we can choose n_i such that $g_i^{n_i} t_i$ extends to $\tilde{t_i} \mapsto g_i^{n_i} s \in F''(X)$. Then, we write $1 = \sum \alpha_i g_i^{n_i}$ (since $(g_1^{n_1}, \ldots, g_N^{n_N}) = (g_1, \ldots, g_N) = A$), so $\sum \alpha_i g_i^{n_i} s = s$ in $F''(X)$.

So, choose $(g_1, \ldots, g_N) = A$ and $t_i \in F(D(g_i))$ lifting $s|_{D(g_i)}$. Now, we have that $t_i - t \in F(D(g_i f))$ maps to $0 \in F''(D(g_i f)) = (F''(D(g_i)))_f$ (since $t_i \mapsto s$ and $t \mapsto s$ on this restricted domain, and by quasicoherence). So $u_i = t_i - t \in F'(D(g_i f))$. Therefore, there is some m such that $f^m u_i$ extends to $\tilde{u_i} \in F'(D(g_i))$: if we have a section inside a basic open, we can clear the denominator to get to a section defined on a bigger basic open. So $t = t_i - u_i$, and we now multiply everything by f^m to get $t \mapsto f^m t$, $t_i \mapsto f^m t_i - \tilde{u_i}$. So we can actually assume that $t_i|_{D(g_i f)} = t|_{D(g_i f)}$. Over $D(g_i g_j)$, $t_i - t_j \in F(D(g_i g_j))$ maps to $0 \in F(D(g_i g_j f))$ and $0 \in F''(D(g_i g_j))$. Since presheaf-kernel is sheaf-kernel, this means that $F'(D(g_i g_j f)) \hookrightarrow F(D(g_i g_j f))$, So $t_i - t_j \in \text{Ker}(F'(D(g_i g_j)) \to F'(D(g_i g_j f)))$. So for each pair i, j there is some $q_{ij} \geq 0$ such that $f^{q_{ij}}(t_i - t_j) = 0$, and assuming that q is the maximum of these we obtain that the system $(f^q t_i) \in F(X)$ maps to $f^q s \in F''(X)$. $\square$

5.2 Vector bundles

Definition 38. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called *locally free of finite rank* if there is a cover $X = \bigcup_i U_i$ such that $\mathcal{F}|_{U_i} \cong (\mathcal{O}_X|_{U_i})^{n_i}$. (This is the algebro-geometric version of a *vector bundle*.) The number n_i is called the *rank*, which we will see is a locally constant function on X.

Remark 26. The rank is perhaps better defined as the function $|X| \to \mathbb{Z}$, $x \mapsto \dim_{k(x)}(\mathcal{F}(x))$. (Recall that a point $x \in X$ induces a map Spec $k(x) \to X$, and $\mathcal{F}$ pulls back to $\mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x =: \mathcal{F}(x)$.) From here, it is clear that the rank is indeed locally constant.

Remark 27. A locally free sheaf of rank 1 is often called a *line bundle*, an *invertible sheaf*, or a *line sheaf*, depending on your taste. (To say that a sheaf is invertible means that the tensor functor that it induces on the category of sheaves is an equivalence of categories.)

Remark 28. If $X = \text{Spec } A$ and A is noetherian (although this hypothesis can probably be dropped if we're careful), then under the correspondence $Qcoh(\text{Spec } A) \simeq \mathbf{Mod}_A$, locally free sheaves of finite rank correspond to finitely generated projective A-modules.

Theorem 8. *$\mathbb{P}^n_{\mathbb{Z}}$ represents the functor sending a scheme T to isomorphism classes of pairs (L, π), where L is an invertible sheaf on T and $\pi : \mathcal{O}_T^{n+1} \twoheadrightarrow L$ is a surjection of $\mathcal{O}_T$-modules.*

Remark 29. Thus L locally corresponds to $\mathcal{O}_T$ up to a choice of basis, and throwing away the choice of basis is what allows this to be a general sheaf instead of just a copy of $\mathcal{O}_T$ as an $\mathcal{O}_T$-module.

Proof. Define a functor $F_n : \mathbf{Schemes}^{op} \to \mathbf{Set}$ by sending T to the set $\{\pi : \mathcal{O}_T^{n+1} \twoheadrightarrow L : L \text{ an invertible sheaf on T}\}/\simeq$. First, observe that $(\mathcal{O}_T^{n+1} \xrightarrow{\pi} L) \simeq (\mathcal{O}_T^{n+1} \overset{\pi'}{\twoheadrightarrow} L')$ if there is some $\sigma : L \xrightarrow{\sim} L'$ such that the diagram

$$\begin{tikzcd}
\mathcal{O}_T^{n+1} \arrow[r, two heads, "\pi'"] \arrow[d, two heads, "\pi"'] & L' \\
L \arrow[ur, "\sim", "\sigma"'] &
\end{tikzcd}$$

commutes. Note that if σ exists, it is unique. Moreover, σ exists if and only if $\ker(\pi) = \ker(\pi')$.

We observe that a map $\pi : \mathcal{O}_T^{n+1} \twoheadrightarrow L$ with L an invertible sheaf is surjective if and only if for all points $t \in T$, the induced map $\pi(t) = k(t)^{n+1} \to L(t) = L_t/\mathfrak{m}_t L_t$ is surjective, as the diagram

$$\begin{tikzcd}
\mathcal{O}_{T,t}^{n+1}/\mathfrak{m}_t\mathcal{O}_{T,t}^{n+1} = k(t)^{n+1} \arrow[r, "\pi(t)"] & L(t) = L_t/\mathfrak{m}_t L_t \\
\mathcal{O}_{T,t}^{n+1} \arrow[r, "\pi_t"'] \arrow[u, twoheadrightarrow] & L_t \arrow[u, twoheadrightarrow]
\end{tikzcd}$$

commutes. Indeed, if π_t is surjective then $\pi(t)$ is surjective by Nakayama's lemma: if A is a local ring and M is a finitely presented A-module, then for any map $N \to M$ of A-modules such that $N/\mathfrak{m}_A N \to M/\mathfrak{m}_A M$ is surjective, $N \to M$ itself is surjective. In our easier case, we can see directly why this is true. Choose an isomorphism $L_t \xrightarrow{\sim} \mathcal{O}_{T,t}$. Let $\operatorname{im}(\pi_t) = N \subset \mathcal{O}_{T,t}$. Then in the diagram

$$\begin{tikzcd}
N \arrow[r, hookrightarrow] \arrow[dr] & \mathcal{O}_{T,t} \arrow[d] \\
 & \mathcal{O}_{T,t}/\mathfrak{m}_t,
\end{tikzcd}$$

since the diagonal arrow is surjective then there is some $x \in N \backslash \mathfrak{m}_t$, and hence $x \in \mathcal{O}_{T,t}^\times$. Thus $N = \mathcal{O}_{T,t}$.

Now, given $g : T' \to T$ and $(\pi : \mathcal{O}_T^{n+1} \twoheadrightarrow L) \in F_n(T)$, we can define $(g^*\pi : \mathcal{O}_{T'}^{n+1} \twoheadrightarrow g^*L) \in F_n(T')$. We must verify that g^*L is invertible and that $g^*\pi$ is surjective. On the level of modules, g^*L is locally just given by tensoring up coefficients, and a free rank 1 module tensors up to another free rank 1 module. Moreover, $g^*\pi$ is indeed have a surjection because tensor product (and hence pullback of quasicoherent sheaves) is right-exact.

We now reach our claim that $F_n \simeq h_{\mathbb{P}^n}$ (where by $\mathbb{P}^n$ we mean $\mathbb{P}^n_{\mathbb{Z}}$). We first warm up with two baby examples.

Example 63 (baby example 1)**.** Take our ring to be $\mathbb{C}$, and consider the point $[3 : 1/2 : \pi] \in \mathbb{P}^2(\mathbb{C}) = \operatorname{Hom}(\operatorname{Spec} \mathbb{C}, \mathbb{P}^2)$. This should correspond to a map $\mathcal{O}^3_{\operatorname{Spec} \mathbb{C}} \twoheadrightarrow L$. What is it?

Well, it must be that $L \simeq \mathbb{C}$ noncanonically, and now our map is $\mathbb{C}^3 \twoheadrightarrow \mathbb{C}$, which is given by $(a, b, c) \mapsto 3a + b/2 + \pi c$. But really, we want to describe our map without choosing these noncanonical isomorphisms. If we have two bases $\{e\}$ and $\{u \cdot e\}$ on the 1-dimensional $\mathbb{C}$-vector space L (for some unit $u \in \mathbb{C}^\times$), we get

$$\begin{tikzcd}
 & \mathbb{C} \\
\mathbb{C}^3 \arrow[ur, "{(3,1/2,\pi)}"] \arrow[r] \arrow[dr, "{(u^{-1}3,u^{-1}/2,u^{-1}\pi)}"'] & L \arrow[u, "e\mapsto 1"'] \\
 & \mathbb{C}. \arrow[u, "1\mapsto u\cdot e"']
\end{tikzcd}$$

The map from the lower copy of $\mathbb{C}$ to the upper copy is given by $1 \mapsto u$, which justifies the indicated transformation of $\mathbb{C}^3 \twoheadrightarrow \mathbb{C}$.

More generally, for $T = \operatorname{Spec} k$ we have

$$\{\vec{a} \in k^{n+1} \backslash \{\vec{0}\}\}/k^\times \xrightarrow{\sim} \{k^{n+1} \twoheadrightarrow L \colon L \in \mathbf{Mod}_k,\ \dim_k L = 1\}/\sim .$$

Example 64 (baby example 2)**.** Consider $\mathbb{P}^1 \to \mathbb{P}^2$ given by $[a : b] \mapsto [a^2 : b^2 : ab]$. This is supposed to be equivalent to some $\mathcal{O}^3_{\mathbb{P}^1} \twoheadrightarrow L$. What is this map?

On the level of points, over a field k we think of $\pi : k^2 \overset{(a,b)}{\twoheadrightarrow} k = L$. We want $k^3 \overset{(a^2,b^2,ab)}{\twoheadrightarrow} k$. We must build the latter in a canonical way from the former. Namely, we have $k^2 \otimes_k k^2 \overset{\pi\otimes\pi}{\twoheadrightarrow} L \otimes_k L \simeq k$. We don't want to choose bases, but we need to. If $x, y \in k^2$ form a basis, then a basis for $k^2 \otimes_k k^2 \simeq k^4$ is given by $x \otimes x, x \otimes y, y \otimes x, y \otimes y$.

Then we include $k^3 \hookrightarrow k^2 \otimes_k k^2$ as the first three of these, which we map naturally as $x \otimes x \mapsto a^2$, $y \otimes y \mapsto b^2$, $x \otimes y \mapsto ab$. But the question is, why did we choose $x \otimes y$ instead of $y \otimes x$? In general, given a scheme T and a surjection $\pi : \mathcal{O}_T^2 \twoheadrightarrow T$, we can define $\mathcal{O}_T^3 \hookrightarrow \mathcal{O}_T^2 \otimes_{\mathcal{O}_T} \mathcal{O}_T^2 \overset{\pi\otimes\pi}{\twoheadrightarrow} L \otimes_{\mathcal{O}_T} L = L^{\otimes 2}$, and this defines a natural transformation $h_{\mathbb{P}^1} \to h_{\mathbb{P}^2}$.

Remark 30. If you believe the theorem, then $\mathrm{id} : \mathbb{P}^n \to \mathbb{P}^n$ corresponds to a line bundle $\mathcal{O}_{\mathbb{P}^n}^{n+1} \twoheadrightarrow L$ over $\mathbb{P}^n$, called the *tautological line bundle.* This is often denoted $\mathcal{O}_{\mathbb{P}^n}(1)$, Serre's *twisting sheaf.* We will return to this.

