# $TMF$ doctoral student seminar

## Contents

# 1 Elliptic curves and modular forms over $\mathbb{C}$ – Alexander Körschgen

## 1.1 Complex lattices

We begin with a few definitions.

- A *lattice* over $\mathbb{C}$ is a set $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ where $\{\omega_1, \omega_2\}$ is an $\mathbb{R}$-basis of $\mathbb{C}$.
- A *complex torus* is a quotient of the form $\mathbb{C}/\Lambda$ for a lattice $\Lambda$; this is a Riemann surface.
- The *upper half plane* is $\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.
- $SL_2(\mathbb{Z})$ is the *modular group*.
- The *Riemann sphere* $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ carries an action of the modular group via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

  This action restricts to an action on $\mathcal{H}$.
- A non-zero holomorphic homomorphism between tori is called an *isogeny*.

Let's begin to analyze these. Let $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$. We can assume without loss of generality that $\tau = \omega_1/\omega_2 \in \mathcal{H}$. Let us write $\Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau$. Then $\mathbb{C}/\Lambda \cong \mathbb{C}/\Lambda_\tau$.

We claim that $\tau$ is unique up to the action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$:

**Theorem 1.** $SL_2(\mathbb{Z})\backslash\mathcal{H} \xrightarrow{\cong} \{\text{complex tori}\}/\cong$.

## 1.2 Elliptic curves

Recall that the map $\mathbb{C}^2 \to P^2(\mathbb{C})$ given by $(x, y) \mapsto [x : y : 1]$ has image everything but a copy of $\hat{\mathbb{C}}$. We give the following ad hoc but sufficient definition: an *elliptic curve* is a subset $E \subset P^2(\mathbb{C})$ such that

$$E \cap (P^2(\mathbb{C}) - \hat{\mathbb{C}}) = \{\infty\} = \{[0 : 1 : 0]\}$$

and

$$E \cap \mathbb{C}^2 = \{(x, y) : y^2 = 4x^3 - g_2 x - g_3\}$$

for some $g_2, g_3 \in \mathbb{C}$. We define the *discriminant* to be $\Delta_E = g_2^3 - 27g_3^2$. This only depends on $E$ (and not the choices of $g_2$ and $g_3$). $E$ is called *smooth* (or *nonsingular*) if $\Delta_E \neq 0$. In the smooth case, we define the *$j$-invariant* as $j_E = 1728g_2^3/\Delta_E$. This also only depends on $E$. One generally thinks of an elliptic curve as just a cubic affine variety in $\mathbb{C}^2$ along with the point $\infty$. We will only consider smooth curves.

### 1.3 Elliptic curves and lattices

We want to associate an elliptic curve to each lattice. So, fix a lattice $\Lambda$. There is a $\Lambda$-periodic meromorphic function $\mathfrak{P} : \mathbb{C} \to \hat{\mathbb{C}}$, called the *Weierstraß $\mathfrak{P}$-function*, given by

$$\mathfrak{P}_\Lambda(z) = \frac{1}{z^2} + \sum_{w \in \Lambda - 0} \frac{1}{(z-w)^2} - \frac{1}{w^2}$$

for $z \in \mathbb{C} \backslash \Lambda$ and $\mathfrak{P}_\Lambda(z) = \infty$ for $z \in \Lambda$. Its complex derivative is

$$\mathfrak{P}'_\Lambda(z) = -2 \sum_{w \in \Lambda} \frac{1}{(z-w)^3}$$

In a certain sense, $\mathfrak{P}_\Lambda$ and $\mathfrak{P}'_\Lambda$ are essentially all of the $\Lambda$-periodic meromorphic functions: the algebra of these is $\mathbb{C}(\mathfrak{P}_\Lambda, \mathfrak{P}'_\Lambda)$.

For $k > 2$ even, we define the $k^{th}$ *Eisenstein series* of $\Lambda$ by

$$G_k(\Lambda) = \sum_{w \in \Lambda - 0} \frac{1}{w^3}.$$

We have the following result.

**Theorem 2.** *We have the relations*

$$\mathfrak{P}_\Lambda(z) = \frac{1}{z^2} + \sum_{n=2 \text{ even}}^{\infty} (n+1) G_{n+2}(\Lambda)$$

*and*

$$\mathfrak{P}'_\Lambda(z) = 4(\mathfrak{P}_\Lambda(z))^3 - g_2(\Lambda)\mathfrak{P}(z) - g_3(\Lambda),$$

*where $g_2(\Lambda) = 60G_4(\Lambda)$ and $g_3(\Lambda) = 140G_6(\Lambda)$. Moreover, the map $C/\Lambda \to \mathbb{C}^2 \cup \{\infty\} \subset P^2(\mathbb{C})$ given by $z \mapsto (\mathfrak{P}_\Lambda(z), \mathfrak{P}'_\Lambda(z))$ for $z \in \mathbb{C} - \Lambda$ and $z \mapsto \infty$ for $z \in \Lambda$ is a bijection from the torus to the elliptic curve given by $g_2(\Lambda), g_3(\Lambda) \in \mathbb{C}$. This curve is smooth. Moreover, given a smooth elliptic curve defined by $g_2, g_3 \in \mathbb{C}$, there is a lattice $\Lambda \subset \mathbb{C}$ such that $g_2(\Lambda) = g_2$ and $g_3(\Lambda) = g_3$. In fact, we have a bijection*

$$\{\text{tori}\}/\cong \leftrightarrow \{\text{smooth elliptic curve}\}/\sim,$$

*where the equivalence relation on the right is "admissible change of variables".*

**Remark 1.** The bijection actually defines an abelian group structure on $E$.

### 1.4 Modular forms

It turns out that $SL_2(\mathbb{Z})$ is generated by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and the corresponding elements of $\mathrm{Aut}(\hat{\mathbb{C}})$ are $\tau \mapsto \tau + 1$ and $\tau \mapsto -1/\tau$.

For any $k \in \mathbb{Z}$, we say that a meromorphic function $f : \mathcal{H} \to \hat{C}$ is weakly modular of weight $2k$ if $f(\gamma(\tau)) = (c\tau + d)^k f(\tau)$ (for

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

our generic element of $SL_2(\mathbb{Z})$). By our generators, we can see that this is equivalent to saying that $f(\tau+1) = f(\tau)$ and $f(-1/\tau) = \tau^k f(\tau)$. Of course, weak modularity of weight 0 means $SL_2(\mathbb{Z})$-invariant.

**Remark 2.** 0 is the only modular form of odd weight (or more precisely, which can be taken to have odd weight).

We want to have a notion of "holomorphic at $\infty$". So, let $f$ be any weakly modular function. The first relation tells us that $f$ is $\mathbb{Z}$-periodic. So, set $D = B_1(0)$ and $D' = D - \{0\}$. We have a $\mathbb{Z}$-periodic holomorphic map $\varphi : \mathcal{H} \to D'$ given by $\tau \mapsto \exp(2\pi i \tau)$, and from this we define the holomorphic map $g : D' \to \hat{\mathbb{C}}$ by $g(q) = f(\log(q)/2\pi i)$ (which is well defined by the periodicity of $f$). Now, $g$ is holomorphic on $D'$. Furthermore, note that $\lim_{\mathrm{Im}(\tau)\to\infty} \varphi(\tau) = 0$. Thus, we say that $f$ is *holomorphic at* $\infty$ if $g$ extends holomorphically to $D$. This is equivalent to the statement that $f(\tau)$ is bounded as $\mathrm{Im}(\tau) \to \infty$. So finally, we say that $f : \mathcal{H} \to \mathbb{C}$ is a *modular form of weight* $k$ if it is holomorphic on $\mathcal{H}$, holomorphic at $\infty$, and weakly modular of weight $k$.

We write $\mathcal{M}_k(SL_2(\mathbb{Z}))$ for the set of weight $k$ modular forms, and we write $\mathcal{M}(SL_2(\mathbb{Z})) = \bigoplus_{k\in\mathbb{Z}} \mathcal{M}_k(SL_2(\mathbb{Z}))$. In fact, this is a graded $\mathbb{C}$-algebra, and each $\mathcal{M}_k(SL_2(\mathbb{Z}))$ is a finite-dimensional $\mathbb{C}$-vector space.

If $f$ is a modular form, the associated function $g : D \to \mathbb{C}$ is holomorphic, so $g$ may be written as $g(q) = \sum_{n\geq 0} a_n q^n$; this is called the *Fourier expansion* of $f$.

**Example 1.** 0 is a modular form of every weight. Every constant function is a modular form of weight 0. For $k > 2$,

$$G_k(\tau) = G_k(\Lambda_\tau) = \sum_{(c,d)\in\mathbb{Z}^2-0} \frac{1}{(c\tau+d)^k}$$

is a modular form of weight $k$. $\Delta(\tau) = \Delta_{\Lambda_\tau}$ is a modular form of weight 12. $j(\tau) = j_{\Lambda_\tau}$ is a modular form of weight 0. So in particular, $j$ is invariant under the action of $SL_2(\mathbb{Z})$, and in fact it allows us to complete the diagram of isomorphisms

$$\begin{tikzcd} \{\text{tori}\}/\cong \arrow[r] \arrow[d] & \{\text{elliptic curves}\}/\sim \arrow[d] \\ SL_2(\mathbb{Z})\backslash\mathcal{H} \arrow[r, "j"'] & \mathbb{C}. \end{tikzcd}$$

**Theorem 3.** *$\mathcal{M}_k(SL_2(\mathbb{Z}))$ has a basis given by $\{G_4^\alpha G_6^\beta : \alpha, \beta \geq 0, 4\alpha + 6\beta = k\}$. That is, $\mathcal{M}(SL_2(\mathbb{Z})) \cong \mathbb{C}[G_4, G_6]$ as graded $\mathbb{C}$-algebras.*

## 1.5 The rational Witten genus

We define $\Omega = \Omega^{SO}$ to be the oriented bordism ring. For us, a *genus* is a (unital) ring homomorphism $\Omega \otimes \mathbb{Q} \to R$, for $R$ an integral domain.

We begin with a general construction, beginning with a power series and giving a genus. Let $Q(x) \in R[[x]]$ be an even power series with constant term 1, so $Q(x) = \sum_{i\geq 0} a_i x^{2i}$ with $a_0 = 1$. We introduce indeterminates $x_1, \ldots, x_n$ with $|x_i| = 2$. Consider the function $Q(x_1) \cdot \ldots \cdot Q(x_n) \in R[[x_1, \ldots, x_n]]$; this is symmetric in the variables $x_i^2$, so if we write $p_j$ (for $1 \geq j \leq n$) for the $j^{th}$ elementary symmetric function in the $x_i^2$, then we can write

$$Q(x_1) \cdot \ldots \cdot Q(x_n) = 1 + K_1(p_1) + K_2(p_1, p_2) + \ldots.$$

For $1 \leq r \leq n$, $K_r$ does not depend on $n$. Now, we define a genus $\varphi_Q : \Omega \otimes \mathbb{Q} \to R$ by $[M] \mapsto K_n(p_1, \ldots, p_n)[M]$ for $\dim M = 4n$, and $[M] \mapsto 0$ otherwise. The element $p_i = p_i(M) \in H^{4i}(M; \mathbb{Z})$ is called the $i^{th}$ *Pontrjagin class*.

Now, let us specialize. Suppose we have a lattice $\Lambda$. We define the Weierstraß $\sigma$-function

$$\sigma_\Lambda(z) = z \cdot \prod_{w \in \Lambda - 0} \left(1 - \frac{z}{w}\right) \cdot \exp\left(\frac{z}{w} + \frac{z^2}{2w^2}\right).$$

Then $Q_\Lambda(x) = x/\sigma_\Lambda(x)$ is an even power series, so the above construction yields a genus, the *Witten genus* $\varphi_W^\Lambda$ associated to $\Lambda$.

**Theorem 4.** *If $M^{4k}$ is a compact oriented smooth manifold such that $p_i(M) = w_2(M) = 0$ (for instance, if $M$ is a string manifold), then the map $z \mapsto \varphi_W^{\Lambda_\tau}([M])$ is a modular form of weight $2k$ with integral Fourier series.*

Thus we obtain a map $\varphi_W : \Omega^{String} \otimes \mathbb{Q} \to \mathcal{M}(SL_2(\mathbb{Z}))$ given by $[M] \mapsto (\tau \mapsto \varphi_W^{\Lambda_\tau}([M])$. This induces $MString_* \otimes \mathbb{Q} \to MF_* \otimes \mathbb{Q}$, where $MF_*$ is the graded ring of integral modular forms (i.e. those with integral Fourier series).

# 2 Generalized elliptic curves and modular forms – Sebastian Thyssen

## 2.1 Schemes

A *ringed space* is a pair $(X, \mathcal{O}_X)$ of a topological space $X$ and a sheaf $\mathcal{O}_X$ of rings on $X$. This is a *locally ringed space* if all stalks $\mathcal{O}_{X,p}$ are local rings. A *scheme* is a locally ringed space which is locally isomorphic to $\mathrm{Spec}(R)$. Recall that $\mathrm{Spec}(R)$ is the locally ringed space whose points are prime ideals $\mathfrak{p} \subset R$, with a basis of open sets given by $\mathcal{U}_f = \{\mathfrak{p} : f \notin \mathfrak{p}\}$ for any $f \in R$, and the sheaf is determined by $\mathcal{O}(\mathcal{U}_f) = R[f^{-1}]$. Morphisms of schemes are just morphisms of locally ringed spaces; that is, a pair $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ where $f : X \to Y$ is a continuous function and $f^\# : f^*\mathcal{O}_Y \to \mathcal{O}_X$ is a morphism of sheaves over $X$ which is a local homomorphism on the stalks (i.e. the maximal ideal of the source gets mapped into the maximal ideal of the target).

## 2.2 Elliptic curves

### 2.2.1 Definitions

An *elliptic curve* over a field $K$ is a nonsingular curve $C$ in $\mathbb{P}^2$ defined by a cubic equation such that $C \cap \mathbb{P}^1_\infty = \{[0 : 1 : 0]\}$ (where $\mathbb{P}^1_\infty = \{[* : * : 0]\}$). Any such curve can be expressed in projective coordinates $[X : Y : Z]$ (up to rescaling $X$ and $Y$) as $Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$ for some $a_1, a_2, a_3, a_4, a_6 \in K$. (We will write $a_1, \ldots, a_6$ in the future, omitting any further mention of the omission of $a_5$.) Conversely, any such equation cuts out a (possibly singular) elliptic curve such that $C \cap \mathbb{P}^1_\infty = [0 : 1 : 0]$. This is called *Weierstraß form.* (In affine coordinates (i.e. putting $x = X/Z$ and $y = Y/Z$), this recovers the usual Weierstraß form.

**Example 2.** The $\mathbb{R}$-points of $y^2 = x^3 - x$ give one of the usual pictures, with an ellipse-like component and a 1-sheet-of-a-hyperbola-like component. The $\mathbb{R}$-points of $y^2 = x^3$ give a 1-component curve with a cusp at the origin.

We need to talk about *families* of elliptic curves, which may include singular elliptic curves. For instance, a family of elliptic curves over $\mathrm{Spec}(\mathbb{Z})$ may have singular reduction at certain primes. For instance, $y^2 = x^3 + 2z + 6$ is usually nonsingular, but it degenerates to the singular elliptic curve $y^2 = x^3$ at the prime 2. Thus, we define a *Weierstraß curve* over a scheme $S$ is a scheme $C$ of the form

$$C = C(a_1, \ldots, a_6) = \{([X : Y : Z], s) \in \mathbb{P}^2 \times S : Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_1XZ^2 + a_6Z^3\}$$

for some system of functions $a_i \in \mathcal{O}_S(S)$. (This should actually be interpreted as a vanishing locus.) For such a curve, we have two obvious maps: the *zero section* $0 : S \to C$ given by $s \mapsto ([0:1:0], s)$ and the *projection* $p : C \to S$. Finally, a *Weierstraß elliptic curve* over a scheme $S$ is a scheme $C$ with maps $S \xrightarrow{0} C \xrightarrow{p} S$ such that $S$ can be covered by affine schemes $S_i$ such that $C_i = C \times_S S_i$ is a Weierstraß curve.

**Remark 3.** We can rephrase this in a more invariant way: a Weierstraß elliptic curve is just an $S$-scheme $C \to S$ which is flat, proper, and finitely presented, along with a section $S \to C$ which lands in the smooth locus of $C$, such that the geometric fibers are all either elliptic curves, or nodally or cuspidally singular cubics in $\mathbb{P}^2$. (Fibers with a nodal singularity are called *multiplicative*, and fibers with a cuspidal singularity are called *additive*.)

**Example 3.** Let's return to $C : y^2 = x^3 + 2x + 6$, considered as an elliptic curve over $\operatorname{Spec}(\mathbb{Z})$. This has discriminant $\Delta = 18624 = 2^6 \cdot 3 \cdot 97$. So, these are the primes where $C$ has singularities: a cusp over 2, a node over 3, and something (maybe also a node?) over 97. But other than that, the fibers of $C$ are nonsingular elliptic curves.

Finally, an *elliptic curve* is a Weierstraß elliptic curve which is locally isomorphic to a smooth Weierstraß curve. A *generalized elliptic curve* is a Weierstraß elliptic curve which is only allowed to have nodal singularities. We write $\omega_{C/S} = I_S/I_S^2$ for the cotangent space of $C$ along $S$ (i.e. along the zero section); here, $I_S$ denotes the ideal sheaf. This is a sheaf on $C$, in fact a line bundle, concentrated on the zero section. We'll often consider it as pulled back to $S$.

### 2.2.2 Changes of coordinates

Let's talk about isomorphism-preserving changes of coordinates; we'll restrict to the affine case in which we're over $\operatorname{Spec}(R)$. There are two types of changes of coordinates, translations (with shearing) and dilations. The first take the form

$$\begin{cases} x \mapsto x + r \\ y \mapsto y + sx + t \end{cases}$$

for some $r, s, t \in R$ while the second take the form

$$\begin{cases} x \mapsto \lambda^{-2} x \\ y \mapsto \lambda^{-3} y \end{cases}$$

for some $\lambda \in R^\times$ (and then we rescale the whole equation by multiplication by $\lambda^6$ to remain in Weierstraß form). These form an algebraic group $\operatorname{Spec}(G) = \operatorname{Spec}(\mathbb{Z}[r, s, t, \lambda^{\pm}])$, which acts on $\operatorname{Spec}(A) = \operatorname{Spec}(\mathbb{Z}[a_1, \ldots, a_6])$, the scheme carrying the universal Weierstraß equation. The action $\operatorname{Spec}(G) \times \operatorname{Spec}(A) \to \operatorname{Spec}(A)$ is given by $C(a_1, \ldots, a_6) \mapsto C(a_1', \ldots, a_6')$, where e.g. $a_1' = a_1\lambda - 2s$, $a_2' = a_2\lambda^2 + a_1 s\lambda - 3r - s^2$, etc. The precise equations don't concern us so much; our real goal now is to define modular forms.

## 2.3 Modular forms

A *modular form of weight $k$ over $\mathbb{Z}$* is a rule $g$ that assigns to each generalized elliptic curve $C/S$ a section of $\omega_{C/S}^{\otimes k}$ over $S$, such that for any pullback diagram of elliptic curves

$$\begin{tikzcd} C \arrow[r, "\hat{f}"] \arrow[d, "p"'] & C' \arrow[d, "p'"] \\ S \arrow[r, "f"'] & S' \end{tikzcd}$$

we have $f^*(g(C'/S')) = g(C/S)$. As before, we write $MF_k$ for the group of modular forms of weight $k$. More generally, one defines modular forms over any ring $R$ by restricting to $R$-schemes and morphisms thereof.

**Theorem 5.** *There is an isomorphism of graded rings $MF_* \cong \mathbb{Z}[c_4, c_6, \Delta]/(1728\Delta - c_4^3 - c_6^2)$, where $c_4 \in MF_4$, $c_6 \in MF_6$, and $\Delta \in MF_{12}$.*

# 3 Elliptic curves and formal groups – Benjamin Kuester

In addition to what has been done so far, we emphasize that any elliptic curve over a scheme $S$ has the unique structure of a group scheme, i.e. a group object in the category $\mathtt{Sch}/S$.

But today, we will be talking about formal groups. To introduce these, we must first introduce formal schemes.

## 3.1 Formal schemes

Let $S$ be a scheme. Then any $X \in \mathtt{Sch}/S$ corresponds to a representable functor $(\mathtt{Sch}/S)^{op} \to \mathtt{Set}$ via the Yoneda embedding $Y : \mathtt{Sch}/S \to [(\mathtt{Sch}/S)^{op}, \mathtt{Set}]$, given by $Y(X) = \mathrm{Hom}(-, X)$.

**Definition 1.** A *formal scheme* over $S$ is a small filtered colimit in the functor category $[(\mathtt{Sch}/S)^{op}, \mathtt{Set}]$ of objects in $Y(\mathtt{Sch}/S)$. These define a subcategory $\mathtt{FSch}/S \subset [(\mathtt{Sch}/S)^{op}, \mathtt{Set}]$. (So morphisms are just natural transformations.)

One might think this is quite weird, because $\mathtt{Sch}/S$ already has filtered colimits. But actually it only has finite filtered colimits, and so usually this contruction takes us out of $\mathtt{Sch}/S$ in an essential way.

**Example 4.** Let $X$ be a scheme, $Y \subset X$ a closed subscheme with ideal sheaf $I(Y)$, i.e. $Y = V(I(Y))$. We have canonical maps ("inclusions") $V(I(Y)^n) \hookrightarrow X$ (note that the underlying space maps homeomorphically onto its image $|Y|$, but we get an inclusion in the opposite direction on structure sheaves). These actually form a tower, and we define $\widehat{X} = \mathrm{colim}_n V(I(Y)^n)$ (the colimit in the functor category), the *completion* of $X$ along $Y$. We have a canonical morphism $\widehat{X} \to X$.

This is an interesting technique that only arises in scheme theory. For instance, if $Y$ is an algebraic variety then $I(Y)$ has no nilpotent elements, but $I(Y)^n$ might.

**Example 5.** Let $A$ be a ring, and let $I \subset A$ be an ideal. Then we have maps $\cdots \to A/I^3 \to A/I^2 \to A/I$. These induce maps $\mathrm{Spec}(A/I) \to \mathrm{Spec}(A/I^2) \to \mathrm{Spec}(A/I^3) \to \cdots$, and so we obtain the *formal spectrum* $\mathrm{Spf}(A) = \mathrm{colim}_n \mathrm{Spec}(A/I^n)$. This is also a formal scheme. If we have a map of rings $f : A \to B$ such that $f(I) \subset J \subset B$, then we obtain $f^* : \mathrm{Spf}(B) \to \mathrm{Spf}(A)$.

This is of course a special case of the first example, where $X = \mathrm{Spec}(A)$ and $Y = V(I)$.

**Definition 2.** A scheme $X$ is called *reduced* if for all open $U \subset X$, $\mathcal{O}_X(Y)$ has no nilpotent elements. Given a ring $R$, the *reduction* of $R$ is $R_{red} = R/I_{nil}$; for a sheaf $\mathcal{O}_X$ over $X$, we set $(\mathcal{O}_X)_{red}$ to be the sheafification of $U \mapsto \mathcal{O}_X(U)_{red}$.

So for any scheme $X$, there is a reduced scheme $X_{red} = (|X|, (\mathcal{O}_X)_{red}$, and there is a natural morphism of schemes $\varphi : X_{red} \to X$, which is a homeomorphism on underlying topological spaces.

## 3.2 Formal groups

**Definition 3.** An *abstract formal group* over a scheme $S$ is a commutative group object in $\mathtt{FSch}/S$.

This is rather abstract, but we can also give a way of constructing such an object concretely.

**Definition 4.** A *commutative formal group law* over a ring $R$ is a power series $F \in R[[x, y]]$ such that:

- $F(x, 0) = F(0, x) = x$ [identity];
- $F(x, y) = F(y, x)$ [commutativity];
- $F(F(x, y), z) = F(x, F(y, z))$ [associativity].

In fact, it can be deduced from these properties that there exists an inverse $i \in R[[x]]$, such that $F(i(x), x) = 0$.

Now, here is how we can obtain a formal group from a formal group law. We can define an affine formal scheme by taking $A = R[[x]]$ and $I = (x)$. Let us write $\widetilde{A} = R[[x, y]]$ and $\widetilde{I} = (x, y)$. Then Spf $R[[x]] = \operatorname{colim}_n \operatorname{Spec} R[[x]]/(x)^n = \operatorname{colim}_n \operatorname{Spec} R[x]/(x)^n$, and Spf $R[[x, y]] = \operatorname{colim}_n \operatorname{Spec} R[x, y]/(x, y)^n$. Then, a formal group law $F \in R[[x, y]]$ induces compatible maps $R[x]/(x)^n \to R[x, y]/(x, y)^n$ by $Q(x) \mapsto [Q(F(x, y))]$ for all $n$. Thus $F$ induces a map $\widetilde{F} : \operatorname{Spf} R[[x, y]] \to \operatorname{Spf} R[[x]]$. It's not hard to show that in fact $\operatorname{Spf} R[[x, y]] = \operatorname{Spf} R[[x]] \times_{\operatorname{Spec} R} \operatorname{Spf} R[[y]]$, and so this gives us a commuative group structure on Spf $R[[x]]$.

**Definition 5.** A *formal group* over a scheme $S$ is an abstract formal group $G \in \mathtt{FSch}/S$ such that there is a cover $\{U_i \to S\}$ by affines $U_i = \operatorname{Spec} R_i$ such that $G|_{U_i}$ is isomorphic to an abstract formal group coming from a formal group law as above. (This might also be called a *smooth* formal group.)

