Deligne conjecture

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1 Introduction

Let A be an associative ring. Then the Hochschild cochains are given by $C^p(A) = \mathrm{Hom}(A^{\otimes p}, A)$, with differential

$$\begin{aligned}(d\varphi)(a_1 \otimes \cdots \otimes a_{n+1}) &= a_1\varphi(a_2 \otimes \cdots \otimes a_{n+1}) \\ &\quad + \sum_{i=1}^{p}(-1)^i \varphi(\cdots \otimes a_i a_{i+1} \otimes \cdots) \\ &\quad + (-1)^{p+1}\varphi(a_1 \otimes \cdots \otimes a_p)a_{p+1}.\end{aligned}$$

We then obtain the Hochschild homology $HH^*(A) = H^*(C^*(A))$. This comes with a cup product given by, for $\varphi \in HH^p(A)$ and $\psi \in HH^q(A)$, defining

$$(\varphi \smile \psi)(a_1 \otimes \cdots \otimes a_{p+q}) = \varphi(a_1 \otimes \cdots \otimes a_p) \cdot \psi(a_{p+1} \otimes \cdots \otimes a_{p+q}).$$

This has no reason to be commutative at the cochain level, but amazingly it induces a commutative product on homology.

Gerstenhaber showed in 1962 that HH^*A is a *Gerstenhaber algebra*, meaning that it has a cup product and a compatible Lie bracket (the commutator of the star operation). Compatibility means that the cup product acts by derivations with respect to the Lie bracket.

Then, in 1973 Fred Cohen showed that if X is a $\mathcal{D}_2$-algebra then H_*X is a Gerstenhaber algebra.

Deligne noticed in 1993 that there was a connection between these two, and asked whether these came from the same place. More explicitly, he asked: **Does the Gerstenhaber structure on HH^*A come from an action of an E_2 chain operad (i.e. a chain operad on the Hochschild cochains which is quasi-isomorphic to the singular chains $S_*\mathcal{D}_2$)?** The answer will be yes, although we'll actually normalize our cochains for technical reasons.

There are a number of proofs of this fact. The first was by Getzler-Jones in 1994, but it had a gap; their key ingredient was a cellular decomposition of the Fulton-MacPherson operad, but it wasn't cellular: they had lower cells attached to higher cells. This was fixed by Voronov in 2000. Meanwhile, in 1998 there were proofs in characteristic 0 by Tamarkin and Kontsevich, and then McClure-Smith obtained a proof over $\mathbb{Z}$ that even works for A_∞ ring spectra. There have been many more proofs since then.

It turns out that McClure-Smith's method can be used to something quite a bit more general. One might ask: if one has a space and wants an operad to act, how do we get that to happen? The main one we know is that $\mathcal{D}_n$ acts on $\Omega^n X$ for a space X, and there are Segal Γ-spaces that admit ∞-actions, but what about the $\mathcal{D}_n$ operad acting on a general space? We will obtain a general condition that implies a $\mathcal{D}_n$-action.

McClure-Smith's work also leads to a combinatorial description of something quasi-isomorphic to the chains $S_*\mathcal{D}_2$.

A good reference for almost all this material is the survey paper by McClure-Smith, *Operads and Cosimplicial Spaces: An Introduction* (on the arXiv). This includes a self-contained introduction to the $\mathcal{D}_n$ operads.

2 Cosimplicial objects

Definition 1. A *cosimplicial object* in a category $\mathcal{A}$ is a covariant functor $\Delta \to \mathcal{A}$. These can be described in terms of coface and codegeneracy maps.

Example. The first example, which has $\mathcal{A} = \mathbf{Ab}$, is C^*A. This takes $\{0, \ldots, p\}$ to the Hochschild chains $C^p(A)$. We decompose the summands in the definition of $d : C^p(A) \to C^{p+1}(A)$ into $d^0, \ldots, d^{p+1}$, and these are our codegeneracies.

Example. Let S^*Y denote the singular cochains on a space Y. This takes $\{0, \ldots, p\}$ to $Map_{\mathbf{Set}}(S_pY, \mathbb{Z}) = Ma_{\mathbf{Top}}(\Delta^p, Y)$. This is the key motivating example.

We will mainly think about **Ab** and **Top**, but in the background will be $Ch(R)$ and **Spectra**.