Remark 31. Giving a map $\mathcal{O}_T^{n+1} \twoheadrightarrow L$ is equivalent to choosing $x_0, \dots, x_n \in \Gamma(T, L)$: x_i corresponds to the vector $(0, \dots, 1, \dots, 0) \in \mathcal{O}_T^{n+1}$. One says that these collectively "generate" L (at every point).

Remark 32. Given a line bundle L generated by $x_0, \dots, x_n \in \Gamma(T, L)$, the map $T \to \mathbb{P}^n$ is determined as follows. For each t we choose a basis $L(t) \overset{\sim}{\to} k(t)$, and then we map $t \mapsto [x_0(t) : x_1(t) : \dots : x_n(t)]$.

We finally return to the proof. Define $D_+(X_i) \hookrightarrow \mathbb{P}^n_{\mathbb{Z}}$; recall that $D_+(X_i) \simeq \mathbb{A}^n_{\mathbb{Z}}$, with coordinates $X_0/X_i, X_1/X_i, \dots, X_i\hat{/}X_i, \dots, X_n/X_i$. This defines a subfunctor $H_i \subset F$ by

$$H_i(T) = \{\pi : \mathcal{O}_T^{n+1} \twoheadrightarrow L \ : \ \eta_i : \mathcal{O}_T \overset{i^{th}\text{coord.}}{\hookrightarrow} \mathcal{O}_T^{n+1} \to L \text{ is surjective}\}.$$

Note that $\mathcal{O}_T \to L$ is surjective iff it is an isomorphism; over a point $t \in T$, the induced map is $\mathcal{O}_{T,t} \twoheadrightarrow L_t \overset{\sim}{\to} \mathcal{O}_{T,t}$. This is determined by $1 \mapsto f$, and it is an isomorphism iff $f \in \mathcal{O}_{T,t}^{\times}$, which is true iff the map is surjective. So in the above definition of H_i, we could equivalently say that $\mathcal{O}_T \hookrightarrow \mathcal{O}_T^{n+1} \to L$ is an isomorphism. This implies that H_i is indeed a subfunctor of F_n. Moreover, we claim that $H_i \simeq h_{\mathbb{A}^n}$. We have the transformation $H_i \to h_{\mathbb{A}^n}$ given by

$$(\pi : \mathcal{O}_T^{n+1} \twoheadrightarrow: \eta_i \text{ is an iso.}) \mapsto (a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_n) \in \Gamma(T, \mathcal{O}_T)^n \simeq h_{\mathbb{A}^n}(T),$$

where a_i is the i^{th} coordinate of the composition $\mathcal{O}_T^{n+1} \twoheadrightarrow L \overset{\sim}{\leftarrow} \mathcal{O}_T$. The inverse $h_{\mathbb{A}^n} \to H_i$ is determined by

$$(a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_n) \mapsto (\mathcal{O}_T^{n+1} \xrightarrow{(a_0, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n)} \mathcal{O}_T).$$

(The latter is certainly a surjection, and indeed $\eta_i = \mathrm{id}_{\mathcal{O}_T}$.)

To finish the proof, we must verify that:

- F_n is a big Zariski sheaf.
- $H_i \subset F_n$ are open subfunctors.
- $\coprod_i H_i \to F_n$ is an epimorphism of big Zariski sheaves.

We address these in order.

- Suppose $T = \bigcup_{i \in I} U_i$ is an open cover. We must check that

  $$F_n(T) \to \prod_i F_n(U_i) \rightrightarrows \prod_{i,j} F_n(U_i \cap U_j)$$

  is exact. So suppose $(\pi_i : \mathcal{O}_{U_i}^{n+1} \twoheadrightarrow L_i)_{i \in I} \in \prod_i F_n(U_i)$ such that for all i and j, $\pi_i|_{U_i \cap U_j} \simeq \pi_j|_{U_i \cap U_j}$. Let $K_i = \mathrm{Ker}(\pi_i) \subset \mathcal{O}_i$. Then it is equivalent to say that $K_i|_{U_i \cap U_j} = K_j|_{U_i \cap U_j}$ in $\mathcal{O}_{U_i \cap U_j}^{n+1}$. This system immediately implies the cocycle condition since they are all subsheaves of $\mathcal{O}_{U_i \cap U_j}^{n+1}$, so by gluing the kernels we get $K \subset \mathcal{O}_T^{n+1}$ which agrees with K_i on each U_i.l Then $(\mathcal{O}_T^{n+1} \twoheadrightarrow \mathcal{O}_T^{n+1}/K) \in F_n(T)$ maps to $(\pi_i)_{i \in I}$. This is a line bundle because locally it is. It is clear that the other half of exactness is satisfied, so F_n is indeed a big Zariski sheaf.

- To check that $H_i \subset F$ is an open subfunctor, we must show that given a scheme T and a surjection $(\pi : \mathcal{O}_T^{n+1} \twoheadrightarrow L) \in F_n(T)$, there is an open subset $U_i \subset T$ such that a morphism $g : T' \to T$ factors through U_i iff the composition $\mathcal{O}_T \overset{i^{th}\text{ coord.}}{\hookrightarrow} \mathcal{O}_T^{n+1} \overset{g^*\pi}{\to} g^*L$ is surjective. We have the diagram

  $$\begin{tikzcd} h_{U_i} \arrow[r] \arrow[d] & H_i \arrow[d] \\ h_T \arrow[r, "\pi"'] & F_n. \end{tikzcd}$$

We first consider the case that we have an isomorphism $\sigma : L \xrightarrow{\sim} \mathcal{O}_T$. Then the composite $\mathcal{O}_T \overset{i^{th}}{\hookrightarrow} \mathcal{O}_T^{n+1} \xrightarrow{\pi} L \xrightarrow{\sim} \mathcal{O}_T$ is multiplication by some $f \in \Gamma(T, \mathcal{O}_T)$, and we should take $U_i = D(f) = \{t \in T : f \mapsto \mathcal{O}_{T,t}^{\times} \subset \mathcal{O}_{T,t}\}$. Note that $\eta_i|_{U_i}$ is an isomorphism. Also, if $g : T' \to T$ is a map such that $g^*\eta_i$ is surjective, then for all $t \in T'$, the map $k(t') \overset{g^*\eta(t')}{\to} g^*L(t')$ is surjective. Indeed, $g^*\eta_i : k(t') \to g^*L(t')$ is the same thing as $(k(g(t')) \overset{f(g(t'))}{\to} L(g(t'))) \otimes_{k(t)} k(t')$, and we can always check surjectivity after extending scalars. So, the image of t' in T is a point such that $f \neq 0$ in $k(g(t'))$. Thus $g(t') \in D(f)$.

For the general case, choose $T = \bigcup_j V_j$ such that $L|_{V_j}$ is trivial. By the special case, for each V_j we get an open subset $U_{ij} \subset V_j$. Now let $U_i = \bigcup_j U_{ij}$. Then, given $g : T' \to T$, $g^*\eta_i$ is surjective iff $g^*\eta_i|_{g^{-1}(V_j)}$ is surjective for all j, which is equivalent to saying that $g^{-1}(V_j) \to V_j$ factors through U_{ij} for every j, which is equivalent to saying that $g : T' \to T$ factors through U_i.

- To check that $\coprod_i H_i \to F_n$ is an epimorphism of big Zariski sheaves, it suffices to check that for every T and every $\pi : \mathcal{O}_T^{n+1} \twoheadrightarrow L$, there exists a covering $T = \bigcup_i U_i$ such that $\eta_i|_{U_i}$ is surjective. For this, we can just let $s_0, \ldots, s_n \in \Gamma(T, L)$ be elements corresponding to π and let $U_i = D(s_i) = \{t \in T : s \not\mapsto 0 \in L(t)\}$.

It remains to check that this really gives us $\mathbb{P}^n$ back.

First, we note that in general, if X and Y are schemes, then the composite functor $(\textbf{Schemes}/X)^{op} \to \textbf{Schemes}^{op} \overset{h_Y}{\to} \textbf{Sets}$ will be isomorphic to $h_{X \times_{\text{Spec } \mathbb{Z}} Y} \simeq h_X \times h_Y$. So in particular, for a given n, the composite $(\textbf{Schemes}/X)^{op} \to \textbf{Schemes}^{op} \overset{F_n}{\to} \textbf{Sets}$ is represented by $\mathbb{P}^n_X := \mathbb{P}^n_{\mathbb{Z}} \times X$. More precisely, the projection morphism $\mathbb{P}^n_X \to X$ represents the functor $(T \to X) \mapsto \{\mathcal{O}_T^{n+1} \twoheadrightarrow L\}/\sim$.

Now, recall that we have this functor H_i defined by

$$H_i(T) = \{\mathcal{O}_T \overset{i^{th}}{\hookrightarrow} \mathcal{O}_T^{n+1} \overset{\pi}{\twoheadrightarrow} L \ : \ \text{composition } \eta_i \text{ is iso.}\},$$

and $H_i \simeq h_{\mathbb{A}^n}$. We also had the embedding $H_i \hookrightarrow F_n$, given by taking $(\mathcal{O}_T^{n+1} \overset{\pi}{\twoheadrightarrow} L)$ (over T) to $(\eta_0/\eta_i, \ldots, \eta_n/\eta_i) \in \Gamma(T, \mathcal{O}_T)^n$; the various coordinates are $\mathcal{O}_T \overset{\eta_j}{\to} L \overset{\eta_i^{-1}}{\to} \mathcal{O}_T$. We can compute that

$$(H_i \times_{F_n} H_j)(T) = \{\pi : \mathcal{O}_T^{n+1} \twoheadrightarrow L \ : \ \eta_i, \eta_j \text{ iso's}\},$$

and this embeds into $H_i \simeq h_{\mathbb{A}^n}$ along the "coordinates" $\eta_0/\eta_i, \ldots, \eta_n/\eta_i$. Thus we can reasonably denote $H_i \times_{F_n} H_j \subset H_i$ by $D(\eta_j/\eta_i)$. Similarly, we can denote $H_i \times_{F_n} H_j \subset H_j$ by $D(\eta_i/\eta_j)$. To allay potential confusion, we'll write D_i for distinguished opens in H_i. Then we have

$$\begin{tikzcd}
\text{Spec } \mathbb{Z}[\eta_0/\eta_j, \ldots, \eta_n/\eta_j]_{\eta_i/\eta_j} \arrow[r, "\text{notation}"] \arrow[d, equal] & \text{Spec } \mathbb{Z}[\eta_0/\eta_i, \ldots, \eta_n/\eta_i]_{\eta_j/\eta_i} \arrow[d, equal] \\
D_j(\eta_i/\eta_j) \arrow[r, "\sim"] \arrow[d, hook] & D_i(\eta_j/\eta_i) \arrow[d, hook] \\
H_j = \mathbb{A}^n_{\eta_0/\eta_j, \ldots, \eta_n/\eta_j} & \mathbb{A}^n_{\eta_0/\eta_i, \ldots, \eta_n/\eta_i} = H_i
\end{tikzcd}$$

where by "notation" we mean the morphism induced by the notation: we associate $\eta_s/\eta_j \leftrightarrow (\eta_j/\eta_i)^{-1} \cdot (\eta_s/\eta_i)$. This is the exact same gluing data as we have for $\mathbb{P}^n$. □

Remark 33. We have the following variants of the above construction.