## 3.3 The formal group of an elliptic curve

Suppose we have an elliptic curve $E/S$. This comes equipped with two morphisms $S \xrightarrow{0} E \xrightarrow{p} S$. Set $I$ to be the ideal sheaf of im(0). This yields $\widehat{E}$, the completion of $E$ along the zero-section $\operatorname{im}(0) = V(I)$. This is a formal scheme.

There are two important fact about $V(I)$ and of $\widehat{E}$ which we will need.

1. Any morphism of schemes $f : X \to E$ factors through $V(I)$ iff $f^*I = 0$.
2. Any morphism of schemes $f : X \to E$ factors through $\widehat{E} = \operatorname{colim}_n V(I^n)$ iff $f^*I$ is locally nilpotent. Equivalently, $f^*I$ is locally nilpotent and that $X_{red} \to X \to E$ factoring through $V(I)$.

Using this, we can define a group structure on $\widehat{E}$. Let $g : A \to S$ be an object of $\mathtt{Sch}/S$, and suppose $a, b \in \widehat{E}(A)$. We need to define the product of these two elements in a natural way. By Yoneda, $a, b$ correspond to morphisms $a', b' : \operatorname{Hom}(-, A) \to \widehat{E}$; composing these with the natural map $\widehat{E} \to \operatorname{Hom}(-, E)$ gives $a'', b'' : \operatorname{Hom}(-, A) \to \operatorname{Hom}(-, E)$. By Yoneda these correspond to $\widetilde{a}, \widetilde{b} : A \to B$. By definition, these both factor through $\widehat{E}$. So we can apply the second fact above to conclude that we have a commutative diagram

$$
\begin{tikzcd}
A_{red} \arrow[r, "f_{a,b}"] \arrow[d, "\varphi"'] & V(I) \times V(I) \arrow[r, "p|_{V(I)}"] \arrow[d, hook] & S \arrow[r, "0"] & V(I) \arrow[d, hook] \\
A \arrow[r, "\widehat{a} \times \widehat{b}"'] & E \times_S E \arrow[r] & S \arrow[r, "0"'] & E
\end{tikzcd}
$$

The two compositions $A_{red} \to$ factor through $V(I)$, and hence the morphisms $A \to E$ factor through $\widehat{E}$. By Yoneda, this defines $a +_{\widehat{E}} b \in \widehat{E}(A)$.

This might be rather mysterious; one might expect instead the diagram in question to use the multiplication map $E \times_S E \to E$. Actually, it is the proof that this is a formal group that uses the group structure of $E$. Note that over a sufficiently small affine $S_i \subset S$, we have $E_i$ defined by a Weierstraßequation. Now, $\widehat{E}_i$ is the completion of $S_i \xrightarrow{0} E_i$, which is given by $s \mapsto ([0:1:0])$. So, $\mathrm{im}(0) = \{[x:y:z] : x = z = 0\}$. But note that in the standard Weierstraßequation, if $z = 0$ then $x^3 = 0$. Thus one might say that $\{z = 0\}$ is an *infinitesimal thickening* of $\{x = z = 0\}$.

**Example 6.** To get a handle on this, let us reduce our generality a bit: we take $S = \mathrm{Spec}\ \mathbb{C}$, and we look in the affine space $z = 1$. We apply the change of variables $\tau = -x/y$, $\omega = -1/y$ (so $x = \tau/\omega$ and $y = -1/\omega$). Then the Weierstraßequation becomes $f(\tau, \omega) = \omega = \tau^3 = a_1\tau\omega + a_2\tau^2\omega + a_3\omega^2 + a_6\tau\omega^2 + a_6\omega^3$. Now, we recursively define $f_1(\tau, \omega)$ and$f_{n+1}(\tau, \omega) = f_n(\tau, f(\tau, \omega))$. One can show that this converges to some $\omega(\tau) = \lim_{m\ ra\infty} f_m(\tau, 0)$, and that this is the unique power series such that $\omega(\tau) = f(\tau, \omega(\tau))$. This actually holds for all points on the elliptic curve. (Note that the origin sits at $(\tau, \omega) = (0, -\infty)$, and so this should be seen as taking a Taylor series about the origin $(\tau, \omega) = (0, 0)$.)

Now, take two points $(\tau_1, \omega(\tau_1)), (\tau_2, \omega(\tau_2)) \in E$. We can use the usual group law on $E$ to extract $p_3 = (\tau_1, \omega(\tau_1)) +_E (\tau_2, \omega(\tau_2))$. Then, we can write $p_3 = (\tau_3, \omega(\tau_3))$, so we get $\tau_3 = \tau_3(\tau_1, \tau_2)$. This is given by some concrete formula, and this defines a formal group law $f(\tau_1, \tau_2) = \tau_3(\tau_1, \tau_2) \in \mathbb{C}[[\tau_1, \tau_2]]$. Indeed, this corresponds with the abstract definition of the formal group of $E$ that we defined earlier.

# 4 Sheaves and stacks – Markus Hausmann

## 4.1 Sheaves

The first thing we will do is expand our notion of sheaf a bit. First, we enlarge the notion of where our sheaves live.

**Definition 6.** A *Grothendieck topoogy* on a category $\mathcal{C}$ is a class of sets of morphisms $\{U_i \to U\}_{i \in I}$, called *coverings*, such that:

- isomorphisms are coverings, i.e. if $V \xrightarrow{\cong} U$ then $\{V \to U\}$ is a covering;
- coverings are stable under fiber products, i.e. if $\{U_i \to U\}_{i \in I}$ is a covering and $V \to U$ is arbitrary, then $\{V \times_U U_i \to V\}_{i \in I}$ is a covering;
- coverings are stable under composition, i.e. if $\{U_i \to U\}_{i \in I}$ is a covering and $\{V_{ij} \to U_i\}_{j \in J_i}$ is a covering for all $i \in I$, then $\{V_{ij} \to U\}_{(i,j) \in I \times J_i}$ is a covering.

A category $\mathcal{C}$ equipped with a Grothendieck topology is called a *Grothendieck site*. Then, a *sheaf* on a site is a functor $F : \mathcal{C}^{op} \to \mathcal{D}$ (where $\mathcal{D}$ has products) such that for all covers $\{U_i \to U\}_{i \in I}$,

$$F(U) \xrightarrow{\cong} \mathrm{eq}\left( \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \times_U U_j) \right).$$

(This definition of a Grothendieck topology is also sometimes called a *Grothendieck pretopology*, since different choices can yield the same notions of sheaves.)

**Example 7.** If $X$ is a topological space, we can form the associated category $\mathcal{C}_X$, namely the topology of $X$ considered as a poset considered as a category; covers are by definition the usual notion of covers. Then, a sheaf on $\mathcal{C}_X$ is just a sheaf on $X$.

**Example 8.** We could take the entire category `Top` with the usual notion of open coverings (i.e. maps that are homeomorphic onto their open image). This gives the notion of a sheaf on `Top`.

**Example 9.** We could take the category `Top` where $\{U_i \to U\}_{i\in I}$ is a covering if $\mathrm{im}(U_i) \subset U$ is open and these jointly cover $U$, and each map $U_i \to U$ is a finite covering space of its image. This is the *étale site* of `Top`, denoted $\mathtt{Top}_{\text{ét}}$. (One must restrict to finite coverings to ensure that compositions of covers are covers.)

**Example 10.** We can take `Sch` with the Zariski topology, $\mathtt{Sch}_{Zar}$.

**Example 11.** We can take `Sch` with the faithfully flat topology, $\mathtt{Sch}_{flat}$. This is characterized by being the coarsest topology containing the Zariski topology and maps of the form Spec $B \to$ Spec $A$ where $A \to B$ is faithfully flat. Recall that this means that $B \otimes_A - : \mathtt{Mod}_A \to \mathtt{Mod}_B$ is an "injective" exact functor. Thus, flat covers are those collections of flat maps that detect all quasicoherent sheaves.

**Example 12.** We can take `Sch` with the étale topology, $\mathtt{Sch}_{\text{ét}}$. A collection of morphisms $\{U_i \to U\}$ is a covering if each $U_i \to U$ is étale, and the images cover $U$. Recall that an *étale* map is a map which is flat, unramified, and of finite type. One should think of these as "local-on-the-source homeomorphisms". In particular, every étale covering is an fpqc covering. Thus, this is the strongest topology of the ones we'll consider.

Now, observe that if $E \to X$ is a covering space map, then $\mathrm{coeq}(E \times_X E \rightrightarrows E) \xrightarrow{\cong} X$; the property that makes this so is that the map $E \to X$ is a quotient map. In fact, this a coequalizer for all coverings $E \to X$ precisely if the topology is *subcanonical*, i.e. all representable functors are sheaves.

**Theorem 6** (Grothendieck)**.** *All the Grothendieck topologies we have discussed are subcanonical.*

Thus, we have a way of strengthening the notion of being a Zariski sheaf; it is of course harder to satisfy the sheaf condition for a finer topology.

## 4.2 Stacks

The reason we will care about stacks in this seminar is that the functor taking a scheme $S$ to the set of elliptic curves over $S$ up to isomorphism is *not* representable. One reason this is so is the fact that this doesn't even form a Zariski sheaf. You can't glue isomorphism classes of elliptic curves; if you only know that two locally-defined elliptic curves are isomorphic on some intersection, you need to choose an isomorphism to glue them. But then you run into issues on triple intersections. Thus, one way to get around this is to remember those isomorphisms: the functor which takes $S$ to the *groupoid* of elliptic curves over $S$ will be representable by a stack.

Of course, now we've introduced a new issue. The elliptic curve will be obtained by pullback from this stack, but pullback is only canonical up to canonical isomorphism. There are two ways to get around this. One way is to consider *pseudofunctors*, i.e. functors $F : \mathcal{C} \to \mathtt{Cat}$ (or to any 2-category) is a function on objects and morphisms, together with for every pair of composable arrows $gf$ in $\mathcal{C}$ an isomorphism $F(gf) \xrightarrow{\cong} F(g) \circ F(f)$ (in a way compatible with triple compositions, etc.). The other way would be to strictify our source category, and including the choice of pullbacks there already. We won't pursue that route. Henceforth, whenever we say functor we probably mean to say pseudofunctor.

Now, the category of groupoids has limits (and in particular, products), and so we can again talk about sheaves valued in groupoids. More precisely, we make the following definition, which should remind us of the definition of the sheafification of an ordinary sheaf (e.g. of sets).

**Definition 7.** Let $\mathcal{C}$ be a site, and let $F : \mathcal{C}^{op} \to \mathtt{Grpds}$ be a functor. For any covering $\{U_i \to U\}$ we define the *descent groupoid* $Desc(F, \{U_i \to U\}_{i\in I})$. An object is a choice of object $a_i \in F(U_i)$ for every $i \in I$ along with isomorphisms $\varphi_{ij} : a_i|_{U_{ij}} \to a_j|_{U_{ij}}$ which satisfy the *cocycle condition*, i.e. $\varphi_{ik}|_{U_{ijk}} = \varphi_{jk}|_{U_{ijk}} \circ \varphi_{ij}|_{U_{ijk}}$. Then, a morphism $f : (a_i, \varphi_{ij}) \to (b_i, \psi_{ij})$ is just a collection of maps $f_i : a_i \xrightarrow{\cong} b_i$ which commute with the coherence maps, i.e. $\psi_{ij} \circ f_i = f_j \circ \varphi_{ij}$.

Then, we can make the following definition.

**Definition 8.** A functor $F : \mathcal{C}^{op} \to \texttt{Grpds}$ is called a *stack* if for all coverings $\{U_i \to U\}_{i\in I}$, the map $F(U) \to Desc(F, \{U_i \to U\}_{i\in I})$ is an equivalence of groupoids.

(This canonical map is just given by restriction.) This should look exactly like the ordinary sheaf condition.

The inclusion functor $\texttt{Sets} = 0\text{-}\texttt{Cat} \to \texttt{Grpds} \subset 1\text{-}\texttt{Cat}$ induces a functor from set-valued sheaves on $\mathcal{C}$ to stacks on $\mathcal{C}$. In particular, if the topology on $\mathcal{C}$ is subcanonical, then every object represents a stack.

**Example 13.** Let $X$ be a topological space, and for any open subset $U \subset X$ we can take $U$ to $\{U \times G\}$, the singleton set of trivial (i.e. trivialized) $G$-bundles over $U$, along with automorphisms of $G$-bundles. This gives a functor $\mathcal{C}_X \to \texttt{Grpds}$ which is not a stack, despite satisfying the set-level sheaf condition. However, we can stackify this to $\mathscr{B}G$, the classifying stack of principle $G$-bundles on $\texttt{Top}$ (with the usual topology); this takes any space (in particular an open subset) to the groupoid of all principal $G$-bundles over it.

**Example 14.** On $\texttt{Sch}_{fpqc}$ (and hence on $\texttt{Sch}_{\text{ét}}$ and $\texttt{Sch}_{Zar}$) we have:

- $\mathcal{M}_{ell}$, the stack of elliptic curves over $\texttt{Sch}_{\text{ét}}$;
- $\mathcal{M}_{Weier}$, the stack of Weierstraß elliptic curves;
- $\overline{\mathcal{M}}_{ell}$, the stack of generalized elliptic curves (i.e. we allow nodal but not cuspidal degenerations).

**Theorem 7.** *The forgetful functor* $\texttt{Stacks}/\mathcal{C} \to \texttt{Grpds}^{\mathcal{C}^{op}}$ *has a left adjoint, called stackification.*

Of course, we simply define $(\mathcal{M}F)(U) = \operatorname{colim}_{\{U_i \to U\}_{i\in I}} Desc(F, \{U_i \to U\}_{i\in I})$.

**Example 15.** The stackification of the trivial $G$-bundle functor is $\mathscr{B}G$.

## 4.3 Hopf algebroids

Now that we have the associated stack functor in hand, we can simply reduce to looking at groupoid-valued functors.

**Definition 9.** A *Hopf algebroid* is a pair of commutative rings $(R, S)$ which corepresent a groupoid-valued functor. (Here, $R$ corepresents the objects and $S$ corepresents the morphisms. Of course, we are sweeping certain structure morphisms under the rug. We may simply say that we have a chosen lifting of $\text{Spec}(R) \times \text{Spec}(S) : \texttt{Rings}^{op} \to \texttt{Sets} \times \texttt{Sets}$ to $\texttt{Grpds}$.)

It follows that $(\text{Spec } R, \text{Spec } S)$ represent a groupoid valued functor on affine schemes, and hence on all schemes (since maps off schemes are given as colimits).

This is usually not a stack. However, as a groupoid-valued functor it admits a stackification $\mathcal{M}_{(R,S)}$. Thus we can construct the stacks related to elliptic curves above.

**Example 16.** Let $A = \mathbb{Z}[a_1, \ldots, a_6]$ classify Weierstraß equations and $G = A[r, s, t, \lambda^{\pm}]$ classify reparametrizations thereof. We described the action $\text{Spec } G \times \text{Spec } A \to \text{Spec } A$ previously. Then $\mathcal{M}_{(\text{Spec } A, \text{Spec } G)} = \mathcal{M}_{Weier}$. (The extra input here is that any isomorphism of Weierstraß elliptic curves over a given base scheme is locally given by such a transformation.)

**Example 17.** If we localize this at $\Delta$ (i.e. take $A' = \mathbb{Z}[a_1, \ldots, a_6][\Delta^{-1}]$) and take the same $G$, then we obtain $\mathcal{M}_{(\text{Spec } A', \text{Spec } G)} = \mathcal{M}_{ell}$.

**Example 18.** $\pi_* MU = L$ (Lazard's ring classifying formal group laws) and $\pi_*(MU \wedge MU)$ classifies isomorphisms of formal group laws. Thus $\mathcal{M}_{(\text{Spec}\pi_* MU, \text{Spec } \pi_*(MU\wedge MU))} = \mathcal{M}_{FG}$, the moduli stack of formal groups.

**Definition 10.** Let $M \xrightarrow{g} N \xleftarrow{f} P$ be a diagram in stacks. Then the *2-category fiber product*, denoted $M \times_N P$, is the stack which evaluates on $U$ to give objects the triples $\{(m, p, \varphi) : m \in M(U), p \in P(U), \varphi : g(m) \xrightarrow{\cong} f(p)\}$, and to give morphisms the pointwise morphisms that commute with the structure. Note that the resulting square only commutes up to a natural isomorphism.

This allows us to give the following general definition.

**Definition 11.** A morphism $M \to N$ of stacks is called *representable* if for every representable stack $P$ (assuming subcanonicality) and every map $N \leftarrow P$, the 2-category pullback $M \times_N P$ is equivalent to a representable stack. Then, a representable map $M \to N$ is said to have a property $\mathcal{P}$ (for $\mathcal{P}$ a property of morphisms of schemes) if for every map $N \leftarrow P$ from a representable stack, the map $M \times_N P \to P$ has property $\mathcal{P}$.

For instance we can talk about étale maps of stacks, flat maps of stacks, and coverings of stacks. In particular, this defines a topology on $\texttt{Stacks}_{\mathcal{C}}$. We then again have a notion of sheaves on this site, and given a particular $M \in \texttt{Stacks}_{\mathcal{C}}$ we can define a sheaf on $M$ to be a sheaf on the site $\texttt{Stacks}_{\mathcal{C}}/M$.

# 5 Presheaf of cohomology theories on $\mathcal{M}_{ell}$ – Markus Land

In this talk, our stacks will always be defined on the site of affine schemes with the *flat* topology, in which a family of maps $\{f_i : U_i \to U\}$ is a covering if all maps are flat and they jointly detect all quasicoherent sheaves (i.e modules, since we're in the affine case).

## 5.1 Even periodic cohomology theories and formal group laws

All our cohomology theories will be multiplicative, but not necessarily structured (i.e. nothing more than associative monoid objects in the stable homotopy category).

**Definition 12.** A multiplicative cohomology theory $E$ is called *even periodic* if $\pi_{2k+1}E = 0$ for all $k \in \mathbb{Z}$ and there is a unit $u \in E^2(\mathrm{pt})^\times$.

**Example 19.** This definition is of course based on $KU$, which has the Bott class $\beta \in KU^2(\mathrm{pt})$. But we can also periodify other theories, e.g. $HP\mathbb{Z} = \bigvee_{k\in\mathbb{Z}} \mathbb{S}^{2k} \wedge H\mathbb{Z}$ and $MP = \bigvee_{k\in\mathbb{Z}} \mathbb{S}^{2k} \wedge MU$. This works because the latter two are even-concentrated. So note that e.g. we can consider $MP_0 = \pi_0 MP \cong \bigoplus_{k\in\mathbb{Z}} \pi_k MU$, i.e. this construction takes a graded ring and shifts it all to degree 0.

**Theorem 8.** *Let $E$ be even periodic. Then for all $n \geq 1$, there exist isomorphisms $\alpha_n : E^0((\mathbb{C}P^\infty)^{\times n}) \xrightarrow{\cong} E^0[[x_1, \dots, x_n]]$.*

Thus, the difference between complex-orientable theories and these even periodic theories is really just a choice of $\alpha_1$, which determines the $\alpha_n$ for all $n \geq 1$.

**Definition 13.** Let $m : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ be the map classifying $\gamma \boxtimes \gamma$ (for $\gamma \downarrow \mathbb{C}P^\infty$ the tautological bundle). This induces a map $m^* : E^0(\mathbb{C}P^\infty) \to E^0(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$, which in the even periodic case is $m^* : E^0[[x]] \to E^0[[y, z]]$. We write $F_E = m^*(x) \in E^0[[y, z]]$ for the *formal group law* of $E$ over $E^0$. This satisfies the axioms of a formal group law (commutative, associative, and unital) because $m$ satisfies these up to homotopy, simply because the tensor product of line bundles does.

Recall that the *Lazard ring* $L$ is built to classify (i.e. corepresent the functor of) formal group laws.

We have the following important theorem.

**Theorem 9** (Quillen)**.** *$MU$ has a canonical complex orientation which induces $F_{MU}$ over $MP^0$, and hence there exists a canonical map $L \to MP_0$. This map is an isomorphism.*

Recall that $(MP_0, MP_0MP)$ is a Hopf algebroid, over which any spectrum $X$ yields a comodule $MP_*(X)$. We have the following continuation. (Recall that if $(A, \Gamma)$ is a Hopf algebroid, then a *comodule* is a left $A$-module $M$ equipped with a left $\Gamma$-coaction $M \to \Gamma \otimes_A M$, which is required to be a map of $A$-modules, and such that the composition with the counit induces the identity on $M$.)

**Theorem 10** (Quillen)**.** $(MP_0, MP_0MP)$ *corepresents the groupoid-valued functor of formal group laws and their isomorphisms.*

We will write $(L, W) = (MP_0, MP_0MP)$ for short.

**Corollary 1.** $\mathcal{M}_{(L,W)} \simeq \mathcal{M}_{FG}$.

*Proof sketch.* Quillen's theorem tells us that these induce the same prestacks. All we do to stackify is forget the coordinate and force all descent data to be effective. □

## 5.2 Landweber exact functor theorem and elliptic homology

We have seen that give an even periodic cohomology theory, we obtain a formal group law. We would like to go in the other direction.

The idea is the following. Suppose $F$ is a formal group law over a ring $R$. This is the same as a map $MP_0 \to R$. Then, we can use this to define a functor $X \mapsto MP_*X \otimes_{MP_0} R$. But we need some exactness conditions to ensure that this is a homology theory. (Using homology *introduces* finiteness conditions; there's also a version of this setup that uses cohomology but has finiteness conditions. Given a cohomology theory on finite complexes, we can always extend to a cohomology theory on all spaces, but not canonically.)

So, the first idea might be to assume that the map $MP_0 \to R$ is flat. This is actually too restrictive; we don't need $- \otimes_{MP_0} R$ to be exact on all modules, but rather only on $(L, W)$-comodules.

**Definition 14.** We call $F$ *Landweber exact* if $- \otimes_{MP_0} R$ is exact on $(L, W)$-comodules.

**Remark 4.** There is an intrinsic criterion to check whether $F$ is Landweber exact.

**Remark 5.** A comment of Miller suggests that we might be able to restrict to finitely presented $(L, W)$-comodules, since all are colimits thereof.

**Example 20.** The additive formal group law over $\mathbb{Z}$ is given by $x + y$; this corresponds to $H\mathbb{Z}$ (or $HP\mathbb{Z}$). However, this is Landweber exact over $\mathbb{Q}$ (and in fact it's Landweber exact iff we're over a $\mathbb{Q}$-algebra); this corresponds to $H\mathbb{Q}$.

**Example 21.** The multiplicative formal group law over $\mathbb{Z}$ is Landweber exact, and is given by $x + y + xy$; this corresponds to $KU$.

**Remark 6.** Even if we restrict to even periodic spectra, it's actually quite nontrivial to prove that going back and forth is an equivalence. There's a serious problem of phantom maps, as follows. The Landweber exact functor theorem produces homology theories, and associated to any homology theory there is a class of associated spectra (or equivalently, cohomology theories); they are all equivalent, but the choices of equivalence aren't unique (even up to homotopy). In other words, the moduli space of spectra that represent a given homology theory is connected but may have higher homotopy groups. This obstructs functoriality even in the homotopy category, since now a priori we need to explicitly determine functorial maps between our spectra which represent our Landweber exact homology theories. On the other hand, any two such maps induce the same map in homology (and hence the same map in cohomology for finite spaces), and so their induced difference in these settings must be zero. Such a map is called a *phantom map*. This is the downside of homology: it isn't *literally* represented in the categorical sense by a spectrum (even though we say it is all the time), and so we can't use Yoneda to conclude that homology operations are equivalent to spectrum maps. However, Hovey-Strickland prove in "Morava $K$-theories and localization" that this cannot happen for Landweber exact homology theories and fgl-preserving maps: given two Landweber exact spectra $E$ and $F$ and a natural transformation $f : E_*(-) \Rightarrow F_*(-)$ which when applied to $S^0$ induces a map of $MU_*$-algebras, the moduli space of realizations $\widetilde{f} : E \to F$ of $f$ is contractible. (This is also laid out in Lecture 17 of Lurie's chromatic course notes, 252x.)

Now, we'd like to understand what we can do with stackifications of Hopf algebroids. We will need to following facts.