Definition 2. The *totalization* of a cosimplicial objects is given as follows. First, if $B^\bullet \in \mathbf{Ab}^\Delta$, then its *conormalization* $N^*(B^\bullet) \in Ch^*$ is a chain complex defined by $N^p(B^\bullet) = \bigcap_{i=0}^{p-1} \ker(s^i)$ (where the s^i are the codegeneracies). The differential is given by $d = \sum_{i=0}^{p+1}(-1)^i d^i$ (where the d^i are the coface maps).

Example. The normalization of the Hochschild chochain complex is $N^*(C^\bullet A)$ are the *normalized* Hochschild cochains.

Example. $N^*(S^\bullet W)$ are the normalized singular cochains.

2.0.1 Cosimplicial spaces

Example. A key example of a cosimplicial space is $\Delta^\bullet$, which takes $\{1, \ldots, p\}$ to Δ^p. This is not cofibrant, which makes life interesting in a number of ways.

Example. Note that a based space Y is a lot like a coalgebra. The geometric cobar construction can be made by mimicking the usual cobar construction. Its totalization will be the loopspace of Y. We can also make the geometric cyclic cobar construction, and this will give the free loopspace of Y.

Definition 3. For a cosimplicial space $X^\bullet$, the *totalization* is $Tot(X^\bullet) = \mathrm{Hom}_\Delta(\Delta^\bullet, X^\bullet) \subset \prod_{p=0}^\infty Map_{\mathbf{Top}}(\Delta^p, X^p)$. An element is a sequence $\{f_p : \Delta^p \to X^p\}$ commuting with the coface and codegeneracy maps, i.e.

$$\begin{array}{ccc}
\Delta^{p+1} & \xrightarrow{f_{p+1}} & X^{p+1} \\
\uparrow{\scriptstyle d_i} & & \uparrow{\scriptstyle d_i} \\
\Delta^p & \xrightarrow[f_p]{} & X^p \\
\downarrow{\scriptstyle s_i} & & \downarrow{\scriptstyle s_i} \\
\Delta^{p-1} & \xrightarrow[f_{p-1}]{} & X^{p-1}
\end{array}$$

commutes.

This should be thought of as dual to geometric realization, which for a simplicial space $W_\bullet$ is given by $|W_\bullet| = W_\bullet \otimes_\Delta \Delta^\bullet$. (In particular, totalization is a coend construction.) In fact, there is an adjunction relation between geometric realization and totalization.

Example. $Tot(\Delta^\bullet)$ is contractible.

Example. $Tot(\text{cobar}(X)) \cong \Omega X$ and $Tot(\text{cyclic cobar}(X)) \cong LX$.

Now we can state our goal. **We would like to describe a structure that $(B^\bullet, X^\bullet)$ could have which induces an action of an E_n operad (for $1 \le n \le \infty$).** The plan for doing this is to start with $n = 1$, move to $n = \infty$, and then interpolate between the two. As a bonus, we will also look at the framed little disks $f\mathcal{D}_2$.

2.1 $n = 1$

Note that $N^*S^\bullet Y$ is a dga under the cup product, and there is a corresponding structure on $S^\bullet Y$. The cup product is given by $(x^p \smile y^q)(\sigma^{p+q}) = x(\sigma(0, \ldots, p)) \cdot y(\sigma(p, \ldots, p+q))$. Note that the p is repeated; this will not always be the case in what follows. This gives $\smile: S^pY \otimes S^qY \to S^{p+q}Y$. This satisfies the conditions:

1. $$d^i(x \smile y) = \begin{cases} d^i x \smile y, & i \le p \\ x \smile d^{i-p} y, & i \ge p; \end{cases}$$

2. $(d^{p+1}x) \smile y = x \smile (d^0 y)$;

3.
$$s^i(x \smile y) = \begin{cases} s^i x \smile y, & i \leq p-1 \\ x \smile s^{i-p} y, & i \geq p. \end{cases}$$

Definition 4. A *cosimplicial abelian group/space with a cup product* is a cosimplicial abelian group/space with an associative and unital product satisfying the above properties.

Proposition. *The cup product on $B^\bullet$ induces a dga structure on $N^*(B^\bullet)$.*

Proof. The proof is just checking that the cup product is a chain map. □

Theorem (**Theorem A**, Batanin, McClure-Smith)**.** *The cup product on $X^\bullet$ induces an A_∞ structure on $Tot(X^\bullet)$.*

McClure considers this a *useful theorem*, which he defines as a theorem with at least three applications. This gives, for example, that topological Hochschild cohomology has an A_∞ structure (although in fact it has a $\mathcal{D}_2$ structure).