1. Suppose X is a scheme and E is a locally free sheaf of finite rank on X. We can consider the functor $(\textbf{Schemes}/X)^{op} \to \textbf{Sets}$ given by $(T \overset{f}{\to} X) \mapsto \{f^*E \overset{\pi}{\twoheadrightarrow} L \ : \ L \text{ an invertible sheaf on } T\}$. This is again representable, which can be proved using the same business with big Zariski sheaves. The scheme is denote $\mathbb{P}E$, and is called the *projectivization of* E. If over an open subset $U \subset X$ we choose a trivializing isomorphism $E|_U \simeq \mathcal{O}_U^{n+1}$, then we get $\mathbb{P}E|_U \simeq \mathbb{P}^n_U$.

2. Define the functor $Grass(r,n) : \mathbf{Schemes}^{op} \to \mathbf{Sets}$ by $T \mapsto \{\mathcal{O}_T^n \twoheadrightarrow E \; : \; E\text{locally free of rank r}\}$. This is called the *Grassmannian (of r-planes in n-space*, which is also representable. (As a special case, $Grass(1,n) = \mathbb{P}^{n-1}$.)

3. Combine the previous two generalizations.

Remark 34. Our above construction gives $\mathbb{P}^n_{\mathbb{Z}} \simeq F_n$. Thus the identity map id $\in h_{\mathbb{P}^n}(\mathbb{P}^n)$ corresponds to an element $(\mathcal{O}_{\mathbb{P}^n}^{n+1} \twoheadrightarrow L) \in F_n(\mathbb{P}^n)$, denoted $\mathcal{O}(1)$. We should be able to describe this line bundle explicitly.

In general, given the graded ring $S = \mathbb{Z}[X_0, \dots, X_n] = \bigoplus_{d \geq 0} S_d$ and $M = \bigoplus_{l \in \mathbb{Z}} M_l$ is a graded S-module (so that $S_d \cdot M_l \subset M_{l+d}$), we can cook up a quasicoherent sheaf $\widetilde{M}$ on $\mathbb{P}^n$ as follows. Recall that $\mathbb{P}^n = \bigcup D_+(X_i)$, and over $D_+(X_i)$ we take the $\mathbb{Z}[X_0/X_i, \dots, X_n/X_i]$-module defined by $(M_{X_i})_0$. The key observation is that $((M_{X_i})_0)_{X_j/X_i} \cong ((M_{X_j})_0)_{X_i/X_j}$ as $(\mathbb{Z}[X_0, \dots, X_n, X_i^{-1}, X_j^{-1}])_0$-modules. This gives us gluing data for these modules; $\widetilde{M}$ is defined by taking $\widetilde{(M_{X_i})}_0$ on $D_+(X_i)$.

Note that there is a canonical map $M_0 \to \Gamma(\mathbb{P}^n, \widetilde{M})$, and $\mathcal{O}(1)$ is obtained from the graded S-module $S[-1]$ (obtained by shifting S to the left by 1, $S[-1]_d = S_{d+1}$). Indeed,

$$(S[-1]_{X_i})_0 = (S_{X_i})_1 = (S_{X_i})_0 \cdot X_i,$$

which implies that $\widetilde{S[-1]}$ is a locally free sheaf of rank 1. Moreover, from the way we cooked this up, the map $S[-1]_0 = S_1 \to \Gamma(\mathbb{P}^n, \widetilde{S[-1]})$ takes x_i to the global section s_i. This gives us the projection $\mathcal{O}_{\mathbb{P}^n}^{n+1} \twoheadrightarrow \mathcal{O}(1)$.

It remains to be checked that this really is the universal line bundle, but we leave this as an exercise.

5.3 A linear-algebraic aside (related to various tensor constructions)

Example 65. What is the determinant of a map of vector bundles $u : E \to F$ of rank n on a scheme X?

Well, if say $n = 2$ then locally our map is given by a matrix $u_x = (a, b|c, d) : R^2 \to R^2$, and we had better hope that $\det(u_x) = ad - bc$. A better way to see this is to look for the induced map on *top exterior powers*: we get $\bigwedge^2 R \to \bigwedge^2 R$, a map on 1-dimensional vector spaces, obtained by

$$(e_1 \wedge e_2) \mapsto (ae_1 + ce_2) \wedge (be_1 + de_2) = (ad - bc) e_1 \wedge e_2.$$

Indeed, this generalizes. In general, if V is an n-dimensional k-vector space and $T : V \to V$, then $\bigwedge^n V$ is 1-dimensional and $\det(T)$ is the map $\bigwedge^n(T) : \bigwedge^n V \to \bigwedge^n V$. (We have a canonical isomorphism $\bigwedge^n V \cong k$, so there is no ambiguity here.)

Back in the world of schemes, the morphism $u : E \to F$ of vector bundles should have that $\det(u) : \bigwedge^n E \to \bigwedge^n F$ (a map of line bundles). To make this more canonical, we can tensor with the dual of $\bigwedge^n E$ to get a map $\det(u) : \mathcal{O}_X \to (\bigwedge^n F) \otimes_{\mathcal{O}_X} (\bigwedge^n E)^\vee$, i.e. $\det(u) \in \Gamma(X, (\bigwedge^n F) \otimes_{\mathcal{O}_X} (\bigwedge^n E)^\vee)$. (In the particular case that $E = F$, we get that $\det(u) \in \Gamma(X, \mathcal{O}_X)$, a global function.)

Corollary 3. *The set* $\{x \in X : u(x) : E(x) \to F(x) \text{ is not an iso.}\}$ *is closed, and locally it is the zero locus of a single element of* $\mathcal{O}_X$*, namely the determinant.*

5.4 In-depth example computations

Example 66. Let k be an algebraically closed field. We claim that $\mathrm{Pic}(\mathbb{A}^1_k) = 0$.

Let L be an invertible sheaf. Pick some open $U \subset \mathbb{A}^1_k$ on which there exists a trivialization $\sigma_U : L|_U \xrightarrow{\sim} \mathcal{O}_U$. Since $k[t]$ is a PID, we can actually write $U = D(f)$. Let us write $f = (t - a_1)^{n_1} \cdots (t - a_s)^{n_s}$.

We would like to extend the isomorphism σ_U to a global trivialization, but it may not extend over the missing points $\{a_1, \dots, a_s\} = \mathbb{A}^1_k \backslash D(f)$. For each a_i, consider the local ring $\mathcal{O}_{\mathbb{A}^1_k, a_i}$. This is a dvr with maximal ideal $(t - a_i) \subset \mathcal{O}_{\mathbb{A}^1_k, a_i}$. Denoting by η the generic point, we have the commutative diagram

$$\begin{tikzcd}
k(t) & \mathcal{O}_{\mathbb{A}^1_k, a_i} \arrow[l, hook'] \\
L_\eta \arrow[u, "\sigma_U", "\cong"'] & L_{a_i}. \arrow[l] \arrow[ul, "\alpha"' sloped] \arrow[u, dashed]
\end{tikzcd}$$

Here, L_{a_i} is a free $\mathcal{O}_{\mathbb{A}^1_k,a_i}$-module, so $\alpha(L_{a_i})$ is a free rank 1 $\mathcal{O}_{\mathbb{A}^1_k,a_i}$-submodule of $k(t)$. Thus $\alpha(L_{a_i}) = \mathcal{O}_{\mathbb{A}^1_k,a_i} \cdot (t-a_i)^{n_i}$. (In general, if V is a dvr with maximal ideal (π) and $L \subset \operatorname{Frac}(V) = K$ is a rank 1 V-submodule, then $L = V \cdot f$ for some $f \in K$. If $v(f) = n$ (where v denotes the valuation), then $(f) = (\pi^n)$, and hence $L = V \cdot \pi^n$.)

So, if we denote by $\mathcal{I}_{a_i} = \widetilde{k[t] \cdot (t-a_i)} \subset \mathcal{O}_{\mathbb{A}^1_k}$ the ideal sheaf of a_i, then $L \otimes \mathcal{I}_{a_i}^{-n_i}$ is isomorphic to $\mathcal{O}_{\mathbb{A}^1_k}$ over $U \cup \{a_i\}$. Hence $\sigma_U \otimes s^{-n_i} : L \otimes \mathcal{I}_{a_i}^{-n_i}|_U \to \mathcal{O}_U \otimes \mathcal{O}_U = \mathcal{O}_U$. So ultimately, we obtain that

$$L \cong \mathcal{I}_{a_1}^{n_1} \otimes \cdots \otimes \mathcal{I}_{a_s}^{n_s} \cong (\widetilde{k[t] \cdot \prod_i (t-a_i)^{a_i}}) = (\widetilde{k[t] \cdot f}),$$

and $k[t] \cdot f \cong k[t]$ as $k[t]$-modules so L is trivial.

Example 67. We claim that $\operatorname{Pic}(\mathbb{P}^1_k) = \mathbb{Z}$.

Given L over $\mathbb{P}^1_k$, we have seen that its restriction to both affines is trivial. If we use homogeneous coordinates $[x : y]$, then we have

$$\begin{tikzcd}
\mathbb{A}^1_{x/y} \arrow[d, hook] & \operatorname{Spec} k[(y/x)^{\pm}] \arrow[l] \arrow[d, hook] \\
\mathbb{P}^1 & \operatorname{Spec} k[y/x]. \arrow[l]
\end{tikzcd}$$

(The top-right scheme is missing the point at infinity, $[1:0]$.) Choose a trivialization $\sigma : L|_{\mathbb{A}^1} \to \mathcal{O}_{\mathbb{A}^1}$. Let L_∞ be the stalk of L at ∞. Then we have

$$\begin{tikzcd}
k(y/x) & \mathcal{O}_{\mathbb{P}^1,\infty} \arrow[l, hook'] \\
L_{\infty,\eta} \arrow[u, "\sigma_\eta"] & L_\infty. \arrow[l, hook']
\end{tikzcd}$$

Here, $\mathcal{O}_{\mathbb{P}^1,\infty}$ is a dvr, and we can identify L_∞ with a power of the uniformizer, which we will call $-n_L$. (The isomorphism $\sigma : L|_{\mathbb{A}^1} \to \mathcal{O}_{\mathbb{A}^1}$ is defined up to $k^\times$, but this doesn't affect n_L since multiplying by units doesn't affect the valuation.) Then, σ extends over all of $\mathbb{P}^1$ if $n_L = 0$. So, n_L gives us a function $\varphi : \operatorname{Pic}(\mathbb{P}^1) \to \mathbb{Z}$.

To see that it is a homomorphism, if $L, M \in \operatorname{Pic}(\mathbb{P}^1)$ and $\sigma : L|_{\mathbb{A}^1} \to \mathcal{O}_{\mathbb{A}^1}$ and $\tau : M|_{\mathbb{A}^1} \to \mathcal{O}_{\mathbb{A}^1}$, then

$$L \otimes M|_{\mathbb{A}^1} \xrightarrow{\sigma\otimes\tau} \mathcal{O}_{\mathbb{A}^1} \otimes \mathcal{O}_{\mathbb{A}^1} \xrightarrow{\mu} \mathcal{O}_{\mathbb{A}^1}$$

gives us a product. This induces $L_\infty \otimes M_\infty \to L_{\infty,\eta} \otimes M_{\infty,\eta} \overset{\mu}{ra} k(y/x)$; if we write $L_{\infty,\eta} = (\pi_\infty^{-n_L})$ and $M_{\infty,\eta} = (\pi_\infty^{-n_M})$, then $L_\infty \otimes M_\infty = \mathcal{O}_{\mathbb{P}^1,\infty} \cdot \pi_\infty^{-(n_L+n_M)}$.

$\varphi(L)$ is usually called the *degree* of L. For example, $\deg(\mathcal{O}(1)) = 1$. This shows that φ is surjective. Moreover, φ is injective since σ is unique up to scalars. So φ is indeed an isomorphism.