1. The 2-category of groupoids has 2-pullbacks. Namely, the functor that $\mathrm{holim}(G_1 \xrightarrow{f} G \xleftarrow{g} G_2)$ represents is: an object $x_1 \in G_1(R)$, an object of $x_2 \in G_2(R)$, and an isomorphism $\varphi : f(x_1) \xrightarrow{\cong} g(x_2)$ in $G(R)$.
2. 2-pullbacks commute with stackification.
3. There is a canonical morphism $\mathrm{Spec}(A) \to \mathcal{M}_{(A,\Gamma)}$, given by the following observation. Write $\mathcal{P}$ for the functor corepresented by a Hopf algebroid. Then we have
$$\mathrm{Spec}(A) = \mathcal{P}_{(A,A)} \xrightarrow{(\mathrm{id},\varepsilon)} \mathcal{P}_{(A,\Gamma)} \to \mathcal{M}_{(A,\Gamma)},$$
where the last map is the unit of the adjunction between prestacks and stacks.
4. In general, one cannot expect a map $\mathrm{Spec}(R) \to \mathcal{M}_{(A,\Gamma)}$ to factor through $\mathrm{Spec}(A) \to \mathcal{M}_{(A,\Gamma)}$. However, this is true up to a faithfully flat extension of $R$. That is, there exists a faithfully flat extension $\varphi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ such that we have a 2-commutative diagram
$$\begin{tikzcd} \mathrm{Spec}(R) \arrow[r] & \mathcal{M}_{(A,\Gamma)} \\ \mathrm{Spec}(S) \arrow[u, "\varphi"] \arrow[r, dashed] & \mathrm{Spec}(A). \arrow[u] \end{tikzcd}$$
One may call this a *local* factorization of $\mathrm{Spec}(R) \to \mathcal{M}_{(A,\Gamma)}$ through $\mathrm{Spec}(A)$.
5. Recall that a map of stacks $f : \mathcal{M} \to \mathcal{N}$ is *representable* if any map from a scheme into $\mathcal{N}$ gives a pullback which is a scheme; one can then apply the usual adjectives for morphisms of schemes to morphisms of stacks; in particular, a representable map can be *flat*, or more particularly it can be a *cover*. Moreover, if $\mathcal{N} = \mathcal{M}_{(A,\Gamma)}$, then we can check representability (and flatness) simply by checking on the cover $\mathrm{Spec}(A) \to \mathcal{M}_{(A,\Gamma)}$.
6. Any map $\mathrm{Spec}(R) \to \mathcal{M}_{(A,\Gamma)}$ is representable. In particular, the cover $\mathrm{Spec}(A) \to \mathcal{M}_{(A,\Gamma)}$ is representable. (This follows from an explicit computation about the surrounding discrete Hopf algebroids.)

**Example 22.** The following is a 2-pullback in stacks:
$$\begin{tikzcd} \mathrm{Spec}(\Gamma) \arrow[r, "\eta_R"] \arrow[d, "\eta_L"'] & \mathrm{Spec}(A) \arrow[d, "c"] \\ \mathrm{Spec}(A) \arrow[r, "c"'] & \mathcal{M}_{(A,\Gamma)}. \end{tikzcd}$$
Thus, $\eta_R$ is flat iff $\eta_L$ is flat. In this case, the above observations imply that $c$ is a cover (in the flat topology). (In particular, $\mathrm{Spec}(MP_0) \to \mathcal{M}_{FG}$ is a cover.)

Let $\mathcal{M}$ be a stack on affine schemes in the flat topology. We define its *structure sheaf* $\mathcal{O}_M : (\mathtt{Aff}/\mathcal{M})^{op} \to \mathtt{Rings}$ by $\mathcal{O}_M(x : \mathrm{Spec}(R) \to \mathcal{M}) = R$. If we have a faithfully flat covering $V \to U$, then we need $\mathcal{O}(U) \to \mathcal{O}(V) \rightrightarrows \mathcal{O}(V \times_U V)$ to be an equalizer, i.e. we want $S \to R \rightrightarrows R \otimes_S R$ to be an equalizer whenever $S \to R$ is faithfully flat. But this is always true for faithfully flat morphisms of rings, which implies that our definition really does give a sheaf.

**Definition 15.** A *quasicoherent module sheaf* over $\mathcal{M}$ is a sheaf $M$ of abelian groups such that $M(x : \mathrm{Spec}(R) \to \mathcal{M})$ is an $R$ module for all $x$, which satisfies the compatibility condition that if the diagram

$$\begin{tikzcd} & \mathrm{Spec}(R) \arrow[d, "x"] \\ \mathrm{Spec}(S) \arrow[ur, "\varphi"] \arrow[r, "y"'] & \mathcal{M} \end{tikzcd}$$

2-commutes, then $M(x, R) \otimes_\varphi S \cong M(y, S)$.

**Remark 7.** If we view $\mathrm{Spec}(R)$ as a stack on $\mathtt{Aff}$, then $\mathtt{Qcoh}(\mathrm{Spec}(R)) \simeq \mathtt{Mod}_R$. This also respects localization, just like ordinary quasicoherent sheaves.

**Proposition 1.** *A map $F : \mathrm{Spec}(R) \to \mathcal{M}_{(A,\Gamma)}$ is flat iff $F^* : \mathtt{Qcoh}(\mathcal{M}_{(A,\Gamma)}) \to \mathtt{Qcoh}(\mathrm{Spec}(R))$ is exact.*

**Remark 8.** It is a theorem that $\mathtt{Qcoh}(\mathcal{M}_{(A,\Gamma)}) \simeq \mathtt{Comod}_{(A,\Gamma)}$; this implies that $\mathtt{Qcoh}(\mathcal{M}_{(A,\Gamma)})$ is an abelian category, which is of course necessary for the above proposition to make any sense in the first place.

**Remark 9.** This pullback is evaluated as

$$F^*(M)(x : \mathrm{Spec}(S) \to \mathrm{Spec}(R)) = M(\mathrm{Spec}(S) \xrightarrow{x} \mathrm{Spec}(R) \xrightarrow{F} \mathcal{M}_{(A,\Gamma)}).$$

However, we can only directly see this as extension of scalars if our map actually factors through $\mathrm{Spec}(A)$; otherwise, it's more subtle.

**Corollary 2.** *A formal group law $F : L \to R$ is Landweber exact iff the composite*

$$f : \mathrm{Spec}(R) \to \mathrm{Spec}(L) \to \mathcal{M}_{(L,W)} \simeq \mathcal{M}_{FG}$$

*is flat.*

*Proof sketch.* On the one hand, if $f$ is flat, then $F^* : \mathtt{Qcoh}(\mathcal{M}_{FG}) \to \mathtt{Qcoh}(\mathrm{Spec}(R))$ is exact. But we identified $\mathtt{Qcoh}(\mathcal{M}_{FG}) \simeq \mathtt{Comod}_{(L,W)}$. Since our map factors through $\mathrm{Spec}(L)$, this means that the fuctor really is given by extension of scalars. ☐

We can now formulate the Landweber exactness condition in terms of maps of stacks. Suppose $F : \mathrm{Spec}(R) \to \mathcal{M}_{FG}$ is flat. If $X \in \mathtt{Top}$, then $MP_*X \in \mathtt{Comod}_{(L,W)} \simeq \mathtt{Qcoh}(\mathcal{M}_{FG})$. Hence, we can take $F^*MP_*X \in \mathtt{Qcoh}(\mathrm{Spec}(R)) \simeq \mathtt{Mod}_R$, where the last equivalence is given by evaluation on $\mathrm{id}_{\mathrm{Spec}(R)}$. So, since $F$ is flat, then this is a homology theory.

## 5.3 $\mathcal{M}_{ell} \to \mathcal{M}_{FG}$ is flat

We would like to use this to the formal groups associated to elliptic curves to obtain homology theories.

**Theorem 11.** *$\mathscr{F} : \mathcal{M}_{ell} \to \mathcal{M}_{FG}$ is flat.*

Let us examine the consequences of this. If $C : \mathrm{Spec}(R) \to \mathcal{M}_{ell}$ is a flat map, then it corresponds to a Weierstraß curve over some faithfully flat extension. Then the composition $\mathcal{F}_C : \mathrm{Spec}(R) \xrightarrow{C} \mathcal{M}_{ell} \xrightarrow{\mathscr{F}} \mathcal{M}_{FG}$ is also flat, and hence corresponds to a homology theory $X \mapsto \mathcal{F}_C^* MP_*(X) = \mathrm{Ell}_*^C(X)$; this is the *elliptic homology theory* associated to $C$. This can be rewritten as $\mathscr{F}^* MP_*(X)(\mathrm{Spec}(R) \xrightarrow{C} \mathcal{M}_{ell})$, where we consider $\mathscr{F}^* MP_*(X) \in \mathtt{Qcoh}(\mathcal{M}_{ell})$. We really would like to think of this as the natural object; obtaining a particular $\mathrm{Ell}_*^C(C)$ just corresponds to its evaluation.

**Remark 10.** If $\mathcal{F}_C$ factors through $\mathrm{Spec}(L) \to \mathcal{M}_{(L,W)} \simeq \mathcal{M}_{(F,G)}$, then the coefficients are given by $\mathrm{Ell}_*^C(\mathrm{pt}) \cong R[u^{\pm}]$. But in general, we don't have even periodicity. (However, we will always have *weak* even periodicity.)

Everything we've done so far tells us that this construction assembles into a presheaf on the site of affine schemes which are flat over $\mathcal{M}_{ell}$, in the flat topology. We collect this into the following definition.

**Definition 16.** We define a presheaf $\mathcal{O}^{hom}$ on this site by restricting to those objects $C : \mathrm{Spec}(R) \to \mathcal{M}_{ell}$ such that the map $C$ is flat; then we set $\mathcal{O}^{hom}(C : \mathrm{Spec}(R) \to \mathcal{M}_{ell}) = \mathrm{Ell}_*^C(-)$. This is functorial as follows: given a map $f : C' \to C$, we naturally get $\mathrm{Ell}_*^C(X) \to \mathrm{Ell}_*^{C'}(X)$ as

$$(\mathscr{F}^* MP_*(X))\,(C) \to (\mathscr{F}^* MP_*(X))\,(C'),$$

which comes from the fact that $\mathscr{F}^* MP_*(X) \in \mathtt{Qcoh}(\mathcal{M}_{ell})$.

We note that everything we've done here is presented in the notes of Henning Hohnhold ("The Landweber exact functor theorem, stacks, and the presheaf of elliptic homology theories") given in the references for the seminar.

# 6 Construction of $TMF$ – Aaron Mazel-Gee

## 6.0 You could've invented $tmf$.

We begin by providing some motivation for this entire seminar, in the process of which we'll meet some of the objects that we'll be looking at more closely in this talk – informally at first, but then we'll go back through the relevant parts more carefully. We admit right up front that we'll be ignoring the minor issue of periodification throughout this introduction.

1. $\mathtt{SHC}^{fin}$ is a tensored triangulated category, so we can talk about ideals and thick subcategories. We define $\mathrm{Spec}(\mathtt{SHC}^{fin})$ to be the space of thick triangulated prime ideals.[1] Note that the kernel of any homology theory is a thick ideal: it is thick by the long exact sequence, and it is an ideal because if $X \in \ker(E_*)$ then $E \wedge X \simeq *$ so $E \wedge (X \wedge Y) \simeq * \wedge Y \simeq *$ for any $Y$. This needn't be prime, however, as e.g. stable homotopy illustrates.

2. By the *nilpotence theorem* of Devinatz-Hopkins-Smith, $\mathrm{Spec}(\mathtt{SHC}^{fin}) \cong \mathrm{Spec}(\mathbb{Z}) \wedge (\mathbb{N}_0 \cup \{\infty\})$.[2]

   - For every $[(p)] \in \mathrm{Spec}(\mathbb{Z})$ we have a tower of points $\mathscr{P}_{n,(p)}$ for $1 \leq n \leq \infty$, and there is a generic point $\mathscr{P}_0$ corresponding to $[(0)] \in \mathrm{Spec}(\mathbb{Z})$. These are defined by $\mathscr{P}_{n,(p)} = \ker(K(n,p)_*)$, where $K(n,p)$ is the $n^{\mathrm{th}}$ *Morava $K$-theory* at the prime $p$. (We could also say that $\mathscr{P}_0 = \mathscr{P}_{0,(p)}$ for any prime $p$, since we always have $K(0,p) = H\mathbb{Q}$.)
   - $\overline{\mathscr{P}_{n,(p)}} = \{\mathscr{P}_{m,(p)} : n \leq m \leq \infty\}$ (and $\overline{\mathscr{P}_0} = \mathrm{Spec}(\mathtt{SHC}^{fin})$), and these form a basis of closed sets.

[1] This actually comes with the "opposite Zariski topology": a basis of open sets is $V(X) = \{\mathscr{P} : X \in \mathscr{P}\}$, and the sets $D(X) = \{\mathscr{P} : X \notin \mathscr{P}\}$ are closed. However, when we take our tensored triangulated category to be $K^b(\mathtt{Proj}_R)$, the perfect complexes of $R$-modules, we recover the space $\mathrm{Spec}(R)$; thus, this is a reasonable definition. And after all, $\mathtt{SHC}^{fin} = K^b(\mathtt{Proj}_{\mathbb{F}_1})$!

[2] Cf. Balmer's *Spectra, spectra, spectra – tensor triangular spectra versus Zariski spectra of endomorphism rings*.

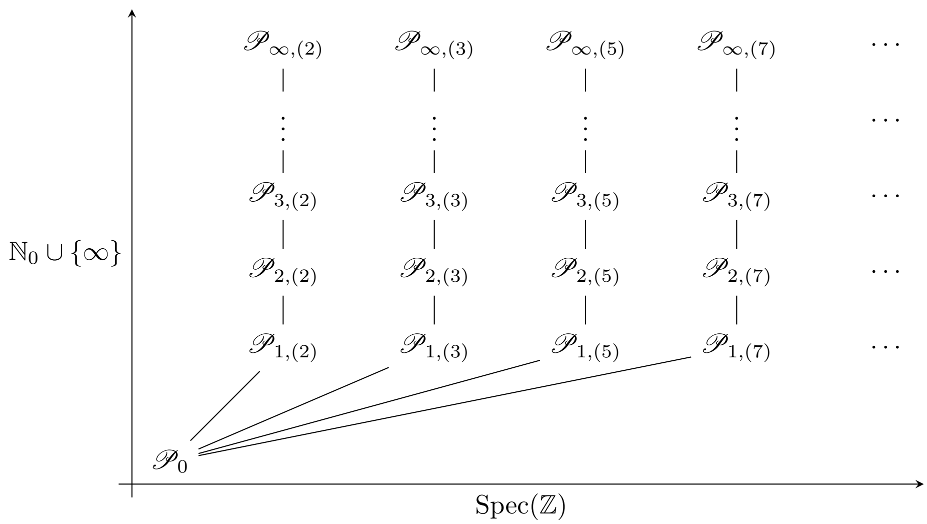


Figure 1: Spec($\texttt{SHC}^{fin}$).

- All closed sets are obtainable as the support of some function $X \in \texttt{SHC}^{fin}$. In particular, $\text{supp}(X) = \text{Spec}(\texttt{SHC}^{fin})$ iff $H\mathbb{Q}_*X \neq 0$. Otherwise, $\text{supp}(X) = \bigcup_{p \text{ prime}} \overline{\mathscr{P}_{\text{type}(X,p),(p)}}$, where the union is over finitely many primes.

This is called the *chromatic filtration* of the (finite) stable homotopy category.

3. The *Morava $K$-theories* are complex-oriented spectra $K(n) = K(n,p)$ for $0 \leq n \leq \infty$. The edge cases are $K(0) = H\mathbb{Q}$, $K(1) = KU/p$, and $K(\infty) = H\mathbb{F}_p$. For $n \geq 1$, $\mathbb{G}_{K(n)} = H_n = H_{n,p}$, the height-$n$ *Honda formal group* over $\mathbb{F}_p$. (If $F$ is a formal group law over a field $k$ of characteristic $p$, then $[p]_F(x) = ux^{p^n} + \text{h.o.t.}$ (with $u \neq 0$) for some $n \geq 1$, called the *height* of $F$. This is an isomorphism invariant.[3] We define $H_n$ by saying that $[p]_{H_n}(x) = x^{p^n}$ with $x$ a $p$-typical coordinate. $H_n$ is defined over $\mathbb{F}_p$, but up to algebraic closure height is a complete isomorphism invariant; thus, up to base change the $H_n$ give all formal groups over $\mathbb{F}_p$.

   The Morava $K$-theories are of central importance in chromatic homotopy theory. In the sense given above, they are the residue fields of Spec($\texttt{SHC}^{fin}$). Further, they are essentially all of the homology theories admitting Künneth isomorphisms. In fact, for any $X \in \texttt{SHC}$, $K(n) \wedge X \simeq \bigvee_j \Sigma^{i_j} K(n)$, and this admits no nontrivial retracts – that is, $K(n)$ is a *field* (i.e. all its modules are free) – and moreover, any field takes the form $\bigvee_j \Sigma^{i_j} K(n)$ (as a spectrum). So, these are also the prime fields of $\texttt{SHC}$ in the same sense that $\mathbb{Q}$ and the $\mathbb{F}_p$ are the prime fields of $\texttt{Mod}_{\mathbb{Z}}$.

   However, as useful as the Morava $K$-theories are for detecting information at the various points of Spec($\texttt{SHC}^{fin}$), they do not tell us how to stitch that information back together.

4. In order to globalize in the *chromatic* direction, there are the complex-oriented *Morava $E$-theories* $E_n = E_{n,p}$ for $0 \leq n < \infty$.[4] The edge cases are $E_0 = H\mathbb{Q}$ and $E_1 = KU_p^\wedge$. For $n \geq 1$, $\mathbb{G}_{E_n} = \widetilde{H}_n = \widetilde{H}_{n,p}$ is a *universal deformation* of $H_n \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n}$ into complete local rings over $\mathbb{F}_{p^n}$, which lives over the

[3] If $G$ is a formal group over an $\mathbb{F}_p$-scheme, then the height of $G$ can also be defined as the number of iterates of the relative Frobenius through which we can factor $[p] : G \to G$.

[4] There isn't an obvious notion of a Morava $E_n$ for $n = \infty$; the universal deformation space of $(\widehat{\mathbb{G}}_a)_{\mathbb{F}_p}$ is a stack, and our usual construction only associates spectra to certain (formal) schemes equipped with formal groups.

*Lubin-Tate deformation space* $LT_n = LT_{n,p} = \mathrm{Spf}((E_n)_0) \cong \mathrm{Spf}(\mathbb{Z}_p[\zeta_{p^n-1}][\![u_1, \ldots, u_{n-1}]\!])$.[5] That is, if $A$ is any complete local ring with residue field $A/\mathfrak{m}_A$ admitting an inclusion $i : \mathbb{F}_{p^n} \to A/\mathfrak{m}_A$ and $\mathbb{G}/A$ is a formal group which reduces to $i^* H_n/(A/\mathfrak{m}_A)$, then there exists a unique map $f : \mathrm{Spf}(A) \to LT_n$ such that $f^* \widetilde{H}_n \cong \mathbb{G}$ and such that this isomorphism reduces to the identity morphism on special fibers.[6] The formal group $\widetilde{H}_n$ is rather complicated, but it ends up that in its $p$-series (in a suitable coordinate), the first nonzero term mod $(p, u_1, \ldots, u_{i-1}, u_i^2, \ldots, u_{n-1}^2)$ is $u_i x^{p^i}$. Thus, over the special fiber a deformation of $H_n$ can have any height up to $n$, and that height is at least $i$ iff the classifying map kills $(p, u_1, \ldots, u_{i-1})$.

Now, $E_n$ has the same *Bousfield class* as $\bigvee_{i=0}^n K(i)$, meaning that $(E_n)_* X = 0$ iff $K(i)_* X = 0$ for $0 \leq i \leq n$.[7] (This should be vaguely plausible based on the above observation, if you believe that chromatic homotopy theory works out as beautifully as one might hope.) So, $E_n$ detects whether $X$ is supported over $\{\mathscr{P}_0, \mathscr{P}_{1,(p)}, \ldots, \mathscr{P}_{n,(p)}\}$. For any homology theory $E$, there is an idempotent unital endofunctor $L_E : \mathtt{SHC} \to \mathtt{SHC}$, called *Bousfield localization*, which can be thought of as very roughly analogous to localization of a ring (but instead, $\mathtt{SHC}$) with respect to some multiplicatively closed set (but instead, the thick subcategory $\ker(E_*)$).[8] Then, we can relate the $L_n X = L_{E_n} X$ as $n$ varies via the *chromatic fracture square*, which is the homotopy pullback square

$$\begin{tikzcd}
L_{n+1}X \arrow[r, "\eta_{L_{K(n+1)}}(L_{n+1}X)"] \arrow[d, "\eta_{L_n}(L_{n+1}X)"'] \arrow[dr, phantom, "h", very near start] & L_{K(n+1)}X \arrow[d, "\eta_{L_n}(L_{K(n+1)}X)"] \\
L_n X \arrow[r, "L_n(\eta_{L_{K(n+1)}}(X))"'] & L_n L_{K(n+1)} X
\end{tikzcd}$$

This gives us the *chromatic tower* $\cdots \to L_2 X \to L_1 X \to L_0 X$, and the *chromatic convergence theorem* says that if $X \in \mathtt{SHC}^{fin}$, then $X_{(p)} \simeq \mathrm{holim}_{n<\infty} L_n X$.

5. But this also suggests how to globalize in the *arithemetic* direction. A *global height-*$(\leq n)$ *theory* should be a homology theory which allows us to recover the $E_{n,p}$ at all primes $p$. Obviously, $H\mathbb{Q}$ is a global height-$(\leq 0)$ theory, since $E_0 = H\mathbb{Q}$ at all primes. Next, $KU$ is a global height-$(\leq 1)$ theory: we just $p$-complete to recover $E_{1,p}$. This should be thought of us the global sections of a quasicoherent spectrum-valued sheaf over $\mathrm{Spec}(\mathbb{Z})$; we obtain $E_{1,p}$ by evaluating on $LT_{1,p} = \mathrm{Spf}(\mathbb{Z}_p) \to \mathrm{Spec}(\mathbb{Z})$, since $E_{1,p} = KU \widehat{\otimes}_{\mathbb{Z}} \mathbb{Z}_p = \lim_n KU \otimes_{\mathbb{Z}} \mathbb{Z}/p^n$. Moreover, the sections of this sheaf are complex-orientable, with formal groups isomorphic to the corresponding sections of the sheaf defined by $\widehat{\mathbb{G}}_m \to \mathrm{Spec}(\mathbb{Z})$. (Note that $\widetilde{H}_{1,p} \cong (\mathbb{G}_m)_{\mathbb{Z}_p}$; we can take the multiplicative formal group law as its own universal deformation.) So, to obtain a global height-$(\leq 2)$ theory, we should look for a scheme or stack $\mathcal{M}$ with a sheaf of formal groups and with maps $LT_{2,p} \to \mathcal{M}$ on which our sheaf evaluates to $\widetilde{H}_{n,p}$ – or equivalently, for a scheme or stack $\mathcal{M}$ with a map $\mathcal{M} \to \mathcal{M}_{FG}$ through which the inclusions of the formal neighborhoods $LT_{2,p}$ all factor.

6. We might think that we could try to define a sheaf of homology theories over $\mathcal{M}_{FG}$, but this is not

[5]The extension of the coefficients of $H_n$ is explained by the fact that we have the sequence of inclusions $\mathrm{Aut}_{\mathbb{F}_p}(H_n) \subseteq \mathrm{Aut}_{\mathbb{F}_{p^2}}(H_n \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}) \subseteq \cdots \subseteq \mathrm{Aut}_{\overline{\mathbb{F}}_p}(H_n \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)$, and this stabilizes precisely at $\mathrm{Aut}_{\mathbb{F}_{p^n}}(H_n \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$.

[6]Note that universal deformations aren't actually universal; there's no canonical choice, although they form a contractible groupoid. For this reason, some people insist on saying *a* universal deformation, as opposed to *the* universal deformation. We may slip up occasionally.

[7]This is a statement about all spectra, not just finite spectra. If $X \in \mathtt{SHC}^{fin}$, then $(E_n)_* X = 0$ iff $K(n)_* X = 0$; this follows from the general fact that if $K(i)_* X = 0$ then $K(i-1)_* X = 0$.

[8]Actually, in this case we're localizing at a relatively open subset in $\mathrm{Spec}(\mathtt{SHC}^{fin}_{(p)})$.

possible with the current cutting-edge technology, and for various reasons probably not possible at all.[9] We may summarily say simply that $\mathcal{M}_{FG}$ isn't sufficiently rigid.

7. However, it turns out there are appropriate natural maps $LT_{2,p} \to \mathcal{M}^{ss}_{ell,p}$, where the target is the completion of $(\mathcal{M}^{ss}_{ell})_{\mathbb{F}_p} \subset \overline{\mathcal{M}}_{ell}$.[10] We can see this as follows.

   (a) Any abelian variety $A/k$ has a *p-divisible group* $A[p^\infty] = \operatorname{colim} A[p^n]$.

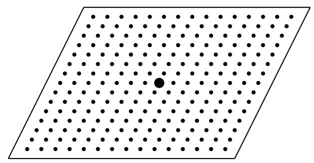

Figure 2: The $p$-divisible group of a complex elliptic curve.

   (b) If $k$ is a field of characteristic $p > 0$, then the *Serre-Tate theorem* says that $\mathrm{Def}_k(A) \cong \mathrm{Def}_k(A[p^\infty])$.

   (c) There is a short exact sequence $0 \to \widehat{A} \to A[p^\infty] \to A_{\text{ét}} \to 0$, under which height is additive; we always have $\mathrm{ht}(A[p^\infty]) = 2 \dim A$.

   (d) Elliptic curves are 1-dimensional abelian varieties, so an elliptic curve $C/k$ has $\mathrm{ht}(C[p^\infty]) = 2$. $C$ can be *ordinary*, meaning that $\mathrm{ht}(\widehat{C}) = 1$, or *supersingular*, meaning that $\mathrm{ht}(\widehat{C}) = 2$.

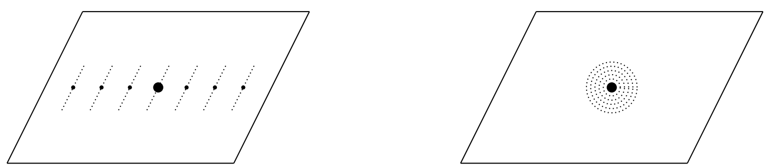

Figure 3: The $p$-divisible groups of ordinary and supersingular elliptic curves in characteristic $p$.

   (e) Thus if $C$ is supersingular then $\widehat{C} = C[p^\infty]$, and hence $\mathrm{Def}_k(\widehat{C}) \cong \mathrm{Def}_k(C[p^\infty]) \cong \mathrm{Def}_k(C)$.

