# Yet another proof of the Deligne conjecture...

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This work is joint with J. Scott, based on work with B. Dwyer. It, like McClure's proof, was an accident. (Nobody seems to have set out to prove the Deligne conjecture!)

If $\mathcal{M}$ is a model category, one can construct the *hammock localization*: for any $X, Y \in \mathcal{M}$, we obtain a simplicial mapping space $Map_{\mathcal{M}}(X, Y) \in$ **sSet**. (This is the nerve of the category of zigzags from $X$ to $Y$ in which the backwards arrows are weak equivalences.) Let $k$ be a field. Let $Ch$ be the category of (unbounded) chain complexes of $k$-vector spaces.

**Theorem** (H.-Scott)**.** *Let $A$ be an associative $k$-algebra. Let $Def(A)$ be the space of "homotopy deformations of the multiplication on $A$". Then $\Omega^2 Def(A) \simeq Map_{Ch}(k, C^*A)$.*

Here, $k$ is seen as a complex concentrated in degree 0, and $C^*A$ denotes the Hochschild cochains. Thus we might say: "The *space* of Hochschild cochain 'is' a double loop-space."

This implies the Deligne conjecture (for associative $k$-algebras). Recall that we're interested in the little disks operads. The theorem tells us that $Map_{Ch}(k, C^*A)$ is a $\mathcal{D}_2$-space. Hence $Map_{Ch}(k, C^*A)_n$ is a $\mathcal{D}_2$-set (or more precisely, its normalized chains $N_*Map_{Ch}(k, C^*A)$ is a $N_*\mathcal{D}_2$-algebra). Thus $C^*A$ is a $N_*\mathcal{D}_2$-algebra.

This theorem is connected to deformation theory, since $H^2A$ classifies infinitesimal deformations and is also the home of obstructions to intrinsic formality. Moreover, in some sense, by proving this result we lift the Gerstenhaber algebra structure on $HH^*A$ all the way up to topology.

The method of proof involves a particular useful fiber sequence.

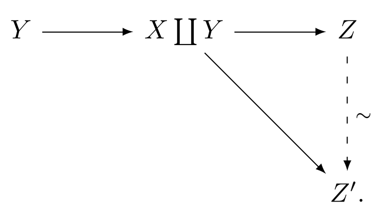
**Theorem** (Dwyer-H.)**.** *Let $(\mathcal{M}, \otimes, I)$ be a monoidal category with "nicely compatible" model category structure. Suppose we have a morphism of monoids $\varphi : R \to S$ in $\mathcal{M}$. Write $y : I \to R$ for the unit. Then there is a fiber sequence of simplicial sets given by*

$$\Omega Map_{\mathbf{Mon}}(R, S)_\varphi \to Map_{R\mathbf{Mod}_R}(R, S_\varphi) \xrightarrow{y^*} Map_{-R\mathbf{Mod}_R}(I, S_\varphi),$$

*where we take the fiber over $y : I \to S$ and we base our loops at $\varphi$.*

The key to the proof is the following result, which generalizes work of Dugger-Rezk.

**Lemma.** *If $\mathcal{C}$ is a left proper model category and $X, Y \in \mathcal{C}$ satisfy that $X^c \coprod Y \to X \coprod Y$ is a weak equivalence (where $X^c$ denotes the cofibrant replacement of $X$), then $Map_{\mathcal{C}}(X, Y)$ is homotopy equivalent to the nerve of the category whose objects are diagrams $Y \hookrightarrow X \coprod Y \longrightarrow Z$ in which the composition is a weak equivalence and whose morphisms are diagrams of the form*




Now we can say that "nicely compatible" means that:

- The model category structure on $\mathcal{M}$ induces compatible model category structures on categories of (bi)modules and monoids. (Compatibility means that a morphism of monoids is a fibration/weak equivalence iff the underlying morphism is a fibration/weak equivalence.)
- If $R$ and $S$ are monoids and $R^c \xrightarrow{\sim}$ is a cofibrant replacement, then $R^c \coprod S \xrightarrow{\sim} R \coprod S$, and if $\hat{R}^c$ is a cofibrant replacement of $R$ as an $R$-bimodule, then $\hat{R}^c \otimes S \xrightarrow{\sim} R \otimes S$.