Example 68 (extending morphisms)**.** Let X and Y be schemes over S, and let $f : X \to Y$ be a morphism of S-schemes. Then the graph Γ_f of f is defined by the fiber product diagram

$$\begin{tikzcd}
\Gamma_f \arrow[r] \arrow[d] & X \times_S Y \arrow[d, "f\times 1"] \\
Y \arrow[r, "\Delta"] & Y \times_S Y.
\end{tikzcd}$$

(If $(y_1, (x, y_2)) \in \Gamma_f$, then $(f(x), y_2) = (y_1, y_1)$, i.e. $y_2 = y_1 = f(x)$.) We can recover f from Γ_f via the diagram

$$\begin{tikzcd}
\Gamma_f \arrow[r, hook] & X \times_S Y \arrow[r, "p_1"] \arrow[d, "p_2"] & X \\
 & Y, &
\end{tikzcd}$$

since the composition along the top is an isomorphism.

Now, let $S = \text{Spec } k$, $Y = \mathbb{P}^n_k$, and suppose X is integral of finite type over k. Let $U \hookrightarrow X$ be a dense open. Given $f_U : U \to \mathbb{P}^n_k$, how can we extend to all of X? Let $Z \subset X \times \mathbb{P}^n$ be the closure of $\Gamma_{f_U} \subset U \times \mathbb{P}^n$. Then we have

$$\begin{tikzcd}
Z_U \arrow[r, hook] \arrow[d, "\cong"'] & Z \arrow[r, hook] \arrow[d, "\text{proper}"] & X \times \mathbb{P}^n \arrow[dr] & \\
U \arrow[r, hook] & X & & \mathbb{P}^n.
\end{tikzcd}$$

So the image of Z in X is closed and hence onto (as it contains U).

Thus we can always extend f_U after making a "birational modification". (As it turns out, this is a *blowup*.)

6 Separatedness & properness

6.1 Definitions

Suppose X is a finite type scheme over $\mathbb{C}$. Let X_{an} denote X endowed with the analytic topology instead of the Zariski topology. For instance, if $X = \text{Spec } \mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_r)$ then $X_{an} = \{\vec{z} \in \mathbb{C}^n : f_i(\vec{z}) = 0\}$ with the topology induced by the usual Euclidean topology on $\mathbb{C}^n$. Since X is over $\mathbb{C}$, this means we have a structure morphism $X \to \text{Spec } \mathbb{C}$.

- This morphism is *proper* iff X_{an} is compact.
- This morphism is *separated* iff X_{an} is Hausdorff.

(For more on this, see Mumford's "Red Book".) We would like to abstract these definitions away from the analytic topology, and talk about them purely in our algebraic world.

Definition 39. A morphism $f : X \to Y$ of schemes is called *separated* if the diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is a closed immersion.

Example 69. Consider the two standard inclusions $\mathbb{A}^1 \hookleftarrow \mathbb{G}_m \hookrightarrow \mathbb{A}^1$, and take X to be the associated gluing. Then X is "the affine line with two origins" (which we will call x and y). We don't want to consider this (in the analytic topology) to be Hausdorff, so hopefully the structure morphism $X \to \text{Spec } \mathbb{C}$ is <u>not</u> separated.

We consider $\Delta_X : X \to X \times_{\text{Spec } \mathbb{C}} X$. Inside of the target, we have the point $(x, y) \in \overline{\Delta_X} \setminus \Delta_X$ (analytically), so we do not expect Δ_X to be a closed immersion. Indeed, it is not. More generally, over any field k, the structure map $X \to \text{Spec } k$ is not separated.

Example 70. Let us recall that the fiber product $X \times_Y X$ is defined by the cartesian diagram

$$\begin{tikzcd}
X \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\
X \arrow[r] & Y.
\end{tikzcd}$$

Then the diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is determined by two copies of the identity morphism $\text{id}_X : X \to X$. In particular, if $X = \text{Spec } B$ and $Y = \text{Spec } B$, then we get $\Delta_{X/Y} : \text{Spec } B \to \text{Spec } B \otimes_A B$, which is opposite to a map $B \otimes_A B \to B$. This is just multiplication in B (which is A-linear). We can check this by checking that the diagram

$$\begin{tikzcd}
B & & \\
& B \otimes_A B \arrow[lu, "\text{mult.}"] & B \arrow[l] \\
& B \arrow[u] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

commutes. In fact, the multiplication map is surjective, and so its opposite $\Delta_{X/Y}$ is indeed a closed immersion. Thus, any morphism of affine schemes is separated. In fact, any affine morphism is separated. (The condition of being spearated is local on Y.)

Example 71. If $f : X \to Y$ is a closed or open immersion, then $X \times_Y X = X$, and $\Delta_{X/Y} : X \to X \times_Y X$ is an isomorphism. Certainly this is a closed immersion. Thus immersions are separated. (Functorially, if h_X is a subfunctor of h_Y, then $h_{X \times_Y X}$ certainly factors through h_X.)

Definition 40. A morphism of topological spaces $f : X \to Y$ is called *closed* if for every closed subset $Z \subset X$, $f(Z) \subset Y$ is closed. Further, f is called *universally closed* if for every morphism $Y' \to Y$, the map $X \times_Y Y' \to Y'$ is closed. (We say that the closedness is "stable under base change".) Finally, if $f : X \to Y$ is a morphism of schemes, then f is called *proper* if it is separated, of finite type, and universally closed.

Example 72. Consider the morphism $\mathbb{A}^1_k \to \operatorname{Spec} k$ opposite to $k \to k[t]$. This is separated and of finite type. It is even closed. But $(\mathbb{A}^1_{\mathbb{C}})_{an} = \mathbb{C}$, which is not compact, so this had better not be universally closed. We must product a base change which is not closed. The standard example is

$$\begin{tikzcd}
\mathbb{A}^2_k \arrow[r] \arrow[d] & \mathbb{A}^1_k \arrow[d] \\
\mathbb{A}^1_k \arrow[r] & \operatorname{Spec} k.
\end{tikzcd}$$

The projection $\mathbb{A}^2_k \to \mathbb{A}^1_k$ is not closed. For instance, if $k = \mathbb{R}$, the vanishing of $y = 1/x$ has non-closed image. Over any field (or even ring) k, the composition $V(xy-1) \hookrightarrow \mathbb{A}^2_{k,x-y} \to \mathbb{A}^1_{k,y}$ factors as $V(xy-1) \xrightarrow{\sim} D(y) = \mathbb{G}_m \hookrightarrow \mathbb{A}^1_{k,y}$, and the latter inclusion is not closed.

Example 73. For any scheme X, The map $\mathbb{P}^n_X = \mathbb{P}^n_{\operatorname{Spec} \mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} X \to X$ is proper. We will see this in a moment.

Remark 35. A closed immersion is proper, and the composition of two proper morphisms is proper. (Thus the above theorem implies that projective morphisms are proper: if $X \to Y$ factors as $X \hookrightarrow \mathbb{P}^n_Y \to Y$, then $X \to Y$ is proper.) By a homework exercise, this implies that finite morphisms are proper.

6.2 Valuative criteria

In general, it is very hard to check if a map is proper. If we glue a scheme up from affine schemes, we could get different results based on the gluing: two copies $\mathbb{A}^1$ glued along $\mathbb{G}_m$ give us either $\mathbb{P}^1$ or the affine line with two origins, depending on whether we glue along the inverse map or the identity, respectively. (These are proper and not proper, respectively.) To abstract this slightly, we have the diagram

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r] \arrow[d, hook] & X \\
\mathbb{A}^1; \arrow[ru, dashed] &
\end{tikzcd}$$

if there are multiply possibilities for the dotted arrow, then X will not be Hausdorff, while if there isn't such an arrow at all, then X will not be compact. (We're talking about "limits of sequences" in the only way we know how.)

Theorem 9 (valuative criterion for separatedness)**.** *Let $f : X \to Y$ be a morphism of schemes. Then f is separated iff for every diagram of solid arrows*

$$\begin{tikzcd}
\text{Spec } K \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
\text{Spec } R \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

with R a valuation ring and K its field of fractions, there exists at most one dotted arrow filling in the diagram.

Remark 36. In most cases (e.g. X and Y are of finite type over a field), it suffices to check when R is a *discrete* valuation ring (dvr).

Remark 37. Recall that a *valuation ring* is an integral domain R such that there exists a group homomorphism $v : K = \text{Frac}(R) - \{0\} \to G$ for G a totally ordered abelian group, such that:

1. $v(xy) = v(x) + v(y)$,
2. $v(x + y) \geq \min v(x), v(y)$,
3. $R = \{x \; inK \; : \; v(x) \geq 0\}$.

A valuation ring is a *discrete valuation ring* if we can take G to be a discrete group. The maximal ideal will always be of the form $\mathfrak{m} = \{x \in K \; : \; v(x) > 0\}$.

Examples of dvr's include $\mathbb{Z}_p = \text{invlim}_n \; \mathbb{Z}/p^n$ (the p-adic integers) and $k[[t]]$. These always have two prime ideals, one of them maximal. So their prime spectra look like a closed point with a single "tail", the generic point. So for R a dvr, our criterion is asking: If we map a "tail" – the generic point – into X (which represents a possibly-convergent sequence), is there more than one place where we can map its closed point (which represents is limit)?

$\mathbb{Z}_p$ has the p-adic valuation, and $k[[t]]$ has the valuation $v(a_n t^n + a_{n+1} t^{n+1} + \text{h.o.t.}) = n$ (for $a_n \neq 0$). The maximal ideal $\mathfrak{m} \subset R$ will be principal, $\mathfrak{m} = (\pi)$, and then for any $f \in A$ we will be able to write $f = u \cdot \pi^{v(f)}$ for $u \in R^\times$. This can be precisified to a structure theorem on dvr's, that they essentially come in these two flavors.

Over an algebraically closed field, any ring is *dominated* by an algebraic field. For more on dvr's, one can (and should!) consult Atiyah-MacDonald.

Theorem 10 (valuative criterion for properness)**.** *Suppose that $f : X \to Y$ is a morphism of schemes of finite type and Y is locally noetherian. Then f is proper iff for all diagrams of solid arrows*

$$\begin{tikzcd}
\text{Spec } K \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
\text{Spec } R \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

where R is a valuation ring and K is its fraction field, there exists a unique dotted arrow filling in the diagram.

Corollary 4. *$\mathbb{P}^n_X \to X$ is proper for all schemes X.*

Proof. It is enough to consider $X = \text{Spec } \mathbb{Z}$, because we have the cartesian diagram

$$\begin{tikzcd}
\mathbb{P}^1_X \arrow[r] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\
X \arrow[r] & \text{Spec } \mathbb{Z},
\end{tikzcd}$$

so to check this for Spec $\mathbb{Z}$ is at least as strong as checking this for any other scheme X.

Now, in our diagram

$$\begin{tikzcd}
\text{Spec } K \arrow[r] \arrow[d, hook] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\
\text{Spec } R \arrow[r] \arrow[ur, dashed] & \text{Spec } \mathbb{Z},
\end{tikzcd}$$

the top arrow corresponds to an equivalence class of surjection $K^{n+1} \twoheadrightarrow L$ for a 1-dimensional K-vector space L, by our functorial description of $\mathbb{P}^n_{\mathbb{Z}}$. So, to extend this map to Spec R is to give a map $R^{n+1} \twoheadrightarrow P$ for a projective R-module, such that the localizing diagram of modules

$$\begin{tikzcd}
R^{n+1} \arrow[r, two heads] & P \\
K^{n+1} \arrow[r, two heads] \arrow[u, hook] & L \arrow[u, hook]
\end{tikzcd}$$

commutes. We have $P \hookrightarrow L$ because for the upper triangle in the first diagram to commute, it must be that we get the equivalence class of $K^{n+1} \twoheadrightarrow L$ by tensoring $R^{n+1} \twoheadrightarrow P$ with K: $[K^{n+1} \twoheadrightarrow L] = [K^{n+1} \twoheadrightarrow P \otimes_R K]$. This means that there exists a unique dotted arrow in the diagram

$$\begin{tikzcd}
R^{n+1} \arrow[r, two heads] \arrow[d, hook] & P \arrow[d, hook] \\
K^{n+1} \arrow[r, two heads] \arrow[dr] & P \otimes_R K \arrow[d, dashed, "\exists!"] \\
& L.
\end{tikzcd}$$

It will turn out that P is actually free; any projective module over a local ring is free.