   In particular, the composition $LT_{2,p} \to \mathcal{M}^{ss}_{ell,p} \to \mathcal{M}_{FG}$ is indeed the canonical inclusion.

8. So, we might hope to get a global height-($\leq 2$) theory by taking global sections on the resulting sheaf of homology theories over $\coprod_p \mathcal{M}^{ss}_{ell,p}$. However, since this stack is disconnected, we can't really expect to get any integral behavior from this construction: as things stand, we'd really just be collecting all our constructions together by taking a coproduct. Rather, we would like some single connected object

[9] In the $E_\infty$-version of the Hopkins-Miller theorem, the subgroup structure of the formal groups determines the $E_\infty$-structure, i.e. the associated power operations. But universal deformations are rather special, and in general formal groups don't contain the information necessary to determine these. Lurie's realization theorem applies to maps of stacks to $\mathcal{M}_{FG}$ which factor through some moduli stack $\mathcal{M}_p(n)$ of $p$-divisible groups of height $\leq n$ with formal component of dimension 1, which sidesteps this problem entirely. Cf. Goerss's *Topological modular forms [after Hopkins, Miller, and Lurie]* and *Realizing families of Landweber exact homology theories*.

[10] Obviously, the map $\mathcal{M}^{ss}_{ell,p} \to \mathcal{M}_{FG}$ factors through $\mathcal{M}_p(2)$.

into which all the $\mathcal{M}_{ell,p}^{ss}$ embed. But there is an obvious choice, namely $\mathcal{M}_{ell}$![11] That is:

$$\boxed{\begin{array}{l}\textbf{We pass to Spec}(\mathbb{Z})\textbf{ to put all primes in the game at once, and then we use}\\ \textbf{the ordinary points to interpolate between the supersingular neighborhoods.}\end{array}}$$

Of course, there are a number of other reasons why people care about *tmf*.

- We can use *tmf* to get at the homotopy groups of spheres via the classical mod 2 Adams spectral sequence. There is a sequence of spectra whose mod 2 cohomologies better and better approximate that of the sphere, as follows.

$$\begin{array}{rcl} H\mathbb{F}_2^* H\mathbb{F}_2 & = & \mathcal{A} \\ H\mathbb{F}_2^* H\mathbb{Z} & = & \mathcal{A} /\!\!/ \langle \mathrm{Sq}^1 \rangle \\ H\mathbb{F}_2^* ko & = & \mathcal{A} /\!\!/ \langle \mathrm{Sq}^1, \mathrm{Sq}^2 \rangle \\ H\mathbb{F}_2^* tmf_{(2)} & = & \mathcal{A} /\!\!/ \langle \mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4 \rangle \\ & \vdots & \\ H\mathbb{F}_2^* S^0 & = & \mathcal{A} /\!\!/ \langle \mathrm{Sq}^{2^n} : n \geq 0 \rangle \end{array}$$

  In fact, all further quotients of $\mathcal{A}$ in this sequence are obstructed by the Hopf invariant 1 theorem, so $tmf_{(2)}$ is is the best possible approximation along this route.

- Based on results in physics, Witten was able to define a genus on string manifolds taking values in integral modular forms, i.e. a ring homomorphism $M\mathrm{String}_{2*} \to MF_*$. This refines to the *string orientation*, which sits in the diagram

$$\begin{tikzcd} S^0 = M\mathrm{Framed} \arrow[r, "\mathrm{holim}"] & \cdots \arrow[r] & M\mathrm{String} \arrow[r] \arrow[d] & M\mathrm{Spin} \arrow[r] \arrow[d] & MSO \arrow[r] \arrow[d] & MO \arrow[d] \\ & & tmf & ko & H\mathbb{Z} & H\mathbb{F}_2 \end{tikzcd}$$

  of factorizations of the unit maps of the ring spectra we saw above. Each factorization is easily seen to be sharp with respect to this tower, which gives a refined sense in which these spectra better and better approximate the sphere.

  But this applies to manifold theory as well. It is a classical result that a Spin manifold is nullbordant iff its *ko*- and $H\mathbb{F}_2$-characteristic classes (i.e. its *ko*-Pontrjagin and Stiefel-Whitney classes) vanish.[12,13] One therefore hopes that the *tmf*-characteristic classes might allow us to completely detect bordism classes of String manifolds.

[11] Actually, in keeping with the number theory, where one demands that modular forms evaluate holomorphically at the Tate curve, we work over the Deligne-Mumford compactification $\overline{\mathcal{M}}_{ell}$ in which we allow nodal degenerations but still require our geometric fibers to be irreducible (i.e. no Néron $n$-gons for $n > 1$). On the one hand, this is reasonable for our purposes, since we really only care about the formal groups, after all, and in fact $tmf_{(p)}$ is an $H\mathbb{F}_p$-spectrum (as we will see shortly for $p = 2$) while $TMF_{(p)}$ is not. But the question remains why number theorists make these same choices in the first place. We're not sure, but e.g. Deligne-Rapoport's *Les schémas de modules de courbes elliptiques* provides some hints. When one allows arbitrary Néron $n$-gons, the resulting stack is not Artin. Working over $\mathbb{F}_p$, one can allow Néron $n$-gons for $(n, p) = 1$, and the resulting stack will be Artin (although not separated). Of course, since we're working over $\mathbb{Z}$, then to ensure that our stack is Artin we must require no Néron $n$-gons for $n > 1$, since any such $n$ is divisible by some prime number.

[12] Cf. Anderson-Brown-Peterson's *The structure of the Spin cobordism ring.*

[13] Note that really we should be writing wedges of suspensions of these ring spectra, or actually even wedges of suspensions of various quotient spectra, if we want to capture all the characteristic classes.

## 6.1 Overview of the construction.

And so without further ado, we now given an illustration of the construction of the sheaf $\mathcal{O}^{top}$ on $\overline{\mathcal{M}}_{ell}$ whose global sections will define *Tmf* (and whose sections over $\mathcal{M}_{ell}$ will define *TMF*). Recall that by definition, $tmf = \tau_{\geq 0} Tmf$.

The stacks in the diagram all stand for their respective étale sites restricted to affine schemes (which fact we'll address in a moment) and have the following denotations:

- $\overline{\mathcal{M}}_{ell}$ is the moduli of generalized (i.e. irreducible and possible admitting cuspidal singularities) elliptic curves over Spec($\mathbb{Z}$);
- $(\overline{\mathcal{M}}_{ell})_{\mathbb{Q}}$ is the pullback of $\overline{\mathcal{M}}_{ell}$ along $\mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(\mathbb{Z})$;
- $(\overline{\mathcal{M}}_{ell})_p$ is the pullback of $\overline{\mathcal{M}}_{ell}$ along $\mathrm{Spf}(\mathbb{Z}_p) \to \mathrm{Spec}(\mathbb{Z})$;
- $\mathcal{M}_{ell}^{ord} \subset (\overline{\mathcal{M}}_{ell})_p$ is the substack of elliptic curves over $p$-complete rings with ordinary reduction;
- $\mathcal{M}_{ell}^{ss}$ is the completion of $(\mathcal{M}_{ell}^{ss})_{\mathbb{F}_p} \subset \overline{\mathcal{M}}_{ell}$.

Localizations are applied sectionwise. A number of comments are in order.

- Lest this construction seem unmotivated or ad hoc, we note that the geometry of sheaves on stacks implies that this necessarily recovers the stack we started with. That is, given a derived module sheaf $\mathcal{F}$ over $\overline{\mathcal{M}}_{ell}$, it must be that $\mathcal{F} \simeq \mathrm{holim}(\mathcal{F}_{\mathbb{Q}} \to (\prod_p \mathcal{F}_p)_{\mathbb{Q}} \leftarrow \prod_p \mathcal{F}_p)$ and that $\mathcal{F}_p \simeq \mathrm{holim}(\mathcal{F}_{ord} \to (\mathcal{F}_{ss})_{ord} \leftarrow \mathcal{F}_{ss})$, where subscripts denote derived completions along the appropriate substack.[14] In fact, it is the same setup applied to $\mathcal{M}_{FG}$ that yields the chromatic fracture squares.
- One of the most important points – indeed, what makes the construction of *tmf* so technically difficult – is that there is no immediate notion of global sections, since (by Yoneda) there is no terminal affine scheme with an étale map to $\overline{\mathcal{M}}_{ell}$.[15] We might instead try to extract a homotopy limit over all affine covers, but unfortunately the category of homology theories isn't complete. However, Brown representability tells us that all homology theories are associated to spectra, and so if we lift our presheaf to this category then we might have some renewed hope of a universal elliptic homology theory.[16]

  In fact, it turned out that it was easier[17] to prove a seemingly stronger result: our presheaf valued in homology theories actually lifts to a *sheaf* valued in *$E_\infty$-ring spectra*. It seems that in this seminar we will mostly take these as a black box[18], but for the moment we will simply say that these are ring spectra which are "commutative up to all possible coherent homotopies". The main point here is that $E_\infty$-rings (and their morphisms) are much more rigid than ordinary spectra (and their morphisms), and so in the immortal words of Lurie: "Although it is much harder to write down an $E_\infty$-ring than a spectrum, it is also much harder to write down a map between $E_\infty$-rings than a map between spectra. The practical effect of this, in our situation, is that it is much harder to write down the *wrong* maps between $E_\infty$-rings and much easier to find the right ones." Indeed, the Goerss-Hopkins obstruction theory for $E_\infty$-rings will dictate that all of our choices will be made from contractible spaces thereof.

[14] Given an ideal sheaf $\mathcal{I}$ on $\mathcal{M}$ we have the inclusions $j_n : \mathcal{M}/\mathcal{I}^n \to \mathcal{M}$, and then the derived completion of $\mathcal{F}$ along $\mathcal{I}$ is given by $\mathcal{F}_{\mathcal{I}} = \mathrm{holim}(j_n)_*(j_n)^*\mathcal{F}$.

[15] In the usual setup, a presheaf on a topological space $X$ is just a contravariant functor on its associated category $\mathtt{Open}(X)$ of open subsets and inclusions. Then, its global sections are by definition given by its evaluation on the initial object $X \in \mathtt{Open}(X)^{op}$.

[16] Actually, one can define an étale morphism of stacks (on $\mathtt{Sch}_{\text{ét}}$), so that one can extend the étale site of $\overline{\mathcal{M}}_{ell}$ to include stacks; this makes it possible to literally take global sections by evaluating on the identity map. Cf. Douglas's *Sheaves in homotopy theory*.

[17] "Easier" is a relative term.

[18] Although we generally work in categories of spectra where the categories of commutative ring spectra and $E_\infty$-rings are Quillen-equivalent, the obstruction theory is built using the precise structure of the operads in question.

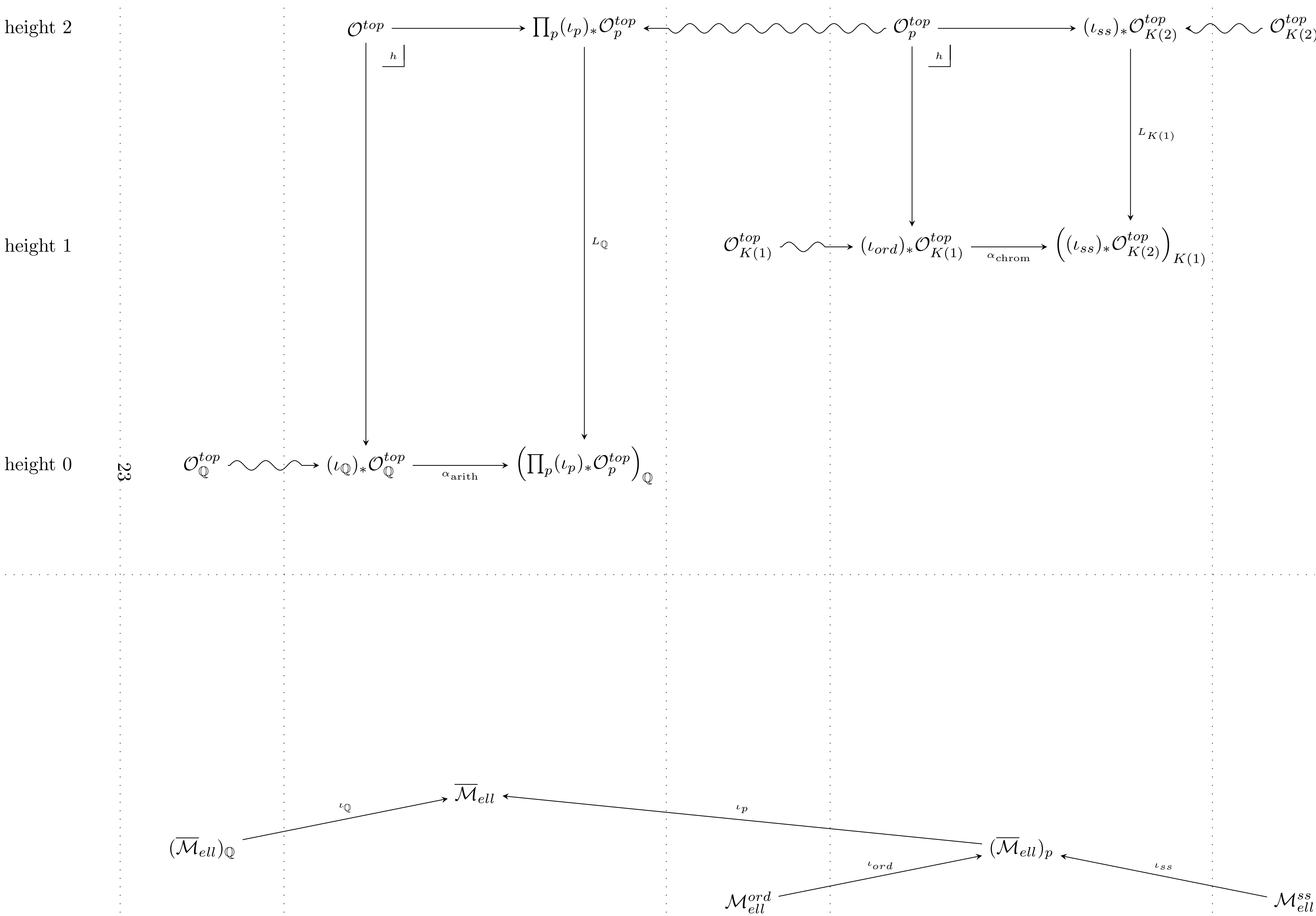


Figure 4: The Diagram.

- What is this sheaf, anyways?
  - If $f : \mathrm{Spec}(R) \to \overline{\mathcal{M}}_{ell}$ classifies a generalized elliptic curve $C/R$, then $E = \mathcal{O}^{top}(\mathrm{Spec}(R))$ is a structured version of the homology theory obtained from the Landweber exact functor theorem. In particular:
    - $\pi_0(E) \cong R$ (so we may call $\mathcal{O}^{top}$ a *derived enhancement* of the ordinary structure sheaf on $\overline{\mathcal{M}}_{ell}$).
    - $E$ is *weakly even-periodic*, i.e. $\pi_2 E \otimes_{\pi_0 E} \pi_n E \xrightarrow{\cong} \pi_{n+2}E$ and $\pi_{2n+1}E = 0$ for all $n \in \mathbb{Z}$. In particular, all even homotopy groups are rank-1 projective $\pi_0 E$-modules. (They aren't necessarily free since a formal group over a ring is only guaranteed to have a coordinate Zariski-locally.)
    - $\mathbb{G}_E \cong \widehat{C}$.
  - In fact, the algebraic geometry tells us that the inclusion $\mathbb{C}P^1 \to \mathbb{C}P^\infty$ corresponds to the projection

    $$\widetilde{E}^*(\mathbb{C}P^\infty) \cong x \cdot E^*[\![x]\!] = \mathfrak{m}_{\mathbb{G}_E,0} \to \widetilde{E}^*(\mathbb{C}P^1) = (x)/(x^2) = \mathfrak{m}_{\mathbb{G}_E,0}/\mathfrak{m}^2_{\mathbb{G}_E,0} = \omega_{\mathbb{G}_E/E_*} = \omega_{C/E_*}$$

    (which may be interpreted either as the relative cotangent space at the identity section or as the module of invariant 1-forms). Thus, sectionwise we have that $\pi_{2n}\mathcal{O}^{top} \cong \omega^{\otimes n}$. Since modular forms of weight $n$ are by definition global sections of the line bundle $\omega^{\otimes n}$ over $\overline{\mathcal{M}}_{ell}$, we might therefore expect $\pi_{2n} Tmf$ to agree with $MF_n$. But in fact, $\pi_* Tmf$ is computed via the *descent spectral sequence* (which will be discussed in the final few talks of this seminar), which takes the form $H^s(\overline{\mathcal{M}}_{ell}, \pi_t^\dagger \mathcal{O}^{top}) \Rightarrow \pi_{t-s} Tmf$ (where the dagger denotes sheafification). Note that we have $E_2^{0,*} = MF_*$ by definition, and then the natural map $E_\infty^{0,*} \to E_2^{0,*}$ induces a map $\pi_{2*} Tmf \to MF_*$, which is an isomorphism away from 6 (in nonnegative degrees).[19] One might therefore call $\pi_* tmf$ the ring of *derived modular forms*. (Note that this is no longer even-concentrated.)
  - The derived stack $(\overline{\mathcal{M}}_{ell}, \mathcal{O}^{top})$ is essentially uniquely characterized by the requirements that it enhances $(\overline{\mathcal{M}}_{ell}, \mathcal{O}_{\overline{\mathcal{M}}_{ell}})$ and that its sectionwise homotopy groups recover the tensor powers of the module of invariant 1-forms.
- For $X$ a Deligne-Mumford stack, $i : X_{\text{ét},\mathit{aff}} \to X_{\text{ét}}$ induces a Quillen equivalence $i^* : \mathtt{Pre}_{X_{\text{ét}}}(\mathtt{Sp}) \leftrightarrows \mathtt{Pre}_{X_{\text{ét},\mathit{aff}}}(\mathtt{Sp}) : i_*$ (using the Jardine model structure), and every fibrant presheaf over $X_{\text{ét}}$ (which in particular must satisfy descent for hypercovers) is the pushforward of a fibrant presheaf over $X_{\text{ét},\mathit{aff}}$. And $\mathcal{O}^{top}$ will be constructed as a fibrant presheaf, so the homotopy limit won't need to be corrected, so all you model category nerds can cool your jets.[20]
- We restrict our attention to étale maps because then we get what we want over $\mathcal{M}_{ell}^{ss}$, which as it turns out can be recovered via $K(2)$-localization.[21] Also, this will make the obstruction theory manageable: objects of the étale site will basically look just like pieces of $\overline{\mathcal{M}}_{ell}$, and pullbacks will not be so far from honest intersections. (Recall that an étale morphism should be thought of as a local-on-the-source isomorphism, and an étale cover is a set of étale maps which are jointly surjective on geometric points.) To wit...

[19] Things always go screwy when the geometry (in this case, the orders of automorphisms of elliptic curves) lines up with the characteristic.

[20] Or you can again cf. Douglas's *Sheaves in homotopy theory*.

[21] Actually we won't recover the $E_2$, but rather their homotopy fixedpoints along the automorphism groups of the associated supersingular elliptic curves. These are called *higher real $K$-theories* and denoted $EO_2$, although this notation is somewhat ambiguous since there's some mess regarding precisely which finite subgroup of the Morava stabilizer group we're using to take homotopy fixedpoints. To partially fix this, we can work away from a fixed prime $p$ and study elliptic curves with $p^k$-level structure for sufficiently large $k$; this extra marking will kill off the automorphisms, although to counterbalance we'll end up with more supersingular points. In any case, by analogy the global sections of the resulting sheaves might be called *higher complex $K$-theories*.

## 6.2 The "easy" part: construction of $\mathcal{O}^{top}_{K(2)}$.

Throughout this section, $R$ will be a $p$-complete ring, and $C/R$ will be an elliptic curve.

Recall that the *scheme of $n$-torsion* of $C$ is by definition the pullback of $C \xrightarrow{[n]} C \xleftarrow{0} \mathrm{Spec}(R)$ in the category of schemes. The *$p$-divisible group* of $C$ is then defined to be $C[p^\infty] = \mathrm{colim}_n\, C[p^n]$. This should be thought of as an algebraic (as opposed to naive) intersection.

**Example 23.** Consider the group scheme $(\mathbb{G}_m)_{\mathbb{F}_p}$; recall that this can be presented as $(\mathbb{G}_m)_{\mathbb{F}_p} \cong \mathbb{F}_p[t^\pm]$, with comultiplication determined by $\Delta(t) = t \otimes t$. On geometric points, there are $\ell$ distinct $\ell^{\text{th}}$ roots of unity, but there is only the trivial $p^{\text{th}}$ root of unity: we can ask for roots of the polynomial $t^p - 1$, but already we have $t^p - 1 = (t-1)^p$. So in characteristic $p$, taking the set-theoretic $p$-torsion may not recover the rank of the group. However, passing to schemes of torsion always gives the correct rank. On the one hand, $(\mathbb{G}_m)_{\mathbb{F}_p}[\ell]$ is a constant group scheme on $\mathbb{Z}/\ell$, so this still works out fine. On the other hand, $(\mathbb{G}_m)_{\mathbb{F}_p}[p] \cong \mathbb{F}_p[t^\pm]/(t^p - 1) = \mathbb{F}_p[t]/(t-1)^p$ is unreduced, but it still has rank $p$. These both reflect that this group has rank 1.

Now, (assuming $\mathrm{ht}(\widehat{C}_0)$ is constant over all mod-$p$ reductions $C_0$ of $C$) we have a short exact sequence $0 \to \widehat{C} \to C[p^\infty] \to C_{\text{ét}} \to 0$; this is analogous to the situation where $G$ is a Lie group, and then we have the short exact sequence $0 \to G_0 \to G \to \pi_0 G \to 0$. Heights are additive over short exact sequences, and so we have the defining dichotomy

| type | $\mathrm{ht}(\widehat{C})$ | $\mathrm{ht}(C[p^\infty])$ | $\mathrm{ht}(C_{\text{ét}})$ |
|---|---|---|---|
| ordinary | 1 | 2 | 1 |
| supersingular | 2 | 2 | 0 |

for elliptic curves.

Here is the theorem that inspired us to invent *tmf* in the first place.

**Theorem 12** (Serre-Tate)**.** *If $k$ is a field of characteristic $p$, $C_0/k$ is an elliptic curve, and $\mathrm{Def}_k$ denotes deformations to complete local rings with residue field $k$, then $\mathrm{Def}_k(C_0) \to \mathrm{Def}_k(C_0[p^\infty])$ is an equivalence of categories. In other words, if $A$ is such a ring, then*

$$\begin{tikzcd} \mathtt{Ell}_A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner^{h}"] & p\text{-}\mathtt{div}_A \arrow[d] \\ \mathtt{Ell}_k \arrow[r] & p\text{-}\mathtt{div}_k \end{tikzcd}$$

*is a pullback diagram.*

(Note that we consider deformations as a groupoid.)

In general, deformations of a $p$-divisible group $\mathbb{G}$ yield deformations of its splitting sequence $0 \to \widehat{\mathbb{G}} \to \mathbb{G} \to \mathbb{G}_{\text{ét}} \to 0$; however, étale groups have no deformations (by definition), and so the deformations of $\mathbb{G}$ are determined by deformations of $\widehat{\mathbb{G}}$ along with an extension class. But if $C_0/k$ is a supersingular elliptic curve, then $\mathrm{ht}((C_0)_{\text{ét}}) = 0$, and so there is no extension class to consider. So in this case, $\mathrm{Def}_k(C_0) \cong \mathrm{Def}_k(C_0[p^\infty]) \cong \mathrm{Def}_k(\widehat{C}_0)$.

Next, we have an identification of these formal moduli spaces.

**Theorem 13** (Lubin-Tate)**.** *If $k$ is a perfect field of characteristic $p$ and $\mathbb{G}/k$ is a formal group of height $n < \infty$, then $\mathrm{Def}_k(\mathbb{G}) \cong \mathrm{Spf}(\mathbb{W}(k)[\![u_1, \ldots, u_{n-1}]\!])$.*

Here $\mathbb{W}(k)$ denotes the ring of *Witt vectors* of $k$, which is the initial complete local ring with residue field $k$; for instance, $\mathbb{W}(\mathbb{F}_p) = \mathbb{Z}_p$ and $\mathbb{W}(\mathbb{F}_{p^n}) = \mathbb{Z}_p[\zeta_{p^n-1}]$. Again, this result tells us that we can deform $\mathbb{G}$ into any height $\leq n$. It also tells us that the functor in question is homotopically discrete: deformations of formal groups admit no nontrivial automorphisms.

Write $B(k, \mathbb{G}) = \mathbb{W}(k)[\![u_1, \ldots, u_{n-1}]\!]$, and let $\widetilde{\mathbb{G}}/B(k, \mathbb{G})$ denote the universal deformation of $\mathbb{G}$. The following result lifts this whole story to topology.

**Theorem 14** (Goerss-Hopkins-Miller)**.** *There is a contravariant functor taking the pair $(k, \mathbb{G})$ to the $E_\infty$-ring spectrum $E(k, \mathbb{G})$, where $E(k, \mathbb{G})$ is Landweber exact and even periodic, $\pi_0 E(k, \mathbb{G}) \cong B(k, \mathbb{G})$, and $\mathbb{G}_{E(k,\mathbb{G})} \cong \widetilde{\mathbb{G}}$. This is an equivalence of topological categories onto its essential image.*

This theorem requires a ridiculous amount of work to prove, but we'll take it as a black box for the moment. It actually follows from the more general Goerss-Hopkins obstruction theory, which we'll talk about in a bit.