- If $R$ and $S$ are monoids, the forgetful functor $U : R \coprod S \downarrow \mathbf{Mon} \longrightarrow R \otimes S \downarrow_R \mathbf{Mon}_S$ given by $(R \coprod^{\mathbf{Mon}} S \to T) \mapsto (R \otimes S \to T)$ admits a left adjoint $E$ (the "enveloping monoid" functor). This satisfies that, given $R \otimes S \to M$, if the composite $S \cong I \otimes S \to R \otimes S \to M$ is a weak equivalence, then for all diagrams

$$\begin{array}{ccc} S \otimes R & \longrightarrow & M^c \\ & \searrow & \downarrow \wr \\ & & M, \end{array}$$

  we get that

$$\begin{array}{ccc} S & \longrightarrow & R \coprod S \\ & \overset{\sim}{\searrow} & \downarrow \\ & & E(M^c). \end{array}$$

*Sketch of proof of theorem.* Let $\varphi : R \to S$ be a monoid morphism. Then $Map_{\mathbf{Mon}}(R, S)$ is homotopy equivalent to the nerve of the category described above, whose objects are diagrams

$$S \rightarrowtail R \coprod S \longrightarrow T$$

in which the composition is a weak equivalence and whose morphisms are diagrams

$$\begin{array}{ccccc} S & \rightarrowtail & R \coprod S & \longrightarrow & T \\ & & & \searrow & \vdots \wr \\ & & & & T'. \end{array}$$

This is in turn homotopy equivalent to the nerve of the category with the same diagrams but with tensor product instead of coproduct.

We then "unpack the basepoint" and loop, which gives us the desired fiber sequence. □

We can apply this to nonsymmetric operads of simplicial sets with multiplication.

**Theorem** (Dwyer-H.)**.** *Let $P$ be a nonsymmetric operad of simplicial sets such that $P(0) \simeq * \simeq P(1)$. Let $\varphi : As \to P$ be any operad map, with associated cosymplicial simplicial set $P^bullet$. Then*

$$Tot(P^\bullet) \simeq \Omega^2 Map_{Op}(As, P)_\varphi.$$

**Corollary.** *The space of tangentially straightened long knots in $\mathbb{R}^m$ (for $m \geq 4$) is a double loop space; in particular it has the homotopy type of $\Omega^2 Map_{Op}(As, \mathcal{K}_m)$, where $\mathcal{K}_m$ is the Kontsevich operad (which is homotopy equivalent to $\mathcal{D}_m$).*

This was known to Sinha, who proved that $Tot(\mathcal{K}_m^\bullet)$ is homotopy equivalent to the space of long knots.

*Proof sketch.* We apply the fiber sequence theorem twice.

First, we find that $(\mathbf{sSet}^{\mathbb{N}}, \circ, \mathcal{J})$ is a "nicely compatible" model structure. Then for all operad maps $\varphi : O \to P$, there is a fiber sequence

$$\Omega Map_{Op}(O, P)_\varphi \to Map_{O\mathbf{Mod}_O}(O, P_\varphi) \to Map_{\mathbf{sSet}^{\mathbb{N}}}(\mathcal{J}, P).$$

Hence $P(\Lambda) \simeq *$, from which it follows that $\Omega Map_{Op}(O, P) \simeq Map_{O\mathbf{Mod}_O}(O, P_\varphi)$.

We now take $O = As$. Let $\odot$ denote the graded monoidal product on $\mathbf{sSet}^{\mathbb{N}}$ given by

$$(X \odot Y)(n) = \coprod_{0 \leq k \leq n} X(k) \times Y(n-k).$$

**Exercise 1.** ${}_{As}\mathbf{Mod} \cong \mathbf{Mon}^{\odot}$ (the category of graded monoids), and this restricts to ${}_{As}\mathbf{Mod}_{As}$ (the category of graded monoids in $\mathbf{Mod}_{As}$).

So $\mathbf{Mod}_{As}$ has nicely compatible model structure with respect to $\odot$. So we get the fiber sequence

$$\Omega Map_{{}_{As}\mathbf{Mod}_{As}}(As, P)_{\varphi} \to Map_{{}_{As}\mathbf{Mod}^{\circ}_{As} \cap {}_{As}\mathbf{Mod}^{\odot}_{As}}(As, P) \to Map_{\mathbf{Mod}_{As}}(As, P).$$

If $P(0) \simeq *$, then the base here is contractible. Moreover, $Map_{\text{bimod intersection thing}}(As, P) \simeq Tot(P^{\bullet})$. Hence $Tot(P^{\bullet}) \simeq \Omega^2 Map_{Op}(As, P)$.

□

The work with J. Scott was to extend this to nonsymmetric operads of chain complexes. Then we can apply the fiber sequence twice to $\mu : As \to End(A)$. This brings in $Def(A) = Map_{Op}(As, End(A))_{\mu}$.