Note that there's really no choice for P: it must be that $P = \operatorname{Im}(R^{n+1} \hookrightarrow K^{n+1} \twoheadrightarrow L)$. So as we hoped, there is at most one dotted arrow filling in the first diagram. This implies that $\mathbb{P}^n_{\mathbb{Z}} \to \text{Spec } \mathbb{Z}$ is separated.

Now, properness amounts to showing that there is a dotted arrow. So we must check that the above formula for P really does give us a free module of rank 1. If we choose an identification $L \cong K$ as K-modules, then we have that $P = \operatorname{Im}(R^{n+1} \hookrightarrow K^{n+1} \to K)$, and the arrow $K^{n+1} \to K$ must be given by some $(a_0, \ldots, a_n)$ for $a_i \in K$. If a_0 has a lowest valuation among these (and is nonzero), then $a_i = (a_i/a_0) \cdot a_0$ for all i, so we see that the image is spanned over R by a_0. Thus, P is the R-module spanned by a_0. So P is indeed a free R-module. ☐

Example 74 (standard qual question on separatedness)**.** Let $f, g : Y \to X$ be two morphisms of S-schemes with X/S separated and Y reduced. Suppose that there is a dense open subset $U \subset Y$ such that $f|_U = g|_U$. We claim that $f = g$. (This fails, for instance, with the two maps of the affine line into the affine line with two points: they agree on a dense open $\mathbb{G}_m \subset \mathbb{A}^1$, but they are not equal.)

To see this, we recall that since X/S is separated, this means that $\Delta_X : X \to X \times_S X$ is a closed immersion.

So we have the cartesian diagram

$$\begin{tikzcd}
 & U \arrow[dl, hook'] \arrow[d, hook] \\
Y_\Delta \arrow[r, "\delta"] \arrow[d] & Y \arrow[d, "f\times g"] \\
X \times_{\Delta_X} X \times_S X & {}
\end{tikzcd}$$

and δ is a closed immersion. Thus $|Y_\Delta| = |Y|$. Moreover, if $\mathcal{J} \subset \mathcal{O}_Y$ is the ideal sheaf of $Y_\Delta \hookrightarrow Y$ then $\mathcal{J}|_U = 0$. Since Y is reduced, then $\mathcal{J} = 0$.

We have used the following facts here. First, as $j : U \hookrightarrow Y$ is a dense open, then $\mathcal{O}_Y \to j_*\mathcal{O}_U$ is injective. Moreover, if we write $p_1, p_2 : X \times_S X \to X$ for the projections, a factorization $Y \xrightarrow{h} X \xrightarrow{\Delta_X} X \times_S X$ of $Y \xrightarrow{f\times g} X \times_S X$ gives us that

$$f = p_1 \circ (f \times g) = p_1 \circ \Delta_X \circ h$$
$$g = p_2 \circ (f \times g) = p_2 \circ \Delta_X \circ h.$$

But $p_1 \circ \Delta_X = p_2 \circ \Delta_X$ since these are both just id_X, so $f = g$.

Remark 38. Let K be a field, and write $\Sigma = \{\text{local ring } A \subset K\}$. We can define the relation $A' \geq A$ if $A \subset A'$ and $\mathfrak{m}_{A'} \cap A = \mathfrak{m}_A$. We say that A' *dominates* A. This defines a partial order on Σ, and valuation rings are exactly the maximal elements. In particular, every local integral domain can be dominated by a valuation ring.

Remark 39. An integral domain A with field of fractions K is a valuation ring iff for all (nonzero) $x \in K$, either $x \in A$ or $x^{-1} \in A$. Note that if we do have a valuation $v : K^\times \to \Gamma$, then we have $A = \{x \in K^\times : v(x) \geq 0\} \cup \{0 \in K\}$. On the other hand, if A has this property, then we can define a total order on $\Gamma = K^\times/A^\times$ by saying that $x \geq y$ whenever $xy^{-1} \in A$. (The valuation $v : K^\times \to \Gamma$ is just the projection.)

Lemma 7. *Let R be a valuation ring and K be its field of fractions. Write $U = \mathrm{Spec}\ K$ and $T = \mathrm{Spec}\ R$. Suppose X is a scheme. Then $\mathrm{Hom}(U, X) = \{(x_1 \in X, k(x_1) \hookrightarrow K)\}$, and*

$$\mathrm{Hom}(T, X) = \{(x_0, x_1, k(x_1) \hookrightarrow K) \ : \ x_0 \in \overline{x_1} = Z \text{ and } R \text{ dominates } \mathcal{O}_{Z,x_0}\}.$$

(We are using the diagram

$$\begin{tikzcd}
R \arrow[d, hook] & \mathcal{O}_{Z,x_0} \arrow[d, hook] \\
K & k(x_1), \arrow[l, hook']
\end{tikzcd}$$

so we are talking about domination inside of K.)

Remark 40. As a warmup, we begin with the affine case. If $x_0 \in \overline{x_1} \subset \mathrm{Spec}\ B$ and $x_1 = [\mathfrak{p}]$ with $\overline{x_1} = \mathrm{Spec}\ B/\mathfrak{p} \subset \mathrm{Spec}\ B$ and $x_0 = [\mathfrak{q}]$, so that $x_0 \in \overline{x_1}$ corresponds to $\mathfrak{q} \subset \mathfrak{p}$ and we get $\overline{\mathfrak{q}} \subset B/\mathfrak{p}$. Then $\mathcal{O}_{Z,x_0} = (B/\mathfrak{p})_{\overline{\mathfrak{q}}}$, and $k(x_1) = \mathrm{Frac}(B/\mathfrak{p})$.

Proof. To check the first claim, since $\mathrm{Spec}\ K$ is a singleton, a map $f : \mathrm{Spec}\ K \to X$ is given by a point $x_1 \in X$ along with a local ring homomorphism $f^{-1}\mathcal{O}_X \to K$. Of course $f^{-1}\mathcal{O}_X = \mathcal{O}_{X,x_1}$, and to say that $\mathcal{O}_{X,x_1} \to K$ is a local ring homomorphism means that the preimage of the maximal ideal of K (i.e. $(0) \subset K$) is the maximal ideal of $\mathcal{O}_{X,x_1}$, which is exactly the statement that we have a factorization $\mathcal{O}_{X,x_1} \twoheadrightarrow k(x_1) \hookrightarrow K$.

Now, suppose we have a map $g : T \to X$. Let us write $x_1 = g(\eta)$ and $x_0 = g(s)$, where $\eta \in T$ is the generic point and $s \in T$ is the closed point. Then $\overline{x_1} \subset X$ is closed, so $g^{-1}(\overline{x_1}) \subset T$ is closed. But it contains η, so it must be all of T. Thus $x_0 \in \overline{x_1}$. So, we get a local ring homomorphism $g^\sharp : \mathcal{O}_{X,x_0} \to R = \mathcal{O}_{T,s}$. This gives us the solid diagram

$$\begin{tikzcd}
\mathcal{O}_{X,x_0} \arrow[rr] \arrow[dr, dashed] \arrow[dd, "\text{localization}"'] & & R \arrow[dd, hook] \\
& \mathcal{O}_{Z,x_0} \arrow[ur, dashed] \arrow[d, dashed] & \\
\mathcal{O}_{X,x_1} \arrow[r] & k(x_1) \arrow[r] & K.
\end{tikzcd}$$

Since $R \hookrightarrow K$ is injective, then we get the indicated factorization. Conversely, if we have a triple $(x_0, x_1, k(x_1) \hookrightarrow K)$ we take Spec $R \to$ Spec $\mathcal{O}_{Z,x_0} \to Z \to X$. ☐

This starts to give us a handle on our valuative criteria. We need the following lemma as well.

Lemma 8. *Let $f : X \to Y$ be a quasicompact morphism of schemes. Then the set $f(X) \subset Y$ is closed if and only if it is stable under specialization, i.e. if $y_0 \in \overline{y_1}$ and $y_1 \in f(X)$, then $y_0 \in f(X)$.*

Example 75. It is obvious that for the image to be closed, it must be stable under specialization. The converse is not true in general. For example, let $X = \coprod_{z \in \mathbb{C}} \text{Spec } \mathbb{C}$, and define a map $f : X \to \mathbb{A}^1_{\mathbb{C}}$. So the generic point $\zeta \in \mathbb{A}^1_{\mathbb{C}}$ is in the closure of $f(X)$ but is not the specialization of any point in $f(X)$. So $f(X)$ is nevertheless closed under specialization, but f is not a quasicompact morphism.

Proof. If $f(X) \subset Y$ is closed, then this is immediate. For the other direction, suppose $f(X)$ is closed under specialization. Given $y_0 \in \overline{f(X)}$, we need to find $y_1 \in f(X)$ such that $y_0 \in \overline{y_1}$.

We reduce to the affine case as follows. We replace Y by an affine neighborhood U of y_0, and then it will suffice to prove the statement for $f|_{f^{-1}(U)} : f^{-1}(U) \to U$. But since $f^{-1}(U)$ is quasicompact, then we can write it as a finite union $f^{-1}(U) = \bigcup_{i=1}^N X_i$ of affine opens, and then y_0 is in the closure of some $f(X_i)$. Thus we can take $X = \coprod_{i=1}^N X_i$, which is affine. So finally, we write $X = \text{Spec } B$ and $Y = \text{Spec } A$.

We make a further reduction: this is a purely topological statement, so we can assume X and Y are reduced. That is, we have a factorization

$$\begin{tikzcd}
\overline{f(X)}_{red} \arrow[r, hook] & Y \\
X_{red} \arrow[u, dashed] \arrow[r, hook] & X; \arrow[u]
\end{tikzcd}$$

on the level of rings, if we write $K = \ker(A \to B/\sqrt{B})$ then we have

$$\begin{tikzcd}
A/K \arrow[d] & A \arrow[l] \arrow[d] \\
B/\sqrt{B} & B. \arrow[l]
\end{tikzcd}$$

And then by replacing Y by $\overline{f(X)}_{red}$ and X by $X \times_Y \overline{f(X)}_{red}$, we can assume that $Y = \overline{f(X)}$. (If $Z \subset |X|$ is a closed subset of a scheme X, Z has a canonical subscheme structure given by locally declaring that if $X = \text{Spec } B$, then we take Z to be cut out by $I = \{f \in B : S \subset V(f)\} = \bigcap_{[\mathfrak{p}] \in S} \mathfrak{p}$.)

So now we have that $X = \text{Spec } B$, $Y = \text{Spec } A$, $Y = \overline{f(X)}$, and Y is reduced. On the level of rings, we're looking at $A \hookrightarrow B$ (since we replace A by A/K). So, let $\mathfrak{p} \subset A$ be the prime ideal corresponding to y_0. Take $\mathfrak{q} \subset \mathfrak{p}$ to be a minimal prime contained in $\mathfrak{p}$. This means that $A_{\mathfrak{q}}$ is a field, and localization is exact so we get $A_{\mathfrak{q}} \hookrightarrow B \otimes_A A_{\mathfrak{q}}$. Thus $[\mathfrak{q}] \in f(X)$. (Geometrically, we're just taking a generic point of an irreducible component containing y_0, and then if this weren't in $f(X)$ then the closure of the image would be strictly smaller than we've assumed.) ☐

When we combine the previous two lemmas, we should be much more comfortable with our valuative criteria: really, everything is about specialization.