With this in hand, we can now construct $\mathcal{O}^{top}_{K(2)}$. So suppose that $C/R$ is the elliptic curve classified by the étale map $f : \mathrm{Spf}(R, I) \to \mathcal{M}^{ss}_{ell}$. This induces an étale map on special fibers $f_0 : \mathrm{Spec}(R/I) \to (\mathcal{M}^{ss}_{ell})_{\mathbb{F}_p}$, which classifies the reduction $C_0/(R/I)$. Observe that the target is zero-dimensional; this is because for any formal group $\mathbb{G}/k$ of height 2, by our identification of its $p$-series we know that the height-2 locus of $\widetilde{\mathbb{G}}$ in $\mathrm{Spf}(B(k, \mathbb{G})) \cong \mathrm{Spf}(\mathbb{W}(k)[\![u_1]\!])$ is precisely $V(p, u_1) \cong \mathrm{Spec}(k)$. This means that $\mathrm{Spec}(R/I)$ is étale over $\mathrm{Spec}(\mathbb{F}_p)$, so we have $R/I \cong \prod_i k_i$ for some finite (i.e. étale) field extensions $k_i/\mathbb{F}_p$. This induces a decomposition $C_0 \cong \coprod_i C_{0,i}$. Since $f$ is étale, then $C$ is a universal deformation of $C_0$, so $R \cong \prod_i B(k_i, \widehat{C}_{0,i})$. Thus we set $\mathcal{O}^{top}_{K(2)}(f : \mathrm{Spf}(R) \to \mathcal{M}^{ss}_{ell}) = \prod_i E(k_i, \widehat{C}_{0,i})$. This is even periodic, and by construction its formal group is isomorphic to $\widehat{C}$. So it is indeed an elliptic $E_\infty$-ring associated to $C/R$.

## 6.3 The not "easy" part: outline of the rest of the construction.

For the remainder of this talk, we'll give a sweeping overview of the rest of the construction.

### 6.3.1 Talk 7: The Igusa tower.

We have a moduli stack $\mathcal{M}^{ord}_{ell}(p^k)$ of generalized elliptic curves $C/R$ (with $R$ $p$-complete) with ordinary reduction, equipped with *$p^k$-level structure*, i.e. an isomorphism $\mathbb{G}_m[p^k] \xrightarrow{\cong} \widehat{C}[p^k]$.[22] These assemble into the *Igusa tower*

$$\mathrm{Spf}(V^\wedge_\infty) = \mathcal{M}^{ord}_{ell}(p^\infty) \xrightarrow{\lim} \cdots \longrightarrow \mathcal{M}^{ord}_{ell}(p^2) \to \mathcal{M}^{ord}_{ell}(p^1) \to \mathcal{M}^{ord}_{ell}(p^0) = \mathcal{M}^{ord}_{ell}$$

Here, $V^\wedge_\infty$ is the ring of *$p$-adic modular functions*, i.e. the universal invariants for generalized elliptic curves $C/R$ (necessarily with no supersingular fibers) equipped with a trivialization $\mathbb{G}_m \xrightarrow{\cong} \widehat{C}$.[23]

Every map $\mathcal{M}^{ord}_{ell}(p^{k+1}) \to \mathcal{M}^{ord}_{ell}(p^k)$ is an étale $\mathbb{Z}/p$-torsor, except at $k = 0$ when this is an étale $(\mathbb{Z}/p)^\times$-torsor; these compose to make the map $\mathrm{Spf}(V^\wedge_\infty) \to \mathcal{M}^{ord}_{ell}$ into an ind-étale $\mathbb{Z}_p^\times$-torsor (via the identification $\mathrm{Aut}_{\mathbb{Z}_p}(\widehat{\mathbb{G}}_m) \cong \mathbb{Z}_p^\times$). In fact, $V^\wedge_\infty$ is canonically a *$\theta$-algebra*, which is the structure naturally present on the $p$-adic $K$-theory of an $E_\infty$-ring. Roughly, this is the data of an action of $\mathbb{Z}_p^\times$, called the *Adams operations*, along with a commuting Frobenius lift. Here, the Adams operations are defined via precomposition of the trivialization with elements of $\mathrm{Aut}((\widehat{\mathbb{G}}_m)_{\mathbb{Z}_p}) \cong \mathbb{Z}_p^\times$, and the Frobenius lift is defined by taking $(C, \phi : \widehat{\mathbb{G}}_m \xrightarrow{\cong} \widehat{C})$ to $(C/C[p], \overline{\phi} : \widehat{\mathbb{G}}_m \cong \widehat{\mathbb{G}}_m/\widehat{\mathbb{G}}_m[p] \xrightarrow{\cong} \widehat{C/C[p]})$.

[22]For $k > 0$, the existence of such a level structure implies that $C$ cannot have any supersingular fibers anyways.

[23]This is immediate by Yoneda's lemma, once we know that $\mathcal{M}^{ord}_{ell}(p^\infty)$ is formally affine. In fact, $\mathcal{M}^{ord}_{ell}(p^k)$ is formally affine for all $k \geq 2$, and is formally affine for $k = 1$ whenever $p > 2$.

### 6.3.2 Talk 8: $\theta$-algebras and $E_\infty$-rings.

Lifts of maps of $\theta$-algebras to maps of $E_\infty$-rings are governed by *Goerss-Hopkins obstruction theory*. Goerss-Hopkins obstruction theory is an extremely general framework for realizing an algebraic map (of algebras over a monad) as a topological map (of algebras in spectra over some operad). They construct a theory of functorial Postnikov towers with respect to the appropriate notion of homotopy groups, which allows them to construct a tower of moduli spaces of maps of simplicial spectra whose realizations yield better and better approximations to the given algebraic map. The Eilenberg-MacLane objects for this theory of Postnikov towers (i.e. the targets of the $k$-invariants) naturally represent André-Quillen cohomology; thus the obstructions to lifting a vertex through the tower live in André-Quillen cohomology groups in the appropriate algebraic category. Fantastically, this means that the obstructions are given entirely at the level of algebra.

### 6.3.3 Talk 9: $K(1)$-local elliptic spectra.

Suppose $R$ is $p$-complete, and suppose $E$ is a $K(1)$-local $E_\infty$ elliptic spectrum associated to a generalized elliptic curve $C/R$. (That $E$ is $K(1)$-local implies that $C$ has ordinary reduction.) Then the $p$-adic $K$-theory of $E$ is given by the pullback diagram

$$\begin{tikzcd} \mathrm{Spf}((K_p^\wedge)_0E) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner h" very near start] & \mathcal{M}_{ell}^{ord}(p^\infty) \arrow[d] \\ \mathrm{Spf}(R) \arrow[r, "f"'] & \mathcal{M}_{ell}^{ord} \end{tikzcd}$$

This is always $\mathbb{Z}_p^\times$-equivariant, i.e. the Adams operations on $(K_p^\wedge)_0E$ coincide with the torsor structure induced from that of $\mathrm{Spf}(V_\infty^\wedge) \to \mathcal{M}_{ell}^{ord}$. When $f$ is étale, this also induces a $\theta$-algebra structure on $(K_p^\wedge)_0E$. This may not coincide with the one already there, but it will coincide on the sections of $\mathcal{O}_{K(1)}^{top}$ by construction.

### 6.3.4 Talk 10: Construction of $\mathcal{O}_{K(1)}^{top}$.

The construction of $\mathcal{O}_{K(1)}^{top}$ proceeds in two steps, both using Goerss-Hopkins obstruction theory.

1. We construct $tmf_{K(1)} = \mathcal{O}_{K(1)}^{top}(\mathcal{M}_{ell}^{ord})$ as follows.
   (a) If $p > 2$ then $\mathcal{M}_{ell}^{ord}(p)$ is formally affine, and so we can relatively easily construct $tmf(p)^{ord} = \mathcal{O}_{K(1)}^{top}(\mathcal{M}_{ell}^{ord}(p))$ along with an action of $(\mathbb{Z}/p)^\times$ through $E_\infty$-ring maps. Then, we set $tmf_{K(1)} = (tmf(p)^{ord})^{h(\mathbb{Z}/p)^\times}$.
   (b) At $p = 2$ we only have that $\mathcal{M}_{ell}^{ord}(4)$ is formally affine. We might like mimic the previous setup and try to construct a spectrum $tmf(4)^{ord}$ with a $(\mathbb{Z}/4)^\times$-action, but this group has order 2 and so the obstructions don't vanish. Instead, we replace $K$ with $KO$; the obstruction theory doesn't carry over entirely, but it carries over enough for us to be able to produce $tmf_{K(1)}$ directly.[24]

2. We construct the sheaf $\mathcal{O}_{K(1)}^{top}$ in the category of $tmf_{K(1)}$-algebras.

[24]Alternatively, Laures gives a construction of $tmf_{K(1)}$ at $p = 2$ by attaching two $K(1)$-local $E_\infty$-cells to the $K(1)$-local sphere, cf. $K(1)$*-local topological modular forms*.

The second step uses crucially the isomorphism $(K_p^\wedge)_0 tmf_{K(1)} \cong V_\infty^\wedge$, which reflects the fact that when $E$ and $F$ are Landweber exact, then $\mathrm{Spec}(\pi_0(E \wedge F)) = \underline{\mathrm{Iso}}(\mathbb{G}_E, \mathbb{G}_F)$.

#### 6.3.5 Not a talk: The chromatic attaching map.

The sheaves $\mathcal{O}_{K(1)}^{top}$ and $\mathcal{O}_{K(2)}^{top}$ interrelate as follows. Write $B = \mathbb{W}(k)[\![u_1]\!]$ with $C/B$ a universal deformation of a supersingular elliptic curve, and let $E = \mathcal{O}_{K(2)}^{top}(\mathrm{Spf}(B))$. Then $C$ restricts to an *ordinary* elliptic curve $C^{ord}$ over the punctured formal disk $\mathrm{Spf}(B^{ord})$, where $B^{ord} = B[u_1^{-1}]_p^\wedge$, and moreover it turns out that $E_{K(1)}$ is an appropriate corresponding elliptic spectrum. This will be the object that receives the *chromatic attaching map* $\alpha_{\mathrm{chrom}} : (\iota_{ord})_* \mathcal{O}_{K(1)}^{top} \to \left((\iota_{ss})_* \mathcal{O}_{K(2)}^{top}\right)_{K(1)}$. This map is also constructed in two steps: we construct the map $tmf_{K(1)} \to (tmf_{K(2)})_{K(1)}$ of global sections, and then we use obstruction theory for $tmf_{K(1)}$-algebras to extend this to a map of sheaves as desired.

#### 6.3.6 Not a talk: Construction of $\mathcal{O}_{\mathbb{Q}}^{top}$ and the arithmetic attaching map.

Note that over a $\mathbb{Q}$-algebra, every formal group is isomorphic to $\widehat{\mathbb{G}}_a$; thus, the sections of $\mathcal{O}_{\mathbb{Q}}^{top}$ are all essentially rational Eilenberg-MacLane spectra. Then, we construct the *arithmetic attaching map* $\alpha_{\mathrm{arith}} : (\iota_{\mathbb{Q}})_* \mathcal{O}_{\mathbb{Q}}^{top} \to \left(\prod_p (\iota_p)_* \mathcal{O}_p^{top}\right)_{\mathbb{Q}}$ as follows. First, we observe that $(\overline{\mathcal{M}}_{ell})_{\mathbb{Q}}$ is covered by $(\overline{\mathcal{M}}_{ell}[\Delta^{-1}])_{\mathbb{Q}}$ and $(\overline{\mathcal{M}}_{ell}[c_4^{-1}])_{\mathbb{Q}}$; note that on singular Weierstraß curves, $c_4$ is invertible precisely if there are no cuspidal singularities. So, we construct $\alpha_{\mathrm{arith}}$ on the sections over these substacks and over their intersection, and we verify that they are compatible; by descent, this induces the desired map of presheaves.

# 7 The Igusa tower – Marcus Zibrowius

Recall The Diagram from Aaron's talk. Let us briefly orient ourselves within the construction. Recall that we have $\mathcal{M}_{ell}^{ord} \to (\overline{\mathcal{M}}_{ell})_{\mathbb{Z}_p^\wedge} \leftarrow \mathcal{M}_{ell}^{ss}$; Aaron indicated how to construct $\mathcal{O}_{K(2)}^{top} \downarrow \mathcal{M}_{ell}^{ss}$, and the next four talks will be dedicated to construction $\mathcal{O}_{K(1)}^{top} \downarrow \mathcal{M}_{ell}^{ord}$. In this talk, we'll construct a $\theta$-algebra $V_\infty^\wedge$, which will eventually be $V_\infty^\wedge = (K_p^\wedge)_0(\mathcal{O}_{K(1)}^{top}(\mathcal{M}_{ell}^{ord})$.

## 7.1 Assorted aspects of characteristic $p > 0$

We collect some different facts that we'll need. Throughout, let $k$ be a field of characteristic $p$ with Frobenius $\sigma$.

#### 7.1.1 Frobenius

There are several sensible ways of defining a Frobenius morphism on a $k$-algebra. For instance, take $k[x]/(x^2 - a)$. We could define a Frobenius by $f \mapsto f^p$, but in fact this will not be a morphism of $k$-algebras (since in particular it doesn't fix $k \subset k[x]/(x^2 - a)$). Instead we could set $x \mapsto x^p$. In fact, this gives a commutative diagram

$$\begin{tikzcd}
k[x]/(x^2-a) & \\
k[x]/(x^2-a^p) \arrow[u, "x\mapsto x^p"] & k[x]/(x^2-a). \arrow[l, "\sigma \text{ on coefficients}"] \arrow[ul, "f\mapsto f^p"']
\end{tikzcd}$$

In general, for a scheme $X/k$, we have the diagram

$$\begin{tikzcd}
X \arrow[d, "\Phi"'] \arrow[dr, "\Phi^{tot}"] & \\
\sigma^* X \arrow[r] \arrow[d] & X \arrow[d] \\
k \arrow[r, "\sigma"] & k,
\end{tikzcd}$$

where $\Phi^{tot}$ is the "total Frobenius" and $\Phi$ is the "relative Frobenius" (and only the latter is $k$-linear).

#### 7.1.2 Morphisms of curves

**Theorem 15.** *Any non-constant morphism of smooth (projective) curves over $k$ factors as*

$$\begin{tikzcd}
C_1 \arrow[r, "f"] \arrow[d, dashed, "\Phi^h"'] & C_2 \\
(\sigma^h)^* C_1 \arrow[ur, dashed, "f_{sep}"'] &
\end{tikzcd}$$

*for some $h \geq 0$, where $\Phi^h$ is purely inseparable and $f_{sep}$ is separable. (That is, the induced extensions of function fields have these properties.)*

#### 7.1.3 Height of fgl's

Similarly, any non-zero morphism of fgl's over $K$ factors as

$$\begin{tikzcd}
(k[[x]], F_1) & (k[[x]], F_2) \arrow[l, "f"'] \arrow[dl, dashed, "f_{sep}"] \\
(k[[x]], (\sigma^h)^* F_1) \arrow[u, dashed, "\Phi^h"] &
\end{tikzcd}$$

for some maximal $h \geq 1$. In fact, $f_{sep}$ will be an isomorphism of fgl's. In particular, for any fgl $F/k$, if we write $[p](x) = x +_F \cdots +_F x$ (with $p$ summands), if this is nonzero then we have $[p](x) = [p]_{sep}(x^{p^h})$. This $h$ is called the *height*. (If $[p](x) = 0$, then we say that the height is $\infty$.)

### 7.1.4 Elliptic curves

For a generalized elliptic curve $C/k$, we can consider the map $[p] : C \to C$ using the group structure. As before, we have a factorization

$$\begin{tikzcd} C \arrow[r, "{[p]}"] \arrow[d, "\Phi^h"'] & C \\ (\sigma^h)^* C. \arrow[ur, "{[p]_{sep}}"'] & \end{tikzcd}$$

In fact, this induces the factorization of $[p]$ on $\widehat{C}$. In particular, $[p]_{sep}$ induces an isomorphism of formal groups, and $h = \text{height}(\widehat{C})$.

But in this case we can actually say something about the height. In general, $\deg([p]) = p^2$ and $\deg(\Phi^h) = p^h$, and so from this we can deduce that $h \in \{1, 2\}$. In particular, we recall that $C$ is called *supersingular* (although it will necessarily be smooth!) if $\text{height}(\widehat{C}) = 2$, and $C$ is called *ordinary* if $\text{height}(\widehat{C}) = 1$. The latter is equivalent to demanding that the Hasse invariant $H(C)$ of $C$ is invertible (but since we're over a field, this is equivalent to being non-zero). Recall that the *Hasse invariant* $H(C)$ is the coefficient of $x^p$ in the power series describing $[p] : \widehat{C} \to \widehat{C}$. (This is not a well-defined function but rather a modular form of weight $p - 1$, but its vanishing is nevertheless well-defined.)

### 7.1.5 Torsion points

Let $C/k$ be an elliptic curve. Then the scheme of $p^l$-torsion is defined by $C[p^l] = \ker([p^l] : C \to C)$ (and similarly $\widehat{C}[p^l] = \ker([p^l] : \widehat{C} \to \widehat{C})$ for the formal group). If $k$ had characteristic 0 we'd just have $\widehat{C}[p^l] = \text{Spec}(k)$ and $C[p^l] = \mathbb{Z}/p^l \times \mathbb{Z}/p^l$. But in characteristic $p$, different things happen. If $C$ is supersingular, then $\widehat{C}[p^l]$ has only one $k$-rational point and $\widehat{C}[p^l] \xrightarrow{\cong} C[p^l]$. If $C$ is ordinary, then there is a short exact sequence $0 \to \widehat{C}[p^l] \to C[p^l] \to \mathbb{Z}/p^l \to 0$, and abstractly $\widehat{C}[p^l] \cong \mu_{p^l} = \text{Spec}(k[x]/(x^{p^l} - 1)$.

In the case that $C = \mathbb{G}_m$ (i.e. a nodal elliptic curve, but without its node) then we get $\widehat{\mathbb{G}}_m[p^l] \xrightarrow{cong} \mathbb{G}_m[p^l]$. So this behaves much like the ordinary case (and this is the reason that Behrens calls these ordinary).

## 7.2 Elliptic curves over $p$-complete rings

Let $R$ be a $p$-complete ring, i.e. the natural map $R \to \lim R/p^i$ is an isomorphism. (The first example is $R = \mathbb{Z}_p^\wedge$.) Let $C$ be a generalized elliptic curve over $\text{Spf}(R)$. This can be viewed equivalently either as a compatible sequence of elliptic curves over the $\text{Spec}(R/p^i)$, or else as a single elliptic curve over $\text{Spf}(R)$.

**Definition 17.** $C$ is called *ordinary* if (the non-singular locus of) every fiber over a field of characteristic $p$ is ordinary (in the sense defined previously).

**Remark 11.** Many results from the previous section carry over to this situation. In particular, there exists a factorization

$$\begin{tikzcd} C \arrow[r, "{[p]}"] \arrow[d, "{[p]_{insep}}"'] & C \\ C^{(p)} \arrow[ur, "{[p]_{sep}}"'] & \end{tikzcd}$$

specializing to the one in characteristic $p$ that we discussed above, and such that $[p]_{sep}$ induces an isomorphism of formal groups. Also, we can detect whether $C$ is ordinary if (any lift of) the Hasse invariant is invertible.

**Definition 18.** A $p^l$*-level structure* on an ordinary curve $C/R$ is an isomorphism $\eta : \mu_{p^l} \xrightarrow{\cong} \widehat{C}[p^l]$.

Recall that there is a moduli stack $\mathcal{M}_{ell}^{ord}$ classifying ordinary elliptic curves over $p$-complete rings. As it turns out, there are also moduli stacks $\mathcal{M}_{ell}^{ord}(p^l)$ these along with a $p^l$-level structure $\eta$.

## 7.3 The Igusa tower

Note that a $p^{l+1}$-level structure induces a $p^l$-level structure by restriction. We can also of course completely forget level structures (which is not the same thing as saying that we can forget about level structures altogether!). This gives us a tower of stacks

$$\cdots \to \mathcal{M}_{ell}^{ord}(p^{l+1}) \to \mathcal{M}_{ell}^{ord}(p^l) \to \cdots \to \mathcal{M}_{ell}^{ord}(p) \to \mathcal{M}_{ell}^{ord},$$

called the *Igusa tower.*

**Theorem 16.**

1. *This is a tower of étale $\mathbb{Z}/p$-torsors (except that the last map is an étale $(\mathbb{Z}/p)^\times$-torsor). That is, if we have any map $\mathrm{Spf}(R) \to \mathcal{M}_{ell}^{ord}(p^l)$, then the pullback of $\mathcal{M}_{ell}^{ord}(p^{l+1}) \to \mathcal{M}_{ell}^{ord}(p^l)$ will be an étale-locally trivial $\mathbb{Z}/p$-torsor.*
2. *For $l \geq 1$ (or for $l \geq 2$ if $p = 2$), $\mathcal{M}_{ell}^{ord}(p^l)$ is actually a formally affine scheme, i.e. $\mathcal{M}_{ell}^{ord}(p^l) \cong \mathrm{Spf}(V_l)$ for some $p$-complete ring $V_l$.*
3. *Let $V_\infty^\wedge = \lim_i \mathrm{colim}_l (V_l/p^i)$. Then $\mathcal{M}_{ell}^{ord}(p^\infty) = \mathrm{Spf}(V_\infty^\wedge)$ classifies pairs $(C, \eta)$, where $C$ is ordinary and $\eta : \mathbb{G}_m \xrightarrow{\cong} \widehat{C}$.*

To prove part 2, Behrens first shows that slightly larger stacks are formally affine. If $C$ is an ordinary curve and $I$ is the ideal sheaf defining the zero section, then we define an *$n$-jet* on $C$ to be a section of $I/I^{n+1}$.

**Lemma 1.** *For $n \geq 4$, ordinary curves with $n$-jets are classified by a formally affine stack $(\mathcal{M}_{ell}^{ord})^n = \mathrm{Spf}(\mathbb{Z}_p^\wedge[a_1, \ldots, a_4, a_6, m_5, \ldots, m_n][H^{-1}])$.*

*Proof sketch.* Recall that the $a_i$ come from a Zariksi-locally-defined Weierstraß equation. Then, an $n$-jet is locally of the form $\mathcal{J} = m_1 T + m_2 T^2 + \ldots + m_n T^n$, where $T = -x/y$ is the canonical coordinate at $\infty$ (the identity) and we don't care about higher-order terms since we've quotiented out by $I^{n+1}$. Rezk proves that there exists a unique automorphism $x \mapsto x'$ and $y \mapsto y'$ fixing $\infty$ and putting $\mathcal{J}$ into the form $\mathcal{J} = T' + m_s'(T')^5 + \ldots + m_n'(T')^n$. (This arises because there are four independent variables in a coordinate change for an Weierstraß curve.) Therefore, $(C, \mathcal{J})$ is Zariski-locally determined up to unique isomorphism by the indicated coordinates. Finally, we invert the Hasse invariant (which is defined in terms of the $a_i$ and the $m_j$) in order to ensure that $C$ is ordinary. □

Then there is another auxiliary result.

**Lemma 2.** *$\mathcal{M}_{ell}^{ord}(p^l)$ is a closed substack of $(\mathcal{M}_{ell}^{ord})^{p^l-1}$.*

From this, we can deduce that $\mathcal{M}_{ell}^{ord}(p^l)$ is also formally affine. Its ring of global functions is just $\mathcal{O}((\mathcal{M}_{ell}^{ord})^{p^l})/I$ for some ideal $I$, and of course we define this to be $V_l$.

## 7.4 The $\theta$-algebra structure on $V_\infty^\wedge$

**Theorem 17.** *$V_\infty^\wedge$ is a $\theta$-algebra, i.e. it is equipped with self maps $\psi^p$, $\psi^k$ for all $k \in (\mathbb{Z}_p^\wedge)^\times$, and $\theta$, such that $\psi^p(x) = x^p + p \cdot \theta(x)$ and such that $\psi^p$ and $\psi^k$ are ring endomorphisms for all $k$.*

We will define these operations indirectly by saying how they act on $R$-points of the stack. In fact, $V_\infty^\wedge$ is torsion-free, so $\theta$ is uniquely determined by $\psi^p$.

### 7.4.1 Definition of $\psi^k$

It turns out that $\mathrm{Aut}_{\mathbb{Z}_p^\wedge}(\widehat{\mathbb{G}}_m) = (\mathbb{Z}_p^\wedge)^\times$, where $\sigma a_i p^i$ acts as $[a_0] +_{\widehat{\mathbb{G}}_m} [a_1 p] +_{\widehat{\mathbb{G}}_m} [a_2 p^2] +_{\widehat{\mathbb{G}}_m} \cdots$. Then, we define $\psi^k$ on the $R$-point $(C, \eta)$ by declaring that

$$(C, \eta : \widehat{\mathbb{G}}_m \xrightarrow{\cong} \widehat{C}) \overset{\psi^k}{\mapsto} (C, \eta \circ [k] : \widehat{\mathbb{G}}_m \xrightarrow{\cong} \widehat{C}).$$

### 7.4.2 Definition of $\psi^p$

Let us recall the diagram

$$\begin{tikzcd} C \arrow[r, "{[p]}"] \arrow[d, "{[p]_{insep}}"'] & C \\ C^{(p)}, \arrow[ur, "{[p]_{sep}}"'] & \end{tikzcd}$$

and recall that $[p]_{sep}$ induces an isomorphism of formal groups. Then we define $\psi^p$ on the $R$-point $(C, \eta)$ by declaring that

$$(C, \eta : \widehat{\mathbb{G}}_m \xrightarrow{\cong} \widehat{C}) \overset{\psi^p}{\mapsto} ((C^{(p)}, (\widehat{[p]_{sep}})^{-1} \circ \eta : \widehat{\mathbb{G}}_m \xrightarrow{\cong} \widehat{C} \xrightarrow{\cong} \widehat{C}^{(p)}).$$

### 7.4.3 Definition of $\theta$

To show that $\theta$ is indeed defined by $\psi^p$, we need to show that $\psi^p(x) \equiv x^p \pmod p$. To show this, we show that $\psi^p$ agrees with the pullback along the Frobenius $\sigma$ for $R$ of characteristic $p$: in this case, we have

$$(C, \eta) \overset{\psi^p}{\mapsto} (\sigma^* C, \widehat{\mathbb{G}}_m = \sigma^* \widehat{\mathbb{G}}_m \xrightarrow{\sigma^* \eta} \sigma^* \widehat{C}).$$

Indeed, one has $C^{(p)} = \sigma^* C$ in characteristic $p$, and then we have the diagrams

$$\begin{tikzcd} \widehat{\mathbb{G}}_m \arrow[r, "{[p]}"] \arrow[d, "\Phi"'] & \widehat{\mathbb{G}}_m \arrow[dl, equal] \\ \sigma^* \widehat{\mathbb{G}}_m & \end{tikzcd}$$

and

$$\begin{tikzcd} \widehat{C} \arrow[r, "{[p]}"] \arrow[d] & \widehat{C} \\ \sigma^* \widehat{C}. \arrow[ur, "\widehat{[p]_{sep}}"'] & \end{tikzcd}$$

The former maps to the latter via $\eta$ and $\sigma^* \eta$, and this proves the claim.

# 8 $\theta$-algebras and $E_\infty$-rings – Uwe Kranz

The goal of this talk is to illustrate how Goerss-Hopkins obstruction theory will allow us to obtain $E_\infty$-rings from the $\theta$-algebras produced by pullbacks of $\mathrm{Spf}(V_\infty^\wedge) \to \mathcal{M}_{ell}^{ord}$ along étale morphisms.

## 8.1 $\lambda$-rings and $\theta$-algebras

The following definition is meant to formally encode the structure of exterior powers.

**Definition 19.** A *$\lambda$-ring* is a commutative $R$ together with operations $\lambda_n : R \to R$ (which are not ring homomorphisms!) for $n \in \mathbb{N}$, such that:

- $\lambda^0(x) = 1$;
- $\lambda^1(x) = x$;
- $\lambda^n(x+y) = \sum_{r=0}^{n} \lambda^r(x)\lambda^{n-r}(y)$.

Moreover, we demand that:

- $\lambda^n((1) =$ for $n > 1$;
- $\lambda^n(xy)$ is given as a particular polynomial $P_n(\lambda_1(x), \ldots, \lambda_n(x), \lambda_1(y), \ldots, \lambda_n(y))$;
- $\lambda^m(\lambda^n(x))$ is given as a particular polynomial $P_{n,m}(\lambda_1(x), \ldots, \lambda_{mn}(x))$.

**Remark 12.** Some people take the first batch of requirements to define a *special* $\lambda$-ring. We won't both to make the distinction.

**Example 24.** We have the following naturally-occurring $\lambda$-rings: $K(X)$, $R_{\mathbb{C}}(G)$, $K(A)$.

**Proposition 2.** *Every natural operation of $\lambda$-rings (i.e. every natural transformation $t : \mathrm{id}_{\lambda\texttt{-rings}} \Rightarrow \mathrm{id}_{\lambda\texttt{-rings}}$) is a polynomial in the $\lambda$-operations.*

In particular, we have the *Adams operations* $\psi^n(x) = \nu_n(\lambda_1(x), \ldots, \lambda_n(x))$, where $\nu_n$ is the $n^{th}$ Newton polynomial. This enjoys the following properties:

1. $\psi^n$ is a ring homomorphism;
2. $\psi^p \equiv x^p \bmod p$ (for prime $p$);
3. $\psi^n\psi^m = \psi^{nm}$.

In fact, for a fixed prime $p$ and given $\lambda$-ring $(R, \lambda)$ with no $p$-torsion, we can define a natural operation $\theta^p$ such that $\psi^p(x) = x^p + p \cdot \theta^p(x)$.

We can use the fact that the Adams operations are ring homomorphisms to figure out what $\theta$ does to addition and multiplication. We will see this in the following definition, athough there are more conditions that are only justified by our particular aims.

**Definition 20.** A *$\theta$-algebra* is a $\mathbb{Z}/2\mathbb{Z}$-graded continuous commutative $\mathbb{Z}_p$-algebra $A$ such that:

1. $A/pA$ is discrete and the natural map $A \to L_0(A)$ is an isomorphism, where $L_0$ is the $0^{th}$ derived functor of $p$-completion (i.e. $A$ is *$L$-complete*);
2. $\psi^k : A \to A$ is linear for all $k \in \mathbb{Z}_p^\times$, and

$$\psi^k(xy) = \begin{cases} \psi^k(x)\psi^k(y), & |x| = 0 \text{ or } |y| = 0 \\ \frac{1}{k}\psi^k(x)\psi^k(y), & |x| = |y| = 1; \end{cases}$$

3. $\theta : A_i \to A_i$ commutes with the $\psi^k$, and

$$\theta(x+y) = \begin{cases} \theta(x) + \theta(y) - \sum_{s=1}^{n-1} \cdot \frac{1}{p}\binom{p}{s} x^s y^{p-s}, & |x| = |y| = 0 \\ \theta(x) + \theta(y), & |x| = |y| = 1, \end{cases}$$

and $\theta(1) = 0$, and

$$\theta(xy) = \begin{cases} \theta(x)y^p + x^p\theta(y) + p\theta(x)\theta(y), & |x| = 0 \text{ or } |y| = 0 \\ \theta(x)\theta(y), & |x| = |y| = 1. \end{cases}$$

**Remark 13.** Note that $p$-completion is neither left- nor right-exact, and the derived exact sequence extends one step in the wrong direction. Thus, $L_0$ is actually the first derived functor of some other functor.

**Remark 14.** What we saw in the previous talk was the same thing, but without the gradings.

This is the algebraic structure we'd like to realize on the level of topology, and this comes from the following result.

**Theorem 18.** *If $X$ is a $K(1)$-local $E_\infty$-ring spectrum, then $(K_p^\wedge)_* X$ can be naturally equipped with the structure of a $\theta$-algebra.*

(The $\psi^k$ are the usual Adams operations, and $\theta$ is a Dyer-Lashof type operation.)

## 8.2 Operads and their algebras

Let us make a digression on the notion of an $E_\infty$-ring spectrum. For a spectrum $X$, the endomorphism operation $\mathrm{End}(X)$ consists of the pointed spaces $T_i = \mathrm{End}(X)_i = \mathrm{map}(X^{\wedge i}, X)$; each $T_i$ comes with a $\Sigma_i$-action, and moreover there are composition maps $T_n \times T_{i_1} \times \cdots T_{i_n} \to T_{\sum i_j}$ given by $(g_n; f_1, \ldots, f_n) \mapsto g(f_1 \wedge \cdots \wedge f_n)$. Moreover, these compositions satisfy certain associativity and unitality conditions. In general, an *operad* is a sequence of spaces with such composition maps satisfying the same conditions. In particular, an *$E_\infty$-operad* is an operad $\{T_i\}$ such that all $T_i$ are contractible and such that $\Sigma_i$ acts freely.

**Definition 21.** An *$E_\infty$-ring spectrum* $X$ is a spectrum $X$ along with a map of operads $T \to \mathrm{End}(X)$, where $T$ is an $E_\infty$-operad.

The idea here is that the contractibility of the $T_i$ encodes the fact that the multiplication determined by $* \in T_2 \to \mathrm{map}(X \wedge X, X)$ is associative and commutative up to all higher homotopies.

**Example 25.** The Thom spectra $MO$, $MSO$, $MU$, $MSU$ are $E_\infty$-ring spectra. The sphere spectrum is also $E_\infty$, as are $KO$ and $KU$.

## 8.3 Obstruction theory

### 8.3.1 André-Quillen cohomology

So, we now address the following question: What are the obstructions to realizing some $\theta$-algebra $A$ as $(K_p^\wedge)_* X$ for some $E_\infty$-ring spectrum $X$? To understand this, we will need to discuss *André-Quillen cohomology.*

We begin with the classical case. Let $A$ be a commutative $R$-algebra, where $R$ is a commutative ring, and let $M$ be an $A$-module. Suppose we have a map $X \to A$ is a map of $R$-algebras. Then we can talk about the $R$-module of *$R$-derivations* $\mathrm{Der}_R(X, M)$. We can consider this as a functor $\mathrm{Der}_R(-, M)$ of $X \in \mathtt{Alg}_R/A$, and then we define the *André-Quillen cohomology* of $X$ with coefficients in $M$ to be $H^n_{AQ}(X, M) = R^n \mathrm{Der}_R(-, M)(X)$.

Let us pass to the $\theta$-algebra setting. We must say what an $A$-$\theta$-module is (for $A$ a $\theta$-algebra). This is a continuous $\mathbb{Z}/2\mathbb{Z}$-graded module $M$ over $A$, equipped with continuous homomorphisms $\psi^k$ for all $k \in \mathbb{Z}_p^\times$ and with an operation $\theta : M \to M$, such that $M$ is a *Morava module* (the non-ring analog of $\theta$-algebra), and such that for all $a \in A$ and $x \in M$, $\psi^k(ax) = \psi^k(a)\psi^k(x)$ and

$$\theta(ax) = \begin{cases} a^p\theta(x) + \theta(a)\theta(x), & |a| = 0 \text{ or } |x| = 0 \\ \theta(a)\theta(x), & |a| = |x| = 1. \end{cases}$$

Now, let us denote by $M \rtimes A$ the augmented $A$-algebra which as a group is $M \otimes A$, and with multiplication $(x, a) \cdot (y, b) = (ay + xb, ab)$; this is called the *square-zero extension* of $A$ by $M$. We turn this into a $\theta$-algebra by setting $\psi^k(x, a) = (\psi^k(x), \psi^k(a))$ and $\theta(x, a) = (\theta(x) - a^{p-1}x, \theta(a))$. As usual, the category of these is equivalent to the category of $A$-$\theta$-modules. Then if $X \in \theta\text{-}\mathtt{alg}/A$, we define $H^n_\theta(X/A, M) = R^n\mathrm{Hom}_{\theta\text{-}\mathtt{alg}/A}(X, M \rtimes A)$.

### 8.3.2 The obstruction theorem

We can now state the following obstruction theorem.

**Theorem 19** (Goerss-Hopkins)**.** *Given a $\theta$-algebra $A$, there are successively defined obstructions to the existence of a $K(1)$-local $E_\infty$-ring spectrum $X$ for which there exists an isomorphism $(K_p^\wedge)_*X \cong A$ of $\theta$-algebras lie in $H^s_\theta(A, \Omega^{s-2}A)$ for $s \geq 3$, and then the obstructions to uniqueness lie in $H^s_\theta(A, \Omega^{s-1}A)$ for $s \geq 2$.*

We still must define the looping operation. We set $(\Omega A)_n = A_{n+1}$, under which we let $\varepsilon x$ correspond to $x$. Then,

$$\psi^k(\varepsilon x) = \begin{cases} k\varepsilon\psi^k(x), & |x| = 0 \\ \varepsilon\psi^k(x), & |x| = 1 \end{cases}$$

and

$$\theta(\varepsilon x) = \begin{cases} \varepsilon(x^p + p\theta(x)), & |x| = 0 \\ \varepsilon\theta(x), & |x| = 1. \end{cases}$$

We can also say how this arises. Namely, if $X$ is an $E_\infty$-ring spectrum, then $(K_p^\wedge)_*\mathcal{F}(S^1_+, X) \cong \Omega(K_p^\wedge)_*X \rtimes (K_p^\wedge)_*X$.

**The next four talks were texed by Markus Land. Thanks, Markus!**

# 9 $K(1)$-local elliptic spectra – Lennart Meier

Goal: Define $Tmf_{K(1)}$ and $\mathcal{O}^{\mathrm{top}}_{K(1)}$ on $\mathcal{M}^{\mathrm{ord}}_{ell}$.

Here $Tmf_{K(1)}$ denotes what will be $K(1)$-local $Tmf$, $\mathcal{M}^{\mathrm{ord}}_{ell}$ denotes the ordinary locus of the moduli stack of generalized elliptic curves and $\mathcal{O}^{\mathrm{top}}_{K(1)}$ will be a sheaf of $E_\infty$-ring spectra on this stack. Note that when we write $\mathcal{M}^{\mathrm{ord}}$ there is always a prime $p$ implicit, most of the time it will be odd.

Recall the following from previous talks.

- $\mathcal{M}^{\mathrm{ord}}(p)$ is formally affine, i.e., $\mathcal{M}^{\mathrm{ord}}(p) \simeq \mathrm{Spf}(V_1)$

where $\mathcal{M}^{\mathrm{ord}}(p)$ is as in the Igusa tower from the previous session.

The first goal now will be to define

$$Tmf(p) = \mathcal{O}^{\mathrm{top}}_{K(1)}(\mathcal{M}^{\mathrm{ord}}_{ell}(p)).$$

The strategy will to to prescribe the $p$-adic $K$-theory $(K_p)_*(Tmf(p))$ as a $\Theta$-algebra and then use Goerss-Hopkins-Miller obstruction theory to realize this algebraic datum topologically.

## 9.1 $p$-adic $K$-theory

Firstly let us define $p$-adic $K$-theory

**Definition 22.** Let $X$ be a spectrum. Then we define

$$(K_p)_*(X) = \pi_*(L_{K(1)}(K \wedge X)) = \lim_k K_*(X \wedge \mathbb{S}(p^k)).$$

Here $K$ is any ordinary periodic $K$-theory spectrum, the limit is an inverse limit, $\mathbb{S}(p^k)$ is a mod-$p^k$ Moore spectrum and $L_{K(1)}(-)$ is a localization functor with respect to the first Morava $K$-theory spectrum $K(1)$.

**Remark 15.** If $X$ is such that $K_*(X)$ is torsionfree, then the $p$-adic $K$-theory of $X$ is just the $p$-completed $K$-theory of $X$, i.e.,

$$(K_p)_*(X) \cong \widehat{(K_*(X))}_p.$$

If $X$ is a commutative ring spectrum (or $E_\infty$-ring spectrum), then $p$-adic $K$-theory allows Adams operations $\psi^k$ for all $k \in (\widehat{\mathbb{Z}_p})^\times$, the units in the $p$-adic integers. Moreover there is an operation $\Theta : (K_p)_*(X) \to (K_p)_*(X)$ which commutes with all $\psi^k$. All of this together shows that $(K_p)_*(X)$ is a $\Theta$-algebra.

## 9.2 What does $p$-adic $K$-theory have to do with the Igusa tower

Recall: If $R$ is a $p$-complete ring then

- $\mathcal{M}^{\mathrm{ord}}_{ell}(p^k)(spf(R))$ = groupoid of generalized elliptic curves $C$ over $R$ together with an isomorphism $\eta : \widehat{\mathbb{G}}_m[p^k] \to \widehat{C}[p^k]$.
- $\mathcal{M}^{\mathrm{ord}}_{ell}(p^\infty)(spf(R))$ = groupoid of generalized elliptic curves $C$ over $R$ together with an isomorphism $\eta : \widehat{\mathbb{G}}_m \to \widehat{C}$

**Proposition 3.** *Let $Spf(R) \to \mathcal{M}^{\mathrm{ord}}$ be an étale map, (in particular flat). This gives (by talk 5) an associated Landweber exact spectrum $E$. Consider the pullback diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{M}^{\mathrm{ord}}(p^\infty) \arrow[d] \\ Spf(R) \arrow[r] & \mathcal{M}^{\mathrm{ord}} \end{tikzcd}$$

*There is a $(\mathbb{Z}_p)^\times$ action on $\mathcal{M}^{\mathrm{odd}}(p^\infty)$ which induces also such an action on $X$. The claim is that there is a $(\mathbb{Z}_p)^\times$-equivariant equivalence $X \simeq Spf((K_p)_*(E))$.*

*Proof.* We have (by Landweber exactness) that

$$\begin{aligned} K_0(E) &\cong K_0 \otimes_{MP_0} MP_0(E) \cong K_0 \otimes_{MP_0} E_0(MP) \\ &\cong K_0 \otimes_{MP_0} MP_0MP \otimes_{MP_0} R \end{aligned}$$

which is claimed to be torsionfree. (Note that at least $R$ is torsionfree, as $\mathcal{M}^{\mathrm{odd}}$ is flat over $\mathbb{Z}_p$ and hence by assumption so is $R$). Here $MP$ denotes the periodic complex bordism spectrum. This implies that:

$$(K_p)_0(E) = (K_0 \otimes_{MP_0} MP_0MP \otimes_{MP_0} R)_p.$$

Now

$$\mathrm{Spec}(MP_0MP) = \mathrm{Spec}MP_0 \times_{\mathcal{M}_{FG}} \mathrm{Spec}MP_0$$

which using the previous implies

$$\begin{aligned} Spf((K_p)_0(E)) &\cong (\mathrm{Spec}K_0 \times_{\mathrm{Spec}MP_0} \mathrm{Spec}MP_0MP \times_{\mathrm{Spec}MP_0} \mathrm{Spec}(R))_p \\ &\cong (\mathrm{Spec}K_0 \times_{\mathrm{Spec}MP_0} (\mathrm{Spec}MP_0 \times_{\mathcal{M}_{FG}} \mathrm{Spec}MP_0) \times_{\mathrm{Spec}MP_0} \mathrm{Spec}(R))_p \\ &\cong (\mathrm{Spec}K_0 \times_{\mathcal{M}_{FG}} \mathrm{Spec}R)_p \\ &\cong Spf((K_p)_0) \times_{\mathcal{M}_{FG}} Spf(R). \end{aligned}$$

Recall that $\mathcal{M}_{FG}$ is the moduli stack of formal groups. Now we can consider the diagram

$$\begin{tikzcd} Spf((K_p)_0(E) \arrow[r] \arrow[d] & \mathcal{M}^{\mathrm{ord}}(p^\infty) \arrow[r] \arrow[d] & Spf((K_p)_0) \arrow[d] \\ Spf(R) \arrow[r] & \mathcal{M}^{\mathrm{ord}} \arrow[r] & \mathcal{M}_{FG} \end{tikzcd}$$

The previous calculation shows that the big square is a pullback diagram. Now in order to show that the left square is a pullback (which is the statement of the proposition) it suffices to show that the right square is one.

The map $Spf((K_p)_0) \to \mathcal{M}_{FG}$ is the map which classifies the multiplicative formal group law. Hence the right square is a pullback, because by definition $\mathcal{M}^{\mathrm{odd}}(p^\infty)$ classifies elliptic curves $C$ (as $\mathcal{M}^{\mathrm{ord}}$) together with an isomorphism $\eta : \widehat{\mathbb{G}}_m \to \widehat{C}$ of formal groups, which is precisely what this diagram encodes. $\square$

## 9.3 What should $\Theta$ be?

Recall that one has the relation

$$\psi^p(x) = x^p + p \cdot \Theta(x).$$

This can (and in our situation will) define $\Theta$. Hence we need only to explain the operation $\psi^p$. For this let again $Spf(R) \to \mathcal{M}^{\mathrm{odd}}$ be an étale map. Define $W$ by the pullback

$$\begin{tikzcd} Spf(W) \arrow[r, "\acute{e}t"] \arrow[d] & \mathcal{M}^{\mathrm{ord}}(p^\infty) \arrow[d] \\ Spf(R) \arrow[r, "\acute{e}t"] & \mathcal{M}^{\mathrm{ord}}. \end{tikzcd}$$

Having this we can consider the diagram

$$\begin{tikzcd} \mathrm{Spec}(W/p) \arrow[r, "\mathrm{Frobenius}"] \arrow[d] & \mathrm{Spec}(W/p) \arrow[r] & Spf(W) \arrow[d, "\acute{e}t"] \\ Spf(W) \arrow[r] \arrow[urr, dashed, "\exists!"] & \mathcal{M}^{\mathrm{ord}}(p^\infty) \arrow[r, "\psi^p"] & \mathcal{M}^{\mathrm{ord}}(p^\infty) \end{tikzcd}$$

which admits a unique lift $Spf(W) \to Spf(W)$ which we again call $\psi^p$. The existence of the lift is just a general argument for commutative diagrams with an étale map as in our situation.

Hence this defines the operation $\psi^p : W \to W$, and since all rings in question are torsionfree, this defines the operation $\Theta : W \to W$ by the formula from before.

**Definition 23.** Let $E$ be a $K(1)$-local commutative ring spectrum whose underlying spectrum is a Landweber exact spectrum associated to an étale map $Spf(R) \to \mathcal{M}^{\mathrm{ord}}$. Then $E$ is called $\Theta$-compatible if

$$(K_p)_0(E) \cong W$$

as $\Theta$-algebras.

## 9.4 André-Quillen Cohomology of $\Theta$-algebras

**Theorem 20.** *There are model structures on simplicial commutative rings (respectively simplicial $\Theta$-algebras) such that a map is*

- *a weak equivalence or a fibration if the underlying map of simplicial sets is.*

**Remark 16.** It is a fact that the forgetful functor $\mathcal{U} : s\Theta\text{-Alg} \to s\text{CRings}$ between simplicial $\Theta$-algebras and simplicial commutative rings preserves cofibrations.

Let $B \to A$ be a map of commutative rings or $\Theta$-algebras. Then there exists a factorization

$$cB \rightarrowtail P \xrightarrow{\sim} \twoheadrightarrow cA$$

in simplicial objects. Here $cA$ and $cB$ denote the constant simplicial objects.

**Definition 24.** Let $\mathbb{L}(A/B) = \Omega_{P/B} \otimes_P cA$ where $\Omega_{P/B}$ is defined degreewise. We call this the cotangent complex.

We have the following observation:

$$\mathrm{Hom}_A(A \otimes_{P_n} \Omega_{P_n/B} \otimes C) \xrightarrow{\cong} \mathrm{Hom}_{P_n}(\Omega_{P_n/B}, C) \xrightarrow{\cong} \mathrm{Der}_B(P_n, C)$$

which implies that

$$\mathrm{Hom}_{cA}(\mathbb{L}(A/B), C) \cong \mathrm{Der}_{cB}(P, C).$$

For $\Theta$-algebras the definition looks formally just the same, we just need to note that there is a $\Theta$-algebra structure on $\Omega_{P/B}$ and on derivations to get a $\Theta$-algebra $\mathbb{L}(A/B)$.

**Definition 25.** Let $M$ be an $A$-module. We define André-Quillen cohomology by

$$H^q_{\mathrm{Alg}_B}(A; M) = H^q(\mathrm{Hom}_A(\mathbb{L}(A/B), M)$$

where $\mathrm{Alg}_B$ denotes the category of algebras over $B$ and we view the simplicial $A$-module $\mathrm{Hom}_A(\mathbb{L}(A/B), M)$ as a chain complex in the usual fashion. For $\Theta$-algebras just use the same formal symbols to obtain a cohomology theory for $\Theta$-algebras over $B$.

**Remark 17.** If we denote the simplicial ring $\mathbb{L}(A/B)$ with a $\Theta$-algebra structure by $\mathbb{L}^{\Theta}(A/B)$ then we have that $\mathcal{U}(\mathbb{L}^{\Theta}(A/B) = \mathbb{L}(A/B)$.

To prove this use that fact that if we factorize the morphism $B \to A$ in simplicial $\Theta$-algebras as we did in the beginning of this section, then applying $\mathcal{U}$ to this diagram yields a factorization in simplicial commutative rings, since $\mathcal{U}$ preserves cofibrations by a previous remark. Moreover by the definition of weak equivalences and fibrations it is clear that $\mathcal{U}$ preserves them as well. (The forgetful functor from simplicial $\Theta$-algebras to simplicial sets of course factors over simplicial commutative rings).

## 9.5 Cotangent complex and smoothness

**Definition 26.** For general algebraic geometry see Hartshorne's book. We need

- étale maps $B \to A$ for $A$ and $B$ commutative rings (morally local homeomorphisms),
- a morphism $f : B \to A$ is called smooth if for every prime ideal $\mathfrak{p} \subset A$ there is a factorization

$$B_{f^{-1}(\mathfrak{p})} \longrightarrow B[x_1, \dots, x_n]_{f^{-1}(\mathfrak{p})} \xrightarrow{\acute{e}t} A_{\mathfrak{p}} .$$

Morally again smoothness says something like $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$ is a submersion.

**Theorem 21** (Quillen)**.** *The theorem has two parts:*

*(i) If $B \to A$ is smooth, then $\mathbb{L}(A/B) \simeq c\Omega_{A/B}$ and $\Omega_{A/B}$ is a projective $A$-module.*

*(ii) if $B \to A$ is étale, then $\mathbb{L}(A/B) \simeq *$.*

**Definition 27.** A morphism $A_1 \to A$ is called ind-étale if it may be factored as a countable composite

$$A_1 \xrightarrow{\acute{e}t} A_2 \xrightarrow{\acute{e}t} A_3 \xrightarrow{\acute{e}t} \dots \longrightarrow A$$

and such that $A \cong \mathrm{colim}(A_i)$.