We use the following result to get ourselves away from having to deal with closed immersions.

Lemma 9. *Let $f : X \to Y$ be a morphism of schemes. Then $\Delta : X \to X \times_Y X$ is a closed immersion iff $\Delta(X) \subset |X \times_Y X|$ is closed.*

Proof. The forward direction is immediate, so suppose that $\Delta(X) \subset |X \times_Y X|$ is closed. Note that we have a "retraction"

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, equal] & X \times_Y X \arrow[d, "p_1"] \\
 & X
\end{tikzcd}$$

which on topological spaces induces a diagram

$$\begin{tikzcd}
{|X|} \arrow[r, "\delta"] \arrow[dr, equal] & \Delta(X) \arrow[d] \\
 & {|X|.}
\end{tikzcd}$$

This implies that δ is a homeomorphism onto its image. This means that on the level of sheaves we have

$$\begin{tikzcd}
\mathcal{O}_X \arrow[dr, equal] & \Delta^{-1}\mathcal{O}_{X \times_Y X} \arrow[l, "\Delta^\sharp"'] \\
 & \Delta^{-1}p_1^{-1}\mathcal{O}_X, \arrow[u, "\Delta^{-1}(p_1^\sharp)"']
\end{tikzcd}$$

which means that $\Delta^\sharp$ is surjective. Thus $\Delta : X \to X \times_Y X$ is a closed immersion. ☐

So in fact, we've shown that Δ is always a homeomorphism onto its image and that the above map on schemes is always surjective.

Theorem 11 (valuative criterion for separatedness)**.** *Suppose $f : X \to Y$ is a morphism of schemes with quasi-compact diagonal. Then f is separated iff for any valuation ring R with $K = \mathrm{Frac}(R)$ and if we write $U = \mathrm{Spec}\ K$, $T = \mathrm{Spec}\ R$, for any diagram of solid arrows*

$$\begin{tikzcd}
U \arrow[r] \arrow[d, hook] & X \arrow[d] \\
T \arrow[r] \arrow[ur, dashed] & Y,
\end{tikzcd}$$

there is at most one dotted arrow filling in the diagram.

Proof. For the forward implication, suppose $h_1, h_2 : T \to X$ both fill in the diagram. Then T is reduced, and from the diagram

$$\begin{tikzcd}
U \arrow[r, hook, "\text{dense}"] \arrow[d] & T \arrow[d, "h_1 \times h_2"] \\
X \arrow[r, "\Delta"'] & X \times_Y X
\end{tikzcd}$$

we obtain that $(h_1 \times h_2)^{-1}(\Delta_X) = T$ (even scheme-theoretically). So $h_1|_U = h_2|_U$, so $h_1 = h_2$.

For the backward implication, by our lemmas it will suffice to check that $\Delta(X)$ is closed under specialization. So suppose $\zeta_1 \in \Delta(X)$, and let $\zeta_0 \in \overline{\zeta_1} \subset X \times_Y X$. Write $K = k(\zeta_1)$, and write O for the local ring of ζ_0 in $\overline{\zeta_1}_{red}$. Choose a valuation ring R that dominates O, i.e. $O \subset R \subset K$. Then we obtain the diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & T \arrow[d, "h_1 \times h_2"] \arrow[dl, dashed] \\
X \arrow[r] & X \times_Y X;
\end{tikzcd}$$

since we are assuming the valuative criterion holds, we have the dotted arrow (since $h_1 = h_2$). This sends the closed point to ζ_0, so $\Delta(X)$ is stable under specialization. ☐

There are perhaps more things to check and say about valuative criteria; the interested reader should consult Hartshorne.

7 Divisors and the Picard group

There are two kinds of divisors: Weil divisors and Cartier divisors. In many good cases, these give us information on the Picard group (the group of line bundles under tensor product).

We begin with some definitions.

7.1 Definitions

Definition 41. If $z \in X$ is a point, we say that z has *codimension 1* if $\overline{z} \subset X$ is maximal among proper (i.e. strictly contained in X) closed irreducible subsets.

Example 76. Given a Dedekind domain A (e.g. $A = k[t]$ or $A = \mathbb{Z}[\sqrt{-5}]$), the points of codomension 1 in Spec A are exactly the closed points. More generally, if A is an integral domain, then $[\mathfrak{p}] \in$ Spec A having codimension 1 means that $\mathfrak{p} \neq (0)$ and that if $(0) \subset \mathfrak{q} \subset \mathfrak{p}$ with $\mathfrak{q} \subset A$ prime, then either $\mathfrak{q} = (0)$ or $\mathfrak{q} = \mathfrak{p}$. That is, $[\mathfrak{p}] \in$ Spec A has codimension 1 when $\mathfrak{p}$ is minimal among nonzero prime ideals.

Definition 42. Let X be an integral separated noetherian scheme which is *regular in codimension 1* (i.e., for every $z \in X$ of codimension 1, $\mathcal{O}_{X,z}$ is a dvr). Then a *Weil divisor* on X is a finite formal sum $\sum_{\{z \in X : \mathrm{codim}(z)=1\}} n_z \cdot z$, where $n_z \in \mathbb{Z}$. These obviously form a group, which is just $\mathrm{Div}(X) = \bigoplus_{\{z \in X : \mathrm{codim}(z)=1\}} \mathbb{Z} \cdot z$. (When it is clear, we will omit the index set.)

There is a map div : $k(X)^\times \to \mathrm{Div}(X)$ given by $f \mapsto \sum v_z(f) \cdot z$, where v_z is the valuation on $\mathcal{O}_{X,z}$ (which has fraction field equal to $k(X)$). (Recall that in the affine case, if $z = [\mathfrak{p}] \in$ Spec A has codimension 1 then we have $(0) \subset \mathfrak{p} \subset A$, so we get a localization $\mathcal{O}_{X,z} = A_{\mathfrak{p}} \to k(X) = A_{(0)}$.) This is in fact a homomorphism.

Lemma 10. *This is well-defined, i.e. for $f \in k(X)^\times$, $v_z(f) = 0$ for all but finitely many z.*

Proof. Since we've assumed X is noetherian, it is enough to consider $X =$ Spec A. Suppose $f \in \mathrm{Frac}(A)$. Then there exists a (basic) open $U \subset$ Spec A such that $f \in \Gamma(U, \mathcal{O}_U)$, which is dense because any nonempty open is dense. For $1/f$ there similarly exists a dense open $V \subset$ Spec A such that $1/f \in \Gamma(V, \mathcal{O}_V)$. Let $Z =$ Spec $A \backslash V$. Remember that the valuation on a unit is always zero, so we only get nonzero valuation for those codimension 1 points contained in Z. But since we are in the noetherian case, if $z \in Z$ has codimension 1 in X and z lies in an irreducible component Z_i of Z, then it must be that $\overline{z} = Z_i$ since z has codimension 1 (and $\overline{z} \subset Z \subset$ Spec A). There are only finitely many of these, since Z_{red} is noetherian. ☐

Definition 43. The *class group* of such a scheme X is the cokernel of div : $k(X)^\times \to \mathrm{Div}(X)$. This is denoted $\mathrm{Cl}(X)$.

Remark 41. If K is a number field with ring of integers $\mathcal{O}_K$, then Cl(Spec $\mathcal{O}_K$) agrees with the class group in the sense of number theory.

The problem with Weil divisors is that they only see codimension 1 information. So we have also the notion of a Cartier divisor.

Definition 44. If X is a scheme, an *effective Cartier divisor* on X is a closed subscheme $Z \hookrightarrow X$ such that if $\mathcal{I}_Z \subset \mathcal{O}_X$ is the ideal sheaf of Z, then locally $\mathcal{I}_Z = (f)$, where $f \in \mathcal{O}_X(U)$ is not a zerodivisor. (For example, if X is integral, then we may simply require that $f \neq 0$.) (More generally, the definition should be that $\cdot f : \mathcal{O}_X|_U \to \mathcal{O}_X|_U$ should be injective on every $V \subset U$, but $\mathcal{O}_X$ is quasicoherent so it suffices to check on U itself.)

Let $\mathrm{Ca}^+(X)$ denote the set of effective Cartier divisors. This is a monoid: if $\mathcal{I}_Z$ and $\mathcal{I}_{Z'}$ are the ideal sheaves of two elements, we obtain $\mathcal{I}_Z \otimes \mathcal{I}_{Z'} \to \mathcal{O}_X$, a quasicoherent sheaf also defining a Cartier divisor: locally, the generator is just the product of the generators.

This isn't a group in general, but it nevertheless has a *groupification.* Define $\underline{\mathrm{Ca}}^+$ to be the presheaf on X given by $U \mapsto \mathrm{Ca}^+(U)$. This is a presheaf of monoids, and we define $\underline{\mathrm{Ca}}$ to be the associated *sheaf* of groups (i.e., we take groupification on each open set and then sheafify the result, or since these are both left adjoints we could first sheafify our sheaf of monoids and then groupify over each open set).

Definition 45. The group of *Cartier divisors* is $\mathrm{Ca}(X) = \underline{\mathrm{Ca}}(X)$, the global sections of the sheaf $\underline{\mathrm{Ca}}$.

Remark 42. Certainly there is a map $(\mathrm{Ca}^+(X))^{groupification} \to \mathrm{Ca}(X)$, but it is not always an isomorphism: there are Cartier divisors that are not simply the difference of two effective Cartier divisors.

Example 77. Let $X = \mathrm{Spec}\ k[x, y]$. Define Weil divisors $Z_x = V(x)$ and $Z_y = V(y)$. Then we have $Z_x + Z_y = V(xy)$, and $Z_x + Z_x = V(x^2)$. (We're not just talking about closed subsets, but about sub*schemes*; the latter has some nilpotence in its structure sheaf.)

Now, suppose X is integral. Let $\mathcal{K}$ be the constant sheaf associated to $k(X)$. Then for all open $U \subset X$, there's the restriction map $\mathcal{O}_X(U) \to \mathcal{K}$. This also gives us an inclusion $\mathcal{O}_X^\times \hookrightarrow \mathcal{K}^\times$, and there's a map $\underline{\mathrm{Ca}}^+ \to \mathcal{K}^\times/\mathcal{O}_X^\times$ defined by $\mathcal{I}_Z = (f) \mapsto [f]$. (If over one open set $\mathcal{I}_Z = (f)$ and over another open set $\mathcal{I}_Z = (g)$, then over the overlap f and g differ by a unit.) This map is compatible with the monoid structure.

Proposition 17. *This map induces an isomorphism*

$$\underline{\mathrm{Ca}} \xrightarrow{\sim} \mathcal{K}^\times/\mathcal{O}_X^\times.$$

Thus $\mathrm{Ca}(X) = \Gamma(X, \mathcal{K}^\times/\mathcal{O}_X^\times)$.

Remark 43. The group $\Gamma(X, \mathcal{K}^\times/\mathcal{O}_X^\times)$ is *not* in general isomorphic to $\Gamma(X, \mathcal{K}^\times)/\Gamma(X, \mathcal{O}_X^\times)$; this is the problem with sheaf-cokernel. In fact, we have the following definition.

Definition 46. The *Cartier class group*, denoted $\mathrm{CaCl}(X)$, is the cokernel of the natural map $\Gamma(X, \mathcal{K}^\times)/\Gamma(X, \mathcal{O}_X^\times) \longrightarrow \Gamma(X, \mathcal{K}^\times/\mathcal{O}_X^\times)$.