**Example 26.** Recall that the maps from the Igusa tower

$$\mathcal{M}^{\mathrm{ord}}(p^k) \to \mathcal{M}^{\mathrm{ord}}(p^{k-1})$$

is étale. Writing as in the last talk $\mathcal{M}^{\mathrm{ord}}(p^k) \cong Spf(V_k)$ we get that the map $V_{k-1} \to V_k$ is étale and hence the map $V_1 \to V_\infty$ is ind-étale.

A nice conlusion of Quillen's theorem is the following

**Corollary 3.** *If $B \to A$ is ind-étale, then $\mathbb{L}(A/B) \simeq *$.*

*Proof.* Use the fact that colimits preserve acyclic fibrations in simplicial rings. □

**Lemma 3.** *Let $W \to A$ be a morphism of $\mathbb{Z}/2\mathbb{Z}$-graded $\Theta$-algebras, both of which have trivial odd part, i.e., the grading operator is the identity. Assume that the morphism of underlying commutative rings is étale. Then for all $s \geq 0$ and all $A$-modules $M$ we have that*

$$H^s_{\mathrm{Alg}^\Theta_W}(A, M) = 0.$$

*Proof.* For any chain complex $C$ of projective objects in some category $\mathcal{C}$ there is a spectral sequence for every object $M \in \mathcal{C}$

$$\mathrm{Ext}^s(H_t(C), M) \Longrightarrow H^{s+t}(\mathrm{Hom}(C, M)).$$

Now $\mathbb{L}^\Theta(A/W)$ can be checked to consist of projective modules. Hence we have

$$\mathrm{Ext}^s_{\mathrm{Mod}^\Theta_A}(H_t(\mathbb{L}(A/W), M) \Longrightarrow H^{s+t}_{\mathrm{Alg}^\Theta_W}(A, M).$$

Thus by the Quillen's theorem the spectral sequence collapses at the $E_2$-term, where all modules are zero. □

**Proposition 4.** *Let $A/(K_p)$be a $\mathbb{Z}/2\mathbb{Z}$-graded torsionfree $\Theta$-algebra, $M$ a torsionfree $\Theta$-module. Denote by $A^0$ the fixed points $A^{(\mathbb{Z}_p)^\times}$, and by $\overline{A}$, $\overline{A^0}$ and $\overline{M}$ the mod-$p$ reductions $A/p$, $A^0/p$ and $M/p$. Assume*

*(i) $A_1 = M_1 = 0$,*

*(ii)* $\overline{A}_0^0$ *is smooth over* $\mathbb{F}_p$,

*(iii)* $\overline{A}_0$ *is ind-étale over* $\overline{A}_0^0$,

*(iv) the continuous group cohomology* $H^s_c(\mathbb{Z}_p^\times, \overline{M}_0) = 0$ *vanishes for* $s > 0$.

*Then*

$$H^s_{\mathrm{Alg}_\Theta}(A, M[t]) = 0 \textit{ for all } s > 0$$

*or* $t$ *is odd.*

**Remark 18.** Later this will be $A = V_\infty$ and $A^0 = V_1$.

*Proof.* Let us at least sketch the argument. By filtering by powers of $p$ we can reduce to the mod-$p$ case, i.e. we want to show

$$H^s_{\mathrm{Alg}^\Theta_{\mathbb{F}_p}}(\overline{A}, \overline{M}[t]) = 0.$$

As in the las lemma we have the spectral sequence

$$\mathrm{Ext}^s_{\mathrm{Mod}^\Theta_{\overline{A}_0}}(H_t(\mathbb{L}(\overline{A}_0/\mathbb{F}_p)), M[t]) \Longrightarrow H^{s+t}_{\mathrm{Alg}^\Theta_{\mathbb{F}_p}}(\overline{A}, \overline{M}[t]).$$

But we have a cofiber sequence in the derived category of $\overline{A}_0$

$$\begin{tikzcd} \mathbb{L}(\overline{A}_0^0/\mathbb{F}_p) \otimes_{\overline{A}_0^0} \overline{A}_0 \arrow[r] \arrow[d, "\simeq"'] & \mathbb{L}(\overline{A}_0/\mathbb{F}_p) \arrow[r] & \mathbb{L}(\overline{A}_0/\overline{A}_0^0) \arrow[d, "\simeq"] \\ c\Omega_{\overline{A}_0^0/\mathbb{F}_p} \otimes_{\overline{A}_0^0} \overline{A}_0 & & \star \end{tikzcd}$$

The right vertical arrow is an equivalence since by assumption $\overline{A}_0$ is ind-étale over $\overline{A}_0^0$ and the first arrow is an equivalence by smoothness and again Quillen's theorem. Hence we get a conclusion for the $E_2$-page of our spectral sequence

$$\mathrm{Ext}^s_{\mathrm{Mod}^\Theta_{\overline{A}_0}}(H_t(\mathbb{L}(\overline{A}_0/\mathbb{F}_p)), M[t]) \cong \mathrm{Ext}^s_{\mathrm{Alg}^\Theta_{\overline{A}_0^0}}(c\Omega_{\overline{A}_0^0/\mathbb{F}_p} \otimes_{\overline{A}_0^0} \overline{A}_0, M[t])$$

and by flatness we can continue by

$$\mathrm{Ext}^s_{\mathrm{Alg}^\Theta_{\overline{A}_0^0}}(c\Omega_{\overline{A}_0^0/\mathbb{F}_p} \otimes_{\overline{A}_0^0} \overline{A}_0, M[t]) \cong \mathrm{Ext}^s_{\mathrm{Alg}^\Theta_{\overline{A}_0^0}}(c\Omega_{\overline{A}_0^0/\mathbb{F}_p}, M[t]).$$

Now the $\mathbb{Z}_p^\times$-action on $c\Omega_{\overline{A}_0^0/\mathbb{F}_p}$ is trivial, hence

$$\mathrm{Hom}_{\mathrm{Mod}^\Theta_{\overline{A}_0^0}}(c\Omega_{\overline{A}_0^0/\mathbb{F}_p}, M[t]) \cong \mathrm{Hom}_{\overline{A}_0^0[\Theta]}(c\Omega_{\overline{A}_0^0/\mathbb{F}_p}, (M[t])^{\mathbb{Z}_p^\times}).$$

Now consider the following composite of functors

$$\mathrm{Mod}^\Theta_{\overline{A}_0^0} \xrightarrow{\mathbb{Z}_p^\times} \mathrm{Mod}_{\overline{A}_0^0[\Theta]} \xrightarrow{\mathrm{Hom}(c\Omega, -)} \mathrm{Ab}\ .$$

Applying the Grothendieck spectral sequence to this composite gives us

$$\begin{tikzcd} \mathrm{Ext}^s_{\overline{A}_0^0[\Theta]}(\Omega_{\overline{A}_0^0/\mathbb{F}_p}, H^t_c(\mathbb{Z}_p^\times, \overline{M}[t]) \arrow[r, Rightarrow] \arrow[d, "\cong"'] & \mathrm{Ext}^{s+t}_{\mathrm{Mod}^\Theta_{\overline{A}_0^0}}(\Omega_{\overline{A}_0^0/\mathbb{F}_p}, \overline{M}[t]) \\ \mathrm{Ext}^s_{\overline{A}_0^0[\Theta]}(\Omega_{\overline{A}_0^0/\mathbb{F}_p}, (\overline{M}[t])^{\mathbb{Z}_p^\times}) & \end{tikzcd}$$

But $\Omega_{\overline{A}_0^0/\mathbb{F}_p}$ is a projective $\overline{A}_0^0$-module, hence has a length-1-resolution over the polynomial ring $\overline{A}_0^0[\Theta]$. $\square$

# 10 Construction of $\mathcal{O}^{top}_{K(1)}$ – Justin Noel

The Goal of this talk is to sketch ideas leading to the following

**Theorem 22.** *There exists a sheaf $\mathcal{O}^{\text{top}}$ of $E_\infty$-ring spectra on the étale site over the compactified moduli stack of generalized elliptic curves $\overline{\mathcal{M}}_{ell}$ such that for any étale map $C : \text{Spec}(R) \to \overline{\mathcal{M}}_{ell}$ we have the following properties*

*(i)* $\pi_0 \mathcal{O}^{\text{top}}(\text{Spec}(R)) \cong R$,

*(ii)* $E = \mathcal{O}^{\text{top}}(\text{Spec}(R))$ *is weakly even periodic, and*

*(iii)* $\widehat{\mathbb{G}}_E \cong \widehat{C}$.

*Here $\widehat{\mathbb{G}}_E$ denotes the formal group associated to the weakly even periodic spectrum $E$.*

**Remark 19.** From this theorem it will follow that the homotopy groups of $E$ may be calculated to be

$$\pi_{2t}(E) \cong \Gamma(\omega^{\otimes t}; \text{Spec}(R)).$$

The idea now is to first define a sheaf $\mathcal{O}^{\text{alg}}_*$ of graded commutative rings by

- $\mathcal{O}^{\text{alg}}_{2t} = \omega^{\otimes t}$,
- $\mathcal{O}^{\text{alg}}_{2t-1} = 0$.

Then we want to

(i) Construct the associated pre sheaf on étale affine covers and sheafify,

(ii) Decompose the stack arithmetically, and

(iii) Decompose the stack chromatically (by height).

The last two steps mean that we analyze the following pullback diagrams .

$$\begin{tikzcd}
(\coprod_p \mathcal{M}_{ell,p})_{\mathbb{Q}} \arrow[r] \arrow[d] & \coprod_p \mathcal{M}_{ell,p} \arrow[d] \\
\overline{\mathcal{M}}_{ell,\mathbb{Q}} \arrow[r] & \overline{\mathcal{M}}_{ell}
\end{tikzcd}$$

for the arithmetic decomposition and for ever $p$ individually

$$\begin{tikzcd}
(\mathcal{M}^{\text{ss}}_{ell,p})^{\text{ord}} \arrow[r] \arrow[d] & \mathcal{M}^{\text{ss}}_{ell,p} \arrow[d] \\
\mathcal{M}^{\text{ord}}_{ell,p} \arrow[r] & \mathcal{M}_{ell,p}
\end{tikzcd}$$

being the chromatic decomposition. The idea now is that we have to construct this sheaf over all parts of these pullback diagrams and then make sure we can glue the sheafs together.

## 10.1 The sheaf over $\overline{\mathcal{M}}_{ell,\mathbb{Q}}$

This is the easy part, because over $\overline{\mathcal{M}}_{ell,\mathbb{Q}}$ the sheaf $\mathcal{O}_{\mathbb{Q}}^{\text{alg}}$ is a sheaf of graded commutative $\mathbb{Q}$-algebras. Hence we can lift this to spectra easily by defining

$$\mathcal{O}_{\mathbb{Q}}^{\text{top}} := H(\mathcal{O}_{\mathbb{Q}}^{\text{alg}})$$

where $H(-)$ is the generalized Eilenberg-McLane functor.

We need to check (in order for this sheaf to have all properties of the theorem) that $E = \mathcal{O}_{\mathbb{Q}}^{\text{top}}(\text{Spec}(R))$ has the correct formal group.

But we have $\mathbb{G}_E \cong \mathbb{G}_a$ over $R$ because any formal group law over a $\mathbb{Q}$-algebra (such as $R$) is isomorphic to the additive formal group law via the logarithm and hence

$$\widehat{C} \cong \widehat{\mathbb{G}}_a \cong \widehat{\mathbb{G}}_E$$

as we need.

Furthermore $\mathcal{O}_{ss,p}^{\text{top}}$ over $\mathcal{M}_{ell,p}^{ss}$ is constructed using the GHM-theorem. Recall for this that the super singularity implies height 2.

Also to construct $\mathcal{O}_{K(1)}^{\text{top}}$ requires more work (over $\mathcal{M}_{ell,p}^{\text{ord}}$).

## Further reductions to construct $\mathcal{O}$ on $\mathcal{M}$

Here by $\mathcal{O}$ we just mean a sheaf of $E_\infty$ ring spectra lifting an algebraic sheaf $\mathcal{O}^{\text{alt}}$ over some moduli stack $\mathcal{M}$.

The idea is to first construct $\mathcal{O}(\mathcal{M}) := \Gamma(\mathcal{M}, \mathcal{O})$ and then construct $\mathcal{O}(U)$ in the category of $\mathcal{O}(\mathcal{M})$ algebras, as indicated by the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & \mathcal{M} \\ V \arrow[ur] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathcal{O}(\mathcal{M}) \arrow[r] \arrow[dr] & \mathcal{O}(U) \arrow[d] \\ & \mathcal{O}(V) \end{tikzcd}$$

We do this because there are "fewer" $\mathcal{O}(\mathcal{M})$ algebras and maps between them (in other words, the obstructions to existence will vanish).

Let us consider the example $\mathcal{M}_{ell}[\Delta^{-1}] \hookrightarrow \overline{\mathcal{M}}_{ell}$ the inclusion of the support of the modular form $\Delta$ in the compactified moduli stack. We construct $\mathcal{O}$ on

$$(\mathcal{M}_{ell}^1[\Delta^{-1}])_{\mathbb{Q}} \cong \text{Spec}(\mathbb{Q}[c_4 c_6, \Delta^{-1}])$$

and set

$$\mathcal{O}((\mathcal{M}_{ell}^1[\Delta^{-1}])_{\mathbb{Q}}) \cong H(\mathbb{Q}[c_4, c_6, \Delta^{-1}][\lambda^{\pm 1}]).$$

Recall here that $\mathbb{G}_m \to \mathcal{M}_{ell}^1 \to \mathcal{M}_{ell}$ is a torsor and $\mathcal{M}_{ell}^1$ classifies elliptic curves with specified non-zero tangent vector, giving a trivialization of $\omega^{\otimes t}$. This gives

$$\begin{tikzcd} \mathcal{M}_{ell}^1[\Delta^{-1}]_{\mathbb{Q}} \arrow[r] \arrow[d, "\cong"'] & \mathcal{M}_{ell}[\Delta^{-1}]_{\mathbb{Q}} \\ \text{Spec}(\mathbb{Q}[c_4, c_6, \Delta^{-1}] & \end{tikzcd}$$

Now consider the pullback diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \mathcal{M}^1_{ell}[\Delta^{-1}]_{\mathbb{Q}} \arrow[d] \\ \mathrm{Spec}(R) \arrow[r] & \mathcal{M}_{ell}[\Delta^{-1}]_{\mathbb{Q}} \end{tikzcd}$$

if the pullback $P$ is affine the it follows that $\mathcal{O}(P) \cong \mathcal{O}^1(P)$.

Now in order to construct $\mathcal{O}(\mathrm{Spec}(R))$ we need to construct an $H(\mathbb{Q}[c_4, c_6, \Delta^{-1}][\lambda^{\pm 1}])$ algebra $R'$ such that $\mathcal{O}^{\mathrm{alg}}(\mathrm{Spec}(R)) = \pi_*(R')$ is correct. Obstruction to the existence of such an algebra lie in

$$H^s_{AQ}(\mathcal{O}^{\mathrm{alg}}(\mathrm{Spec}(R), \Sigma^t \mathcal{O}^{\mathrm{alg}}(\mathrm{Spec}(R)) \text{ for } s \geq 3$$

by the GHM-theorem. Here André-Quillen Cohomology is for commutative $\mathbb{Q}[c_4, \dots] = \mathbb{Q}[c_4, c_6, \Delta^{-1}, \lambda^{\pm 1}]$ algebras. Now if we assume the map $\mathbb{Q}[c_4, \dots] \to \mathcal{O}^{\mathrm{alg}}(\mathrm{Spec}(R))$ is étale all of these cohomology groups vanish.

**Corollary 4.** *We have*

*(i) There is a contractible space of realizations,*

*(ii) there is a contractible space of maps realizing an algebraic map $\mathcal{O}^{\mathrm{alg}}(\mathrm{Spec}(R)) \to \mathcal{O}^{\mathrm{alg}}(\mathrm{Spec}(T))$ over $\mathcal{O}^{\mathrm{alg}}(\mathcal{M}_{ell}[\Delta^{-1}]_{\mathbb{Q}})$.*

Hence we can rigidify $\mathcal{O}$ to a pre sheaf on $\overline{\mathcal{M}}^1_{ell}[\Delta^{-1}]$.

But now for the $K(1)$ local part knowing $\mathcal{O}^{\mathrm{alg}}_{\mathrm{ord}}$ is not enough, the obstruction theory is simply to complicated. Hene we will not only demand that $\pi_*(\mathcal{O}^{\mathrm{top}}_{\mathrm{ord}} \cong \mathcal{O}^{\mathrm{alg}}_{\mathrm{ord}}$ (locally) but we will also specify the $p$-adic $K$-theory $(K_p)_* \mathcal{O}^{\mathrm{top}}_{\mathrm{ord}}$ in terms of algebraic data (as $\Theta$-algebra).

For this we will construct a sheaf over $\mathcal{M}^{\mathrm{ord}}_{\mathrm{ell}}(p^i)$ (generalized elliptic curves with ordinary reduction and a specified isomorphism $\widehat{\mathbb{G}}_m[p^i] \to \widehat{C}[p^i]$) first.

For this consider the pullback diagram

$$\begin{tikzcd} Spf(W) \arrow[r, "(\mathbb{Z}/p^i)^\times - torsor"] \arrow[d] & \mathcal{M}^{\mathrm{ord}}_{ell}(p^\infty) \arrow[d, "(\mathbb{Z}_p)^\times - torsor"] \\ \mathcal{M}^{\mathrm{ord}}_{ell}(p^i) \arrow[r, "(\mathbb{Z}/p^i)^\times - torsor"'] & \mathcal{M}^{\mathrm{ord}}_{ell} \end{tikzcd}$$

Recall from the Igusa tower that all maps in this diagram are étale (this is why the top horizontal map is also a torsor) and $W$ is a $\Theta$-algebra (morally $K_p \wedge E$).

Now we have to a find out what are the obstructions to the existence of an $E_\infty$-ring spectrum $tmf(p^i)$ such that

(i) $\pi_*(tmf(p^i)) \cong \mathcal{O}^{\mathrm{alg}}(Spf(V_i))$,

(ii) $(K_p)_*(tmf(p^i)) \cong (K_p)_* \otimes W$ as $\Theta$-algebras.

Fact: they lie in

$$H^*_{AQ}((K_p)_* \otimes W, \Sigma((K_p)_* \otimes W)),$$

and the obstruction for existence and homotopical uniqueness vanish (show $W$ has all properties to apply the theorems of the last talk).

Now suppose we have an étale map $Spf(R) \to \mathcal{M}_{ell}^{\text{ord}}(p^i)$, then the obstructions to realizing $\mathcal{O}^{\text{alg}}(Spf(R))$ as a $tmf(p^i)$-algebra with $(K_p)_*(-)$ a specific $\Theta$-algebra under $(K_p)_*(tmf(p^i))$ will vanish entirely (due to the étaleness of the map). Hence again we get a contractible space of realizations and possible maps between them. Then as before we rigidify this homotopy coherent diagram to a strict one.

To get

$$tmf_{K(1)} = \mathcal{O}^{\text{top}}(\mathcal{M}_{ell}^{\text{ord}})$$

we will set

$$tmf_{K(1)} \cong (tmf(p^i))^{h(\mathbb{Z}/p^i)^\times}$$

(i.e. the homotopy fixed points of the action). If $p = 2$ then $i \geq 4$ is needed, but for odd primes $i$ can be arbitrary.

This works for odd primes where the obstructions to realizing the $(\mathbb{Z}/p)^\times$-action vanishes.

A general idea for this is that one has

$$BG \to BhAut_{E_\infty}(R)$$

which one has to loop down and rigidify to get $G \to hAut_{E_\infty}(R)$, i.e. a $G$-action. Now do this for $G = (\mathbb{Z}/p)^\times$.

# 11 The descent spectral sequence I – Gisa Schäfer

Before coming to the main contents of the talk, let us recall the following notions from earlier talks.

(i) Given a Grothendieck site $\mathcal{C}$ when writing a collection $\{U_i \to U\}_{i\in I}$ we will always mean a covering in the site.

(ii) A sheaf on a site $\mathcal{C}$ with values in a category $\mathcal{D}$ is a functor $F : \mathcal{C}^{op} \to \mathcal{D}$ s.t. for all coverings $\{U_i \to U\}_{i\in I}$ we have

$$F(U) \xrightarrow{\cong} \mathrm{Eq}\Big(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{i,j})\Big).$$

(iii) Here and elsewhere we denote by $U_{i,j} = U_i \times_U U_j$ the pullback over $U$, similarly for more than 2 indices.

(iv) It is a fact that the above equalizer is equal to the following limit

$$\lim_\Delta \Big(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{i,j}) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \prod_{i,j,k} F(U_{i,j,k}) \dots \Big)$$

where by $\lim_\Delta$ we mean the limit of the simplicial object indicated in the brackets.

For this talk we will change the notion of a sheaf as follows. Suppose $\mathcal{D}$ is a category in which we have notions of homotopy limits and weak equivalences (eg. spectra or suitable model categories), we say that a sheaf is a functor $F : \mathcal{C}^{op} \to \mathcal{D}$ such that the induced map

$$F(U) \xrightarrow{\simeq} \mathrm{holim}_\Delta \Big(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{i,j}) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \dots \Big)$$

is a weak equivalence in $\mathcal{D}$.

## 11.1 Description of (co)limits

Suppose now that $\mathcal{C}$ is a simplicial model category, $I$ a small category and $X : I \to \mathcal{C}$ an $I$-diagram in $\mathcal{C}$. Being simplicial means that $\mathcal{C}$ is in particular tensored and cotensored over sSet. So by usings ends and coends we can make the diagram category $\mathcal{C}^I$ cotensored and tonsured over $\text{sSet}^I$ $I$-diagrams in simplicial sets.

(i) let $A$ be an $I^{op}$-diagram in sSet, we can define the tensor of $X$ and $A$ by

$$X \otimes_I A = \int^I X(i) \otimes A(i) = \operatorname{colim}\Big( \coprod_{i \to j} X(i) \otimes A(j) \rightrightarrows \coprod_i X(i) \otimes A(i) \Big),$$

(ii) let $A$ be an $I$-diagram in sSet, then we can define the cotensor of $A$ and $X$ by

$$\hom^I(A, X) = \int_I X(i)^{A(i)} = \lim \Big( \prod_i X(i)^{A(i)} \rightrightarrows \prod_{i \to j} X(i)^{A(j)} \Big).$$

**Definition 28.** Let $*_I$ denote both the trivial $I$-and $I^{op}$-diagram in sSet that sends every object to the terminal object $* \in$ sSet.

**Example 27.** Consider the category $\mathcal{C} = \text{Top}$, then $Y \otimes B = Y \times |B|$ and $Y^B = \text{map}(|B|, Y)$. Here it is a simple calculation using the above explicit description of the tensor and cotensor to verify that

(i) $X \otimes_I *_I \cong \operatorname{colim}_I X$, and

(ii) $\hom^I(*_I, X) \cong \lim_I X$.

Indeed this is true in general:

**Lemma 4.** *Given the assumptions of the beginning of the section we have that*

*(i)* $\operatorname{colim}_I X \cong X \otimes_I *_I$, *and*

*(ii)* $\lim_I X \cong \hom^I(*_I, X)$.

## 11.2 From (co)limits to Homotopy (co)limits

Now let $\mathcal{C}$ be a combinatorial, simplicial model category (any assumptions that guarantee that we have the projective and the injective model structure on $I$-diagrams on $\mathcal{C}$ would suffice). We introduce the notation that $F_{inj}$, $F_{proj}$, $C_{inj}$ and $C_{proj}$ are fibrant and cofibrant replacement functors on $\mathcal{C}^I_{inj}$ and $\mathcal{C}^I_{proj}$.

We want to view the limit as a functor $\lim : \mathcal{C}^I \to \mathcal{C}$ sending $X \mapsto \hom^I(*_I, X)$. The problem with this construction is, that it is not homotopy invariant.

The first solution would be to derive the limit functor:

It is a fact, that $\lim : \mathcal{C}^I_{inj} \to C$ is a right Quillen functor, hence we can build its derived functor

$$\begin{aligned} Ho(C^I_{inj}) &\xrightarrow{\text{Rlim}} Ho(C) \\ X &\longmapsto \hom^I(*_I, F_{inj}(X)) \end{aligned}$$

which is by construction homotopy invariant.

The problem with this approach is, that in general it is complicated to calculate $F_{inj}(X)$, so instead of replacing $X$ we might as well try to replace $*_I$ instead.

Note that $*_I$ is cofibrant in $\mathrm{sSet}^I_{inj}$ and by definition $F_{inj}(X)$ is fibrant in $\mathcal{C}^I_{inj}$ so that the cotensor $\hom^I(*_I, F_{inj}(X))$ has a well-behaved homotopy type.

The idea now is to switch from the injective model structure to the projective model structure and replace $*_I$ cofibrantly in $\mathrm{sSet}^I_{proj}$ instead of replacing $X$ fibrantly in $\mathcal{C}^I_{inj}$.

**Lemma 5.** *If $X$ is objectwise fibrant then $\hom^I(*_I, F_{inj}(X))$ and $\hom^I(C_{proj}(*_I), X)$ are weakly equivalent.*

The advantages in this new approach are, that $C_{proj}(*_I)$ only depends on $I$ and not on $X$ or $\mathcal{C}$. Hence we will make a choice for $C_{proj}(*_I)$ once and for all as follows.

**Lemma 6.** *The nerve $\mathcal{N}(I \downarrow -)$ of the over-categories $I \downarrow i$ is weakly equivalent to $*_I$ and cofibrant in $\mathrm{sSet}^I_{proj}$.*

*Proof.* The category $I \downarrow i$ has a terminal object and is hence contractible. Since the weak equivalences in $\mathrm{sSet}^I_{proj}$ are defined object wise this shows that $\mathcal{N}(I \downarrow -) \simeq *_I$ in $\mathrm{sSet}^I_{proj}$. The fact that $\mathcal{N}(I \downarrow -)$ is cofibrant is proven in Hirschhorn's book. $\square$

**Definition 29.** We define a functor

$$\begin{aligned} \mathcal{C}^I &\xrightarrow{\text{preholim}} \mathcal{C} \\ X &\longmapsto \hom^I(\mathcal{N}(I \downarrow -), X) \end{aligned}$$

Still, this functor is not homotopy invariant if $X$ is not objectwise fibrant. Let us denote an objectwise fibrant functorial replacement $F_{obj}$, and define the homotopy limit by

$$\begin{aligned} \mathcal{C}^I &\xrightarrow{\text{holim}} \mathcal{C} \\ X &\longmapsto \hom^I(\mathcal{N}(I \downarrow -), F_{obj}(X)) \end{aligned}$$

We get the following

**Corollary 5.** $\hom^I(*_I, F_{inj}(X)) \simeq \operatorname{holim}_I X$.

Dually the same holds for colimits. This means that here the idea is to consider $C_{proj}(*_I)$ instead of $C_{proj}(X)$.

**Lemma 7.** *If $X$ is objectwise cofibrant, then*

$$C_{proj}(X) \otimes_I *_I \simeq X \otimes_I C_{proj}(*_I).$$

Again we can choose similarly a cofibrant replacement of $*_I$ to be $\mathcal{N}(I \uparrow -)^{op}$ and can make the following

**Definition 30.** We define the prehocolim by the diagram

$$\begin{aligned} \mathcal{C}^I &\xrightarrow{\text{prehocolim}} \mathcal{C} \\ X &\longmapsto X \otimes_I \mathcal{N}(I \uparrow -)^{op} \end{aligned}$$

and the hocolim as expected by

$$\begin{aligned} \mathcal{C}^I &\xrightarrow{\text{hocolim}} \mathcal{C} \\ X &\longmapsto C_{obj} \otimes_I \mathcal{N}(I \uparrow -)^{op} \end{aligned}$$

Let us calculate one familiar example with these definitions namely let us calculate

$$L = \text{prehocolim}(A \leftarrow C \rightarrow B).$$

Here obviously $I$ is the category

$$\bullet_i \xleftarrow{\alpha} \bullet_j \xrightarrow{\beta} \bullet_k$$

We then get that $\mathcal{N}(I \uparrow j)_0 = \Big\{\alpha, \mathrm{id}, \beta\Big\}$ (the zero simplices are precisely the objects) and that $\mathcal{N}(I \uparrow j)_1 = \Big\{\alpha, \beta\Big\} \cup$ degenerate simplices and all higher simplices are degenerate anyways. This implies that the geometric realization $|\mathcal{N}(I \uparrow j)| \cong [0,1]$ is an interval. It is even easier to compute that $|\mathcal{N}(I \uparrow i)| = * = |\mathcal{N}(I \uparrow k)|$. This in turn implies that

$$L = A \sqcup C \times [0,1] \sqcup B \Big/ \big\{(c,0) \sim f(c), (c,1) \sim g(c)\big\}$$

i.e., $L$ is just the double mapping cylinder.

Similarly if you consider the category $I^{op}$, we get that the preholim of an $I^{op}$-diagram in spaces is just the double mapping path object.

One can ask why we should care about the preho(co)lim instead of the constructed ho(co)lim. One good thing is, that the preho(co)lim is independent of the model structure on the category our diagrams take value in. Moreover the main source (Douglas) even defines sheaves using only the prelim (which he calls holism, what we call holim is the "corrected holim" in Douglas paper).

Recall that the category of orthogonal spectra os tensored and cotensored over pointed topological spaces by the following formulas. Let $X \in Sp^O$, $A \in \mathrm{Top}_*$ and $V$ be some real inner product space. Then we define

(i) $(X \otimes A)(V) = X(V) \wedge A$, and

(ii) $X^A(V) = X(V)^A$.

## 11.3 Sheaf Cohomology

In this section we finally want to define sheaf cohomology over sites. For this recall the following fact.

**Proposition 5.** *Let $\mathcal{D}$ be a site. Denote by $\mathrm{Shv}_{\mathrm{Ab}}(\mathcal{D})$ the category of abelian group valued sheaves on $\mathcal{D}$. Then $\mathrm{Shv}_{\mathrm{Ab}}(\mathcal{D})$ is an abelian category with enough injectives.*

**Definition 31.** Suppose $\mathcal{C}$ is a site, and let $\pi \in \mathrm{Shv}_{\mathrm{Ab}}(\mathcal{C})$ and $X \in \mathcal{C}$. Then we define *sheaf cohomology* of $X$ with coefficients in the sheaf $\pi$ by

$$H^q(X;\pi) = H_q\Big(0 \to I^0 X \to I^1 X \to I^2 X \to \dots\Big)$$

where

$$0 \longrightarrow \pi|_X \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots$$

is an injective resolution of $\pi|_X$ in $\mathrm{Shv}_{\mathrm{Ab}}(\mathcal{C} \downarrow X)$.

**Definition 32.** Let again $\pi \in \mathrm{Shv}_{\mathrm{Ab}}(\mathcal{C})$ and let $\mathcal{U} = \{U_i \to U\}$ be a cover in the site $\mathcal{C}$. We define the *Cech-cohomology* of $U$ with respect to $\mathcal{U}$ and coefficients in $\pi$ by

$$\check{H}^q_{\mathcal{U}}(U;\pi) = H_q\Big(0 \to \prod_i \pi(U_i) \to \prod_{i,j} \pi(U_{i,j}) \to \dots\Big)$$

where the maps in the chain complex on the right hand side are the alternating sum of the restriction maps (recall that this comes from a simplicial abelian group).

We end this talk with a comparison of sheaf and Cech cohomology groups.

**Proposition 6.** *Let $\pi \in \mathrm{Shv}_{\mathrm{Ab}}(\mathcal{C})$ and let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a cover in the site $\mathcal{C}$. Assume that for all $\emptyset \neq J \subset I$ and all $q \geq 1$ we have that $H^q(U_J;\pi) = 0$, then we have*

$$\check{H}^q_{\mathcal{U}}(U;\pi) \cong H^q(U;\pi).$$

# 12 The descent spectral sequence II – Karol Szumilo

This will be the setup for the talk. $\mathcal{M}$ will always be some site, $\mathcal{O}$ will be a sheaf of spectra on $\mathcal{M}$ and $\mathcal{U} = \{U_i \to U\}$ will be a cover in $\mathcal{M}$.

The main problem that will be addressed in this talk is the following.

How can we compute $\pi_*(\mathcal{O}(U))$ from $\pi_*(\mathcal{O}(U_i))$?

And some sense of a solution will be, that there is a spectral sequence

$$E^2_{p,q} = \check{H}^q_{\mathcal{U}}(U;\pi_p(\mathcal{O})) \Longrightarrow \pi_{p-q}(\mathcal{O}(U)).$$

The outline of the construction of the spectral sequence is as follows.

0) Take a cover $\mathcal{U} = \{U_i \to U\}$ in $\mathcal{M}$,

1) form a simplicial object in $\mathcal{M}$ by

$$\mathcal{U}_\bullet = \Big\{ \textstyle\coprod_i U_i \leftleftarrows \coprod_{i,j} U_{i,j} \;\substack{\leftarrow\\\leftarrow\\\leftarrow}\; \coprod_{i,j,k} U_{i,j,k} \;\;\cdots \Big\}$$

2) get a cosimplicial spectrum

$$\mathcal{O}(\mathcal{U}_\bullet) = \Big\{ \mathcal{O}(\textstyle\coprod U_i) \rightrightarrows \mathcal{O}(\coprod U_{i,j}) \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; \cdots \Big\}$$

3) build a tower of spectra

$$\mathrm{Tot}^0\mathcal{O}(\mathcal{U}_\bullet) \xleftarrow{\phi_1} \mathrm{Tot}^1\mathcal{O}(\mathcal{U}_\bullet) \xleftarrow{\phi_2} \mathrm{Tot}^2\mathcal{O}(\mathcal{U}_\bullet) \xleftarrow{\phi_3} \cdots$$

4) get an exact couple formed by long exact sequences of the form

$$\cdots \longleftarrow \pi_{p-q}\mathrm{Tot}^{q-1}\mathcal{O}(\mathcal{U}_\bullet) \xleftarrow{\phi_q} \pi_{p-q}\mathrm{Tot}^q\mathcal{O}(\mathcal{U}_\bullet) \longleftarrow \pi_{p-q}\mathrm{fib}(\phi_q) \longleftarrow \cdots$$

5) construct and understand the descent spectral sequence

## Step 1

Suppose we are given a cover $\mathcal{U} = \{U_i \to U\}_{i\in I}$. Let $\alpha = (i_0, \ldots, i_q) \in I^{q+1}$ be a multi-index of length $q+1$ and define the "intersection" as the limit

$$U_\alpha = \lim \left( \begin{tikzcd} & U_{i_o} \arrow[dl] \\ U & \vdots \arrow[l] \\ & U_{i_q} \arrow[ul] \end{tikzcd} \right)$$

Now assume that $\mathcal{M}$ has coproducts, and consider associating $[q]$ to $\mathcal{U}_q = \coprod_\alpha U_\alpha$ for $\alpha$ as before being a multi-index of length $q+1$. It follows that $\mathcal{U}_\bullet$ is a simlicial object in $\mathcal{M}$.

## Step 2

Of course when applying to a simplicial object in a category $\mathcal{C}$ a contravariant functor $F : \mathcal{C} \to \mathcal{D}$ we get a cosimplicial object in $\mathcal{D}$. Applying this to $\mathcal{C} = \mathcal{M}$ and the simplicial object of Step 1, and the sheaf of spectra $\mathcal{O}$ we get that

$$\mathcal{O}(\mathcal{U}_\bullet) = \left\{ \mathcal{O}(\coprod U_i) \rightrightarrows \mathcal{O}(\coprod U_{i,j}) \overrightarrow{\rightrightarrows} \ \dots \ \right\}$$

is a cosimplicial spectrum.

We want to relate this cosimplicial spectrum to another such object which will occur later in the talk. Namely consider the object

$$\mathcal{O}(\mathcal{U})^\bullet = \left\{ \prod_i \mathcal{O}(U_i) \rightrightarrows \prod_{i,j} \mathcal{O}(U_{i,j}) \overrightarrow{\rightrightarrows} \dots \right\} .$$

Now assume that for any set of objects $\{V_k\}$ the family $\mathcal{V} = \{V_j \to \coprod_k V_k = V\}_j$ is a cover in $\mathcal{M}$. Then we have that

$$\mathcal{O}(\coprod_j V_j) = \mathcal{O}(V) \simeq \operatorname{holim}_\Delta \mathcal{O}(\mathcal{V})^\bullet \simeq \prod_j \mathcal{O}(V_j)$$

where first equivalence is by the fact that $\mathcal{O}$ is a sheaf and the second equivalence is a direct computation.

In particular we get that $\mathcal{O}(\mathcal{U}_\bullet) \simeq \mathcal{O}(\mathcal{U})^\bullet$ are weakly equivalent.

## Digression in Reedy fibrant cosimplicial spectra

Let $\Delta_{\leq n}$ be the full subcategory of $\Delta$ on the objects $[0], \dots, [n]$. We have an adjunction of functors

$$\begin{tikzcd} Sp^\Delta \arrow[r, bend left, "U_n"] & Sp^{\Delta_{\leq n}} \arrow[l, bend left, "R_n"] \end{tikzcd}$$

where $R_n$ denotes the right adjoint to the forgetful functor $U_n$. With this we can define the $n^{th}$ coskeleton of a cosimplicial object $X$ by

$$\operatorname{cosk}^n X = R_n U_n X$$

which receives a canonical map $\eta_X : X \to \operatorname{cosk}^n X$ (the unit of the adjunction). We define the $n^{th}$ matching object of $X$ to be

$$M^n X = (\operatorname{cosk}^{n-1} X)^n \in Sp$$

which receives (through $\eta$) a canonical map $X^n \to M^n X$ of spectra.

For example we have that

(i) $M^0 X = * \leftarrow X^0$,

(ii) $M^1 X = X^0 \leftarrow X^1$, coming from the unique map $[1] \to [0]$,

(iii) $M^2 X = X^1 \times_{X^0} X^1 \leftarrow X^2$, coming from the two maps $[2] \rightrightarrows [1]$.

For the following definition, let us consider some stable model structure on a sufficiently nice category of spectra. Later we will need that fibrant objects are $\Omega$-spectra, so any of the usual stable model structures on orthogonal or symmetric spectra will do the job.

**Definition 33.** A cosimplicial object $X$ is called *Reedy fibrant* if for all $n \geq 0$ the map $X^n \to M^n X$ is a fibration.

## Step 3

Now assume that $\mathcal{O}$ preserves products (takes coproducts to products, recall that $\mathcal{O}$ is a contravariant functor). Assume moreover that it takes values in fibrant spectra.

**Claim 1.** *Then the cosimplicial object $\mathcal{O}(|U_\bullet)$ is Reedy fibrant.*

Propably there are some assumptions missing for this to be true.

Now assume we are given a cosimplicial object $X : \Delta \to Sp$ and consider the standard cosimplicial space $\Delta^\bullet_+ : \Delta \to \mathrm{Top}_*$ which sends $[n] \mapsto \Delta^n_+$, the standard topological $n$-simplex with a disjoint base point.

**Definition 34.** We define the totalization of $X$ by the contensor

$$\mathrm{Tot}X = \hom^\Delta(\Delta^\bullet_+, X),$$

and define

$$\mathrm{Tot}^n X = \mathrm{Tot}\mathrm{cosk}^n X$$

for ever $n \geq 0$.

**Remark 20.** Notice that the functor $U_{n-1}$ defines a map

$$\begin{aligned} Sp^\Delta(R_n U_n X, X) &\to Sp^{\Delta_{\leq n-1}}(U_{n-1} R_n U_n X, U_{n-1} X) \\ &\cong Sp^\Delta(R_n U_n X, R_{n-1} U_{n-1} X) = Sp^\Delta(\mathrm{cosk}^n X, \mathrm{cosk}^{n-1} X) \end{aligned}$$

and this sends $\eta_X : R_n U_n X \to X$ to some map $\mathrm{cosk}^n X \to \mathrm{cosk}^{n-1} X$. Hence we have a tower of totalizations

$$\mathrm{Tot}^0 X \longleftarrow \mathrm{Tot}^1 X \longleftarrow \mathrm{Tot}^2 X \longleftarrow \dots$$

and its inverse limit is isomorphic to $\mathrm{Tot}X$, i.e.,

$$X \cong \lim \mathrm{Tot}^\bullet X.$$

**Claim 2.** *If $X$ is Reedy fibrant then* $\mathrm{Tot}^\bullet X$ *is a tower of vibrations.*

*Proof.* This is just a sketch of a proof. There are Reedy model structures on $Sp^\Delta$ and $Sp^{\mathbb{N}^{op}}$. Here $Sp^{\mathbb{N}^{op}}$ is the category of inverse systems of spectra. In the Reedy model structure on $Sp^{\mathbb{N}^{op}}$ the fibrant objects are precisely the towers of fibrations. Moreover the functor $\mathrm{Tot}^\bullet$ is part of a Quillen adjunction

$$\begin{tikzcd} Sp^\Delta \arrow[r, bend left, "\mathrm{Tot}^\bullet"] & Sp^{\mathbb{N}^{op}} \arrow[l, bend left, "T"] \end{tikzcd}$$

Hence $\mathrm{Tot}^\bullet$ preserves fibrant objects (it is the right Quillen functor). But actually we only need that it preserves fibrant objects (which is a weaker statement then being a right Quillen functor). This is equivalent to the condition that its left adjoint $T$ preserves acyclic Reedy cofibrations, and this indeed can be checked directly. $\square$

Alltogether this implies that

$$\mathrm{Tot}^0\mathcal{O}(\mathcal{U}_\bullet) \longleftarrow \mathrm{Tot}^1\mathcal{O}(\mathcal{U}_\bullet) \longleftarrow \mathrm{Tot}^2\mathcal{O}(\mathcal{U}_\bullet) \longleftarrow \dots$$

is a tower of fibrations.

## Step 4

Now we take the associated long exact sequences in homotopy groups and get

$$\begin{tikzcd}
\cdots & \pi_{p-q-1}\mathrm{fib}(\phi_q) \arrow[l] \arrow[d, equal] & \pi_{p-q}\mathrm{Tot}^{q-1}\mathcal{O}(\mathcal{U}_\bullet) \arrow[l] \arrow[d, equal] & \pi_{p-q}\mathrm{Tot}^{q}\mathcal{O}(\mathcal{U}_\bullet) \arrow[l] \arrow[d, equal] & \pi_{p-q}\mathrm{fib}(\phi_q) \arrow[l] \arrow[d, equal] \\
& E^1_{p-1,q} & A^1_{p-1,q-1} & A^1_{p,q} & E^1_{p,q}
\end{tikzcd}$$

Taking these long exact sequences for all $p$ and $a$ we get an exact double

$$\begin{tikzcd}
A^1_{*,*} \arrow[dr, "{(0,1)}"'] & & A^1_{*,*} \arrow[ll, "{(-1,-1)}"'] \\
& E^1_{*,*} \arrow[ur, "{(0,0)}"'] &
\end{tikzcd}$$

## Step 5

The following theorem is due to Boardman.

**Theorem 23.** *The above exact couple yields a half-plane spectral sequence with entering differentials which converges conditionally to*

$$\lim_i \pi_{p-1}\mathrm{Tot}^i\mathcal{O}(\mathcal{U}_\bullet).$$

In order to "understand" the spectral sequence we want to do the following

(i) check that this spectral sequence converges strongly,

(ii) relate the abutment $\lim_i \pi_{p-q}\mathrm{Tot}^i\mathcal{O}(\mathcal{U}_\bullet)$ with $\pi_{p-q}\mathcal{O}(U)$,

(iii) identify the $E^2_{*,*}$-term.

We identify the $E^2$-term first.

Consider the map $\phi_q : \mathrm{Tot}^q X \to \mathrm{Tot}^{q-1} X$. We have that

$$\mathrm{Tot}^q X = \hom^\Delta(\Delta^\bullet_+, \mathrm{cosk}^q X) \cong \hom^\Delta(\mathrm{sk}^q\Delta^\bullet_+, X)$$

and under this identification the map $\phi_q$ comes from the map $\mathrm{sk}^{q-1}\Delta^\bullet_+ \to \mathrm{sk}^q\Delta^\bullet_+$. Let $C^\bullet_q$ be the cofiber of this map $\mathrm{sk}^{q-1}\Delta^\bullet_+ \to \mathrm{sk}^q\Delta^\bullet_+$. We have that

$$C^m_q = \begin{cases} * & q > m \\ \Delta^q/\partial\Delta^q & q = m \\ \vee S^q & q < m \end{cases}$$

where the last wedge of spheres is indexed over all inclusions $[q] \hookrightarrow [m]$.

Now we have yet another adjunction

$$\begin{tikzcd}
\mathrm{Top}^\Delta_* \arrow[r, bend left, "\mathrm{ev_q}"] & \mathrm{Top}_* \arrow[l, bend left, "\mathrm{Lan}_{[q]}"]
\end{tikzcd}$$

where $\mathrm{Lan}_{[q]}$ is given by Left-Kan extension. It then follows that $C_q^\bullet \cong \mathrm{Lan}_{[q]} S^q$. In particular we have that

$$\mathrm{fib}(\mathrm{Tot}^q X \to \mathrm{Tot}^{q-1} X) = \hom^\Delta(C_q^\bullet, X) \cong \mathrm{Top}_*(S^q, X^q) = \Omega^q X^q.$$

Note that by the symbol $\mathrm{Top}_*(S^q, X^q)$ we again mean the cotensor between a space and spectrum, as considered in Gisa's talk.

Now by the very definition we have

$$E^1_{p,q} = \pi_{p-q}\mathrm{fib}(\phi_q) \cong \pi_{p-q}\Omega^q X^q \cong \pi_p X^q.$$

Hence if we take $X = \mathcal{O}(\mathcal{U}_\bullet)$ we get that

$$E^1_{p,q} = \pi_p(\prod_\alpha \mathcal{O}(U_\alpha)) \cong \prod_\alpha \pi_p(\mathcal{O}(\mathcal{U}_\alpha)).$$

The last equality follows from the fact that we assumed $\mathcal{O}$ to take values in fibrant spectra.

Therefore $E^1_{p,*}$ is the Cech-complex, and hence

$$E^2_{p,q} \cong \check{H}^q_{\mathcal{U}}(U; \pi_p\mathcal{O})$$

the $E^2$ page is given by Cech-cohomology.

Next, let us understand the abutment better. Assume as before that $X \in Sp^\Delta$ is a Reedy fibrant cosimplicial spectrum. Recall from the last talk that

$$\operatorname{holim} X = \hom^\Delta(|\mathcal{N}(\Delta \downarrow -)|, X).$$

For each $m \geq 0$ we have a functor

$$\begin{aligned} \Delta \downarrow [m] &\longrightarrow [m] \\ [k] \xrightarrow{\varphi} [m] &\longmapsto \varphi(k) \end{aligned}$$

where $[m]$ on the right hand side is just viewed as poset. This yields a cosimplicial map $|\mathcal{N}(\Delta \downarrow -)| \to \Delta^\bullet$ which is a weak equivalence between Reedy cofibrant objects. Hence we get

$$\mathrm{Tot} X = \hom^\Delta(\Delta^\bullet_+, X) \simeq \hom^\Delta(|\mathcal{N}(\Delta \downarrow -)|, X) = \operatorname{holim} X.$$

This then implies that

$$\mathcal{O}(U) \simeq \operatorname{holim}\ \mathcal{O}(\mathcal{U}_\bullet) \simeq \mathrm{Tot}\ \mathcal{O}(\mathcal{U}_\bullet),$$

and we get a Milnor exact sequence

$$0 \longrightarrow \lim_i^1 \pi_{p-q-1}\mathrm{Tot}^i\mathcal{O}(\mathcal{U}_\bullet) \longrightarrow \pi_{p-q}\mathcal{O}(\mathcal{U}_\bullet) \longrightarrow \lim_i \pi_{p-q}\mathrm{Tot}^i\mathcal{O}(\mathcal{U}_\bullet) \longrightarrow 0$$

in which we want the $\lim^1$-term to vanish.

We are in the situation where $\mathcal{M} = (\mathcal{M}_{ell})_{et}$ is the étale site of the moduli stack of elliptic curves, $\mathcal{O} = \mathcal{O}^{\mathrm{Top}}$ and $\mathcal{U} = \{U_i \to \mathcal{M}_{ell}\}$ is a cover by affine schemes.

The following fact is again due to Boardman.

For the descent spectral sequence to converge strongly it suffices that for all $p$ and $q$ there are only finitely many indices $r \geq 1$ such that the differential starting at $E^r_{p,q}$ is non-zero. Moreover in this situation the above $\lim^1$-term vanishes.

This fact applies to our situation.

To end this talk, let us give just a few more remarks about the $E^2$-page.

**Proposition 7.** *If $\mathcal{U} = \{U_i \to \mathcal{M}_{ell}\}_{i\in I}$ is a cover by affines and $\pi_p^{\dagger}\mathcal{O}^{\mathrm{Top}}$ is the sheafification of the presheaf $\pi_p\mathcal{O}^{\mathrm{Top}}$, then for all multi-indices $\alpha = (i_o, \cdots_q) \in I^{q+1}$ we have*

$$\pi_p^{\dagger}\mathcal{O}^{\mathrm{Top}}(U_\alpha) \cong \pi_p\mathcal{O}^{\mathrm{Top}}(U_\alpha).$$

*Moreover $U_\alpha$ is acyclic for $\pi_p^{\dagger}\mathcal{O}^{\mathrm{Top}}$, i.e.,*

$$H^q(U_\alpha; \pi_p^{\dagger}\mathcal{O}^{\mathrm{Top}}) = 0 \quad \forall q \geq 1.$$

*Thus*

$$E^2_{p,q} \cong \check{H}^q_{\mathcal{U}}(\mathcal{M}_{ell}; \pi_p^{\dagger}\mathcal{O}^{\mathrm{Top}}) \cong H^q(\mathcal{M}_{ell}; \pi_p^{\dagger}\mathcal{O}^{\mathrm{Top}})$$

*hence the $E^2$-term does not really depend on the affine cover $\mathcal{U}$*

Since similar arguments apply for the case $\mathcal{M} = \overline{\mathcal{M}}_{ell}$ of the compactified moduli stack, we obtain strongly converging spectral sequences

(i) $E^2_{p,q} \cong H^q(\mathcal{M}_{ell}; \pi_p^{\dagger}\mathcal{O}^{\mathrm{Top}}) \Longrightarrow \pi_{p-q}\mathrm{TMF}$, and

(ii) $E^2_{p,q} \cong H^q(\overline{\mathcal{M}}_{ell}; \pi_p^{\dagger}\mathcal{O}^{\mathrm{Top}}) \Longrightarrow \pi_{p-q}\mathrm{Tmf}$.

**The last two talks were essentially an outline of the material contained in Bauer's "Computation of the homotopy groups of the spectrum `tmf`", which paper the interested reader is enthusiastically encouraged to peruse.**

# 13 Calculations in the homotopy of $tmf$ I – Irakli Patchkoria

# 14 Calculations in the homotopy of $tmf$ II – Irakli Patchkoria