Remark 44. This works more generally even if X is not integral, with $\mathcal{K}$ replaced by the "sheaf of total quotients".

Remark 45. When we talk about cohomology, we will see that we have an exact sequence $0 \to \mathcal{O}_X^\times \to \mathcal{K}^\times \to \mathcal{K}^\times/\mathcal{O}_X^\times \to 0$ of sheaves, which will give us an exact sequence

$$0 \to \mathcal{O}_X^\times(X) \to k(X)^\times \mathrm{Ca}(X) \to H^1(X, \mathcal{O}_X^\times) \to 0.$$

of groups. (Since $\mathcal{K}$ is constant, it has no higher cohomology.) This cohomology group $H^1(X, \mathcal{O}_X^\times)$ is most naturally intepreted as $\mathrm{Pic}(X)$, the *Picard group.*

7.2 Weil divisors vs. Cartier divisors

We will obtain a map $\mathrm{Ca}(X) \to \mathrm{Div}(X)$, which will enable us to compare these two groups of divisors.

We begin by defining a sheaf $\underline{\mathrm{Div}}$ on X by

$$\underline{\mathrm{Div}}(U) = \mathrm{Div}(U) = \bigoplus_{\{z \in U : \mathrm{codim}(z) = 1\}} \mathbb{Z} \cdot z.$$

Actually, $\underline{\mathrm{Div}} \simeq \bigoplus_z z_*\mathbb{Z}$, where $z : * \to X$ is the inclusion of a point and $\mathbb{Z}$ denotes the trivial sheaf $\mathbb{Z}$ over $*$. It must be checked, but this really is a sheaf. Of course, $\mathrm{Div}(X) = \Gamma(X, \underline{\mathrm{Div}})$.

We understand $\underline{\mathrm{Ca}}^+$ very well, but the problem is that we don't necessarily know much about $\underline{\mathrm{Ca}}$ since it's a sheafification. So to an effective Cartier divisor $Z \subset U$ we associate $\mathcal{I}_Z \subset \mathcal{O}_U$. Thus for all codimension 1 points

$z \in U$, we get $\mathcal{I}_{Z,z} \subset \mathcal{O}_{U,z}$, a principal ideal in the dvr $\mathcal{O}_{U,z}$ defined by $(\pi_z^{v_z(\mathcal{I}_Z)})$, where π_z is the generator of the maximal ideal in $\mathcal{O}_{U,z}$. So, we get a map $\underline{\mathrm{Ca}}^+ \to \underline{\mathrm{Div}}$ by

$$z \mapsto \sum_z v_z(\mathcal{I}_Z) \cdot z.$$

(This is a finite sum, and is independent of the choice of π_z.) But as $\underline{\mathrm{Div}}$ is a sheaf, this defines a map $\underline{\mathrm{Ca}} \to \underline{\mathrm{Div}}$. Taking global sections, we finally obtain our map $\Phi : \mathrm{Ca}(X) \to \mathrm{Div}(X)$.

Concretely, we're looking at a commutative diagram

$$\begin{tikzcd}
\mathrm{Ca}(X) \arrow[d] & \Gamma(X, \mathcal{K}^\times/\mathcal{O}_X^\times) \arrow[l, "\sim"'] & \Gamma(X, \mathcal{K}^\times) \arrow[l] \arrow[d, equal] \\
\mathrm{Div}(X) & & k(X)^\times. \arrow[ll, "\mathrm{div}"']
\end{tikzcd}$$

Theorem 12. *If every local ring $\mathcal{O}_{X,x}$ is a UFD, then Φ is an isomorphism.*

Idea of proof. It is a theorem of commutative algebra that $\mathcal{O}_{X,x}$ is a UFD iff it is a normal domain (i.e. it is an integral domain which is integrally closed in its field of fractions) and $\mathrm{Cl}(\mathrm{Spec}\ \mathcal{O}_{X,x}) = 0$. So, we attempt to define a backwards map $\mathrm{Div}(X) \to \mathrm{Ca}(X)$ as follows. Suppose $z \in X$ is a codimension 1 point. We'd like to send $z \mapsto \overline{z} \subset X$. So the question is, when is the ideal sheaf $\mathcal{I}_{\overline{z}}$ locally generated by a single element? The answer is that this holds iff $\mathcal{O}_{X,x}$ is a UFD for all $x \in X$. At least in one direction, we can see that $\mathcal{I}_{\overline{z},x} \subset \mathcal{O}_{X,x}$ is the closure of $z \in \mathrm{Spec}\ \mathcal{O}_{X,x}$. ☐

Example 78. In fact, any ring is a UFD iff it is a normal domain with trivial class group. For example, $k[t]$ is a UFD (for k a field), and as we have seen, $\mathrm{Cl}(\mathbb{A}^1_k) = 0$. The proof is closely related to the PID property, and uses the fact that the ideal sheaves of points are all principal.

Example 79. The standard non-example of a UFD is $\mathbb{Z}[\sqrt{-5}]$, in which $6 = 2 \cdot 3 = (1+\sqrt{5})(1-\sqrt{-5})$. It turns out that $\mathrm{Cl}(\mathbb{Z}[\sqrt{-5}]) = \mathbb{Z}/2$. At the very least, we know it's nonzero because we have a nonprincipal ideal.

We have, at this point, obtained the following table:

Weil divisors	$\mathrm{Div}(X) = \bigoplus_{\mathrm{codim}(z)=1} \mathbb{Z} \cdot z$	$\mathrm{Cl}(X) = \mathrm{Div}(X)/\text{principal}$
Cartier divisors	$\mathrm{Ca}(X) = \Gamma(X, \mathcal{K}^\times/\mathcal{O}^\times)$	$\mathrm{CaCl}(X) = \{\mathit{coker}(\Gamma(X, \mathcal{K}^\times) \to \Gamma(X, \mathcal{K}^\times/\mathcal{O}^\times))$
line bundles	?	$\mathrm{Pic}(X) = (\{\text{invertible sheaves}/\sim, \otimes)$.

There is always a diagram

$$\begin{tikzcd}
\mathrm{Ca}(X) \arrow[r] \arrow[d] & \mathrm{Div}(X) \arrow[d] \\
\mathrm{CaCl}(X) \arrow[r] & \mathrm{Cl}(X),
\end{tikzcd}$$

and the horizontal maps tend to be injective, though this is not always the case. We can write down a Weil divisor, and then the condition of being a Cartier divisor is that the ideal can be written globally as a principal ideal (not just locally); this is the question of surjectivity.

Example 80 (the standard counterexample to surjectivity)**.** Let $X = \mathrm{Spec}\ k[x,y,z]/(xy - z^2)$. This is a cone centered about the z-axis. Consider the ideal $I = (x, z)$. Writing $P = (0,0,0)$ for the unique closed point at which $z = 0$, we have that $X - P = \mathrm{Spec}\ k[x^\pm, z] \cup \mathrm{Spec}\ k[y^\pm, z]$.

Note that $\mathfrak{m}_P/\mathfrak{m}_P^2$ is a 3-dimensional k-vector space with basis x, y, z (because our relations are only in dimension 2, which we're quotienting by anyways). The image of I in $\mathfrak{m}/\mathfrak{m}^2$ is the 2-dimensional space spanned by x and z. Thus I cannot be a principal ideal. This means that we get an effective Weil divisor which is not Cartier.

To compare I with the Cartier divisor (x), we can compare valuations. In the decomposition of $X - P$, we get that I is (1) and (z), whereas (x) is (1) and (z^2) The punchline here is that $\mathrm{Cl}(X) = \mathbb{Z}/2$, generated by I. On the other hand, $\mathrm{CaCl}(X) = 0$.

7.3 Cartier divisors vs. invertible sheaves

What should the question mark in our table be?

Suppose that X is integral. Let $\mathcal{K}$ be the constant sheaf associated to $k(X)$. We want a map $\mathrm{Ca}(X) \to \mathrm{Pic}(X)$. We have the diagram

$$\begin{tikzcd}
\mathrm{Ca}(X) \arrow[r, dashed] & \mathrm{Pic}(X) \\
\mathrm{Ca}^+(X). \arrow[u] \arrow[ur, "(Z\subset X)\mapsto \mathcal{I}_Z^\vee"'] &
\end{tikzcd}$$

Unfortunately, the vertical map isn't so easily understandable, although it's okay in the case that $\mathrm{Div}(X) \cong \mathrm{Ca}(X)$. However, note that $\mathcal{I}_Z$ comes equipped with an embedding $\mathcal{I}_Z \hookrightarrow \mathcal{O}_X$, and we can postcompose to $\mathcal{I}_Z \hookrightarrow \mathcal{O}_X \hookrightarrow \mathcal{K}$.

Now, consider the set

$$\mathcal{P}\mathrm{ic}(X) = \{(L, \sigma) \ : \ L \text{ an invertible sheaf, } \sigma : L \hookrightarrow \mathcal{K} \text{ as } \mathcal{O}_X\text{-submodules}\}.$$

We claim that this is what should fill in the last spot. First of all, we certainly have a map $\mathcal{P}\mathrm{ic}(X) \to \mathrm{Pic}(X)$, and the sequence $0 \to k(X)^\times \to \mathcal{P}\mathrm{ic}(X) \to \mathrm{Pic}(X) \to 0$ (where $f \in k(X)^\times$ is sent to $(\mathcal{O}_X, \mathcal{O}_X \xrightarrow{f} \mathcal{K})$) is exact. In particular, we have a surjection $\mathcal{P}\mathrm{ic}(X) \twoheadrightarrow \mathrm{Pic}(X)$, i.e. if L is an invertible sheaf on X then there is an embedding $\sigma : L \hookrightarrow \mathcal{K}$. (For this to hold in general, we need at least that $\mathcal{K}^\times$ is *flasque*, i.e. restriction maps are surjective. As this is just a constant sheaf, this is true whenever we restrict to irreducible components.)

To see this, note that $L \hookrightarrow L \otimes_{\mathcal{O}_X} \mathcal{K}$, so it suffices to show that $L \otimes_{\mathcal{O}_X} \mathcal{K} \simeq \mathcal{K}$ as $\mathcal{O}_X$-modules. If we have a trivializing open cover $X = \bigcup_i U_i$ and trivializations $\sigma_i : L|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}$, then on $U_i \cap U_j$ we have cocycles

$$u_{ij} : \mathcal{O}_{U_{ij}} \xrightarrow{\sigma_i^{-1}} L|_{U_{ij}} \xrightarrow{\sigma_j} \mathcal{O}_{U_{ij}}$$

(i.e. $u_{jk}u_{ij} = u_{ik}$). So, L is trivial if and only if there is some $v_i \in \Gamma(U_i, \mathcal{O}_{U_i}^\times)$ such that $u_{ij} = v_j^{-1}v_i$; that is, we just chose the "wrong" trivializations. (This should smell a whole lot like computations of cohomology.)

So, it suffices to show that $L \otimes_{\mathcal{O}_X} \mathcal{K}$ is trivial as a rank 1 $\mathcal{K}$-module. (If X is an irreducible scheme and $\mathcal{K}$ is a constant sheaf of rings, then any invertible $\mathcal{K}$-module $\mathcal{M}$ is trivial. We can anchor our trivialization at say the generic point $\eta \in X$, and then for any nonempty open $U \subset X$ such that $\mathcal{M}|_U$ is trivial, then the restriction map $\mathcal{M}(U) \to \mathcal{M}_\eta$ is an isomorphism. Also, we have an isomorphism $\mathcal{K}(U) \to \mathcal{K}_\eta$. So if we fix an isomorphism $\mathcal{M}_\eta \simeq \mathcal{K}_\eta$, then there exists a unique isomorphism $\mathcal{M}(U) \simeq \mathcal{K}(U)$.) So we get our isomorphism $L \otimes_{\mathcal{O}_X} \mathcal{K} \xrightarrow{\sim} \mathcal{K}$.

Remark 46. If we knew about cohomology, we could say that $H^i(X, \mathcal{K}) = 0$ for $i > 0$.

One might ask: What does this strange equivalence relation have to do with functions? This was actually known quite classically (way before the definition of schemes). Suppose X is an integral separated scheme of finite type over k that is regular in codimension 1, and $f \in k(X)^\times$. We should think of $\mathrm{div}(f)$ as follows. First of all, there is some dense open $U \subset X$ such that $f \in \Gamma(U, \mathcal{O}_U)$, which corresponds to a map $f : U \to \mathbb{A}^1_k$. If we embed $\mathbb{A}^1_k \subset \mathbb{P}^1_k$, then we can try to extend to a map $f : X \to \mathbb{P}^1_k$. This may not be possible, but nevertheless there is some $V \subset X$ containing U and an extension $f : V \to \mathbb{P}^1_X$ such that V contains all codimension 1 points of X. (That is, we can extend over all points of codimension at least 2.) Indeed, if $\mathrm{codim}(z) = 1$, when we look at the local ring at that point we get a diagram

$$\begin{tikzcd}
\mathrm{Spec}\ k(X) \arrow[r] \arrow[d] & U \arrow[r] & \mathbb{P}^1_k \arrow[d] \\
\mathrm{Spec}\ \mathcal{O}_{X,z} \arrow[rr] & & \mathrm{Spec}\ k.
\end{tikzcd}$$

Since $\mathcal{O}_{X,z}$ is a dvr and $\mathbb{P}^1_k$ is proper, we get a diagonal lift $\mathrm{Spec}\ \mathcal{O}_{X,z} \to \mathbb{P}^1_k$. Now, we claim that $\mathrm{Div}(V) \simeq \mathrm{Div}(X)$. Indeed, if we have the extension $F : V \to \mathbb{P}^1_k$, then $\div(f) = F^{-1}(0) - F^{-1}(\infty) = (V \times_{\mathbb{P}^1_{k,0}} \mathrm{Spec}\ k) - (V \times_{\mathbb{P}^1_{k,\infty}} \mathrm{Spec}\ k)$ (the scheme-theoretic fiber).

7.4 A few geometric computations of Picard groups

Example 81. We claim that for a (separated) integral scheme X, $\operatorname{Pic}(\mathbb{A}^1_X) \cong \operatorname{Pic}(X)$ (recall that $\mathbb{A}^1_X = \mathbb{A}^1_Z \times_{\text{Spec } \mathbb{Z}} X$). We have a projection $\pi : \mathbb{A}^1_X \to X$ admitting a section $s : X \to \mathbb{A}^1_X$ given by $s : \text{Spec } \mathbb{Z} \times_{\text{Spec } \mathbb{Z}} X \to \mathbb{A}^1_{\mathbb{Z}} \times_{\text{Spec } \mathbb{Z}} X$, i.e. $\pi \circ s = \mathrm{id}_X$. So these give us maps $\pi^* : \operatorname{Pic}(X) \to \operatorname{Pic}(\mathbb{A}^1_X)$ and $s^* : \operatorname{Pic}(\mathbb{A}^1_X) \to \operatorname{Pic}(X)$ such that $\mathrm{id}_{\operatorname{Pic}(X)} = \mathrm{id}^*_X = (\pi \circ s)^* = s^* \circ \pi^*$.

So, let L be an invertible sheaf on $\mathbb{A}^1_X$. We claim that $L = \pi^* M$, where M is an invertible sheaf on X (so it must be that $M = s^* L$). It suffices to show that there is an open cover of $X = \bigcup_i \text{Spec } R_i$ such that $L|_{\mathbb{A}^1_{R_i}}$ is trivial. For then, let us represent L as a cocycle $(u_{ij}) \in \prod_{i,j} \Gamma(\mathbb{A}^1_{\text{Spec } R_i \cap \text{Spec } R_j}, \mathcal{O}^\times)$. If we define R_{ij} by $\text{Spec } R_{ij} = \text{Spec } R_i \cap \text{Spec } R_j$ (by separatedness, though this is probably not strictly necessary) then we have $\Gamma(\mathbb{A}^1_{R_{ij}}, \mathcal{O}^\times) \cong (R_{ij}[t])^\times = R_{ij}^\times$. So, we can just take M to correspond to the same cocycle (u_{ij}).

Now, choose a point $x \in X$. Then we have the rings $\mathcal{O}_{X,x} \hookrightarrow \mathcal{O}_{X,x}[t]$, and we have a projective $\mathcal{O}_{X,x}[t]$-module P of rank 1; we'd like to say that P is free. If we write $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ for the maximal ideal, then we have $P/\mathfrak{m}_x P \simeq k(x)[t] \cdot \overline{p}$. Let p be a lifting of $\overline{p}$. By Nakayama's lemma, p is a basis $\mathcal{O}_{X,x}[t] \to P$; at every point of $\mathcal{O}_{X,x}[t]$ lying over $\mathfrak{m}_x$, it gives an isomorphism.

Example 82. Let P be a finitely generated integral saturated monoid. For example, we could take a polytope in Euclidean space, then take the integral points of the cone, e.g. $\{(x,y) \in \mathbb{Z}^2 : x > 0,\ |y| < x\}$. But actually, we generate this as $\{x = (1,1), y = (1,-1), z = (1,0)\}$, with the relation $x + y = 2z$. Recall that *integral* means that the map $P \to P^{gp}$ is integral, and *saturated* means that $P = \{p \in P^{gp} : \exists n > 0,\ np \in P\}$. Or alternatively, we could generate by the vertices $\{x = (0,0), w = (1,0), y = (1,1) = z = (0,1)\} \in \mathbb{Z}^2 \subset \mathbb{R}^3$ of a unit square, giving the relation $x + y = z + w$. We want to consider the monoid-ring $k[P]$, so for example the former monoid gives $k[x,y,z]/(xy - z^2)$ and the latter gives the ring $k[x,y,z,w]/(xy - zw)$.

We claim that $\operatorname{Pic}(\text{Spec } k[P]) = 0$. The idea here is that there is a torus action on $\text{Spec } k[P]$. We'll stick to the first example. Let $X = \text{Spec } k[x,y,z]/(xy - z^2)$. Then we have the action $\mathbb{G}_m \times X \to X$ via $(u,(a,b,c)) \mapsto (ua, ub, uc)$. Moreover, we have an embedding $\mathbb{G}_m^2 = \text{Spec } k[x^\pm, z^\pm] \hookrightarrow X$. Now, our $\mathbb{G}_m$-action corresponds to a ring homomorphism $k[x,y,z]/(xy - z^2) \to (k[x,y,z]/(xy - z^2))[t^\pm]$, and this is given by $x \mapsto tx$, $y \mapsto ty$, $z \mapsto tz$. So this actually extends to a map $\varphi : \mathbb{A}^1 \times X \to X$ by the exact same formula. (This should be thought of as a *homotopy* of maps $X \to X$, or alternatively of X as a *retract* of $\mathbb{A}^1 \times X$.)

Now, we have $\operatorname{Pic}(\mathbb{A}^1 \times X)$, and we can pull back for different projections $\mathbb{A}^1 \times X$, giving us

$$\begin{tikzcd}
\operatorname{Pic}(\{1\} \times X) & & \\
& \operatorname{Pic}(\mathbb{A}^1_X) \arrow[ul] \arrow[dl] & \operatorname{Pic}(X) \arrow[l, "\varphi^*"'] \\
\operatorname{Pic}(\{0\} \times X) & \operatorname{Pic}(X). \arrow[u, "\pi^*"'] &
\end{tikzcd}$$

All but one of these maps (and some others) are isomorphisms by the previous example, but there's the one strange map $\rho_0^* : \operatorname{Pic}(\{0\} \times X) \to \operatorname{Pic}(X)$ from the bottom left to the far right. This is given by $\rho_0 : k[x,y,z]/(xy - z^2) \to k[x,y,z]/(xy - z^2)$ via

$$\begin{tikzcd}
k[x,y,z]/(xy - z^2) \arrow[r, "\rho_0"] \arrow[d, "(x,y,z)\mapsto(0,0,0)"'] & k[x,y,z]/(xy - z^2) \\
k, \arrow[ur] &
\end{tikzcd}$$

so

$$\begin{tikzcd}
\mathrm{Pic}(X) \arrow[r, "\rho_0^*"] \arrow[d, "0^*"'] & \mathrm{Pic}(X) \\
\mathrm{Pic}(\mathrm{Spec}\ k) \arrow[ur] &
\end{tikzcd}$$

and Pic(Spec k) $= 0$, so $\rho_0^* = 0$. On the other hand, ρ_0^* is an isomorphism by the big diagram, so Pic$(X) = 0$!

Example 83. We will compute Pic(Proj $k[x, y, z, w]/(xy - zw)$). The monoid P here is given by the integer points of the cone over the vertices of a unit square $\{(0, 0, 1), (1, 0, 1), (1, 1, 1), (0, 1, 1)\} \subset \mathbb{Z}^3$ (from the previous example).

We understand $X = \mathrm{Proj}\ k[x, y, z, w]/(xy - zw)$ by covering it by charts, e.g. $U_x = \mathrm{Spec}(k[x^{\pm}, z, w])_0 = \mathrm{Spec}\ k[z/x, w/x] = \mathrm{Spec}\ k[(P_x)_0]$; here P_x is the localization of the monoid P at x. Similarly, the rest of the charts are $U_y = \mathrm{Spec}\ k[(P_y)_0]$, $U_z = \mathrm{Spec}\ k[(P_z)_0]$, $U_w = \mathrm{Spec}\ k[(P_w)_0]$. Now, these are all just affine planes, so they have trivial Picard groups, so we can understand line bundles on X just by transition maps for this cover. Let us denote $X^0 = U_x \cap U_y \cap U_z \cap U_w = U_x \cap U_y$ (by looking at the ring: if we've inverted x and y, then we've inverted w and z too). Let us denote $U_\alpha \cap U_\beta = V_{\alpha\beta}$, so $X^0 = V_{xy}$. Then we also have $V_{xz} = \mathrm{Spec}\ k[(P_{xz})_0]$, so e.g. the u_{xz} piece of the cocycle $(u_{\alpha\beta})$ will live in $k[(P_{xz})_0]^\times = (k[(z/x)^{\pm}, w/x])^\times = k^\times \cdot (z/x)^{\mathbb{Z}}$.

It will turn out that we'll only have to worry about intersections corresonding to faces of our square: let us call $F_1 = [x, z] = [(0, 0, 1), (0, 1, 1)]$, $F_2 = [x, w] = [(0, 0, 1), (1, 0, 1)]$, $F_3 = [w, y] = [(1, 0, 1), (1, 1, 1)]$, $F_4 = [z, y] = [(0, 1, 1), (1, 1, 1)]$. To compute the Picard group, we need to compute the cohomology of

$$\bigoplus_{\text{single intersections}} k^\times \longrightarrow \bigoplus_{\text{double intersections } F} k^\times \cdot (P_F)_0^\times \longrightarrow \bigoplus_{\text{triple intersections}} k^\times (P^{gp})_0$$

where the second map is given by $(u_{ij}) \mapsto u_{ij} u_{jk} u_{ik^{-1}}$. But in fact, this complex precisely computes the (honest topological) cellular cohomology of the original square! So in particular, the cohomology at the middle group is zero.

8 Projective morphisms

9 Differentials