An interesting open problem is whether this is a Quillen equivalence. McClure conjectures that it does.
Q. Where's the $\mathcal{D}_2$ coming from in Deligne's conjecture?
A. A $\mathcal{D}_n$ action gives a Lie bracket on degree $n-1$, which is often called the *Browder operation*. In the Hochschild complex we have a bracket of degree 1, which suggests that we want $n = 2$. More generally, the primary obstruction to refining a $\mathcal{D}_n$ action to a $\mathcal{D}_{n+1}$ action is the Lie bracket. In fact, there are higher Hochschild cohomologies $HH^{(n)}(A)$ for A an E_n-ring, and this has a $\mathcal{D}_{n+1}$ action.

Exercise 1. An *operad with multiplication* is an operad $\mathcal{O}$ along with a map $As \to \mathcal{O}$. For example, in the endomorphism operad of a ring A we have $\mathcal{O}_A(p) = \mathrm{Hom}(A^{\otimes p}, A)$, and the map $As \to \mathcal{O}_A$ is determined by $1 \in \mathcal{O}_A(0) = A$ while the multiplication is determined by $M \in \mathcal{O}_A(2) = \mathrm{Hom}(A \otimes A, A)$. Show that $1 \in \mathcal{O}(0) = A$ induces $As \to \mathcal{O}_A$, and show that an operad with multiplication induces a cosimplicial opject (with cup product).

3 Generalizing Theorem A (which sits at $n = 1$) to higher n

One can continue along the path of $\smile_i$-products, but these end up being somewhat messy to work with, especially because of their signs. Rather, we will generalize in a different direction.

Recall that (up to signs), $(x \smile y)(\sigma) = x(\sigma(0, \ldots, p)) \cdot y(\sigma(p, \ldots, p+q))$: the cup product has an interesting repetition. This leads us to the following reformulation of Theorem A.

For $x \in S^p Y$, $y \in S^q Y$, $\sigma \in S_{p+q+1} Y$, define $(x \sqcup y)(\sigma) = x(\sigma(0, \ldots, p)) \cdot y(p+1, \ldots, p+q+1))$. This is not a chain map (and hence does not descend to cohomology), but it has obvious relations with the coface and codegeneracy maps. We call a *cosimplicial abelian group/space with* $\sqcup$ a cosimplicial space with operations as above that satisfy those same relations with the coface and coboundary maps.

Proposition. *These two kinds of structure on a cosimplicial abelian group/space are equivalent.*

Thus, there must be some $\sqcup$ in the Hochschild complex $C^\bullet A$; this is given by $(\varphi \sqcup \psi)(a_1 \otimes \cdots \otimes a_{p+q+1}) = \varphi(a_1 \otimes \cdots \otimes a_p) \cdot a_{p+1} \cdot \psi(a_{p+2} \otimes \cdots \otimes a_{p+1+q})$.

3.1 The E_∞ case

Note that so far we have just been partitioning the vertices of σ; this is an associative sort of gadget. If we allow ourselves to mix the ordering, we will get a commutative sort of gadget.

We first need some notation.

Definition 5. The *Delta category* Δ_+ is the category of all finite totally-ordered sets, with morphisms bijections. (These will be denoted by the letters S and T.) If $T \in \Delta_+$, we define the *convex hull* of the elements in T by $\Delta^T = \{h : t \to \mathbb{R} : \sigma h(t) = 1,\ h(t) \geq 0\ \forall t\} \subset Map_{\mathbf{Set}}(T, \mathbb{R})$. We have the simplices $S_T Y = Map_{\mathbf{Top}}(\Delta^T, Y)$, and we have the cosimplices $S^T Y = Map_{\mathbf{Set}}(S_T Y, \mathbb{Z})$. (Note that $S^\emptyset Y \cong \mathbb{Z}$.)

Definition 6. Given any $f : T \to \{1, 2\}$, we define $\langle f \rangle : S^{f^{-1}(1)} Y \otimes S^{f^{-1}(2)} Y \to S^T Y$ by, for $\sigma : \Delta^T \to Y$, setting $\langle f \rangle(x, y)(\sigma) = x(\sigma(f^{-1}(1))) \cdot y(\sigma(f^{-1}(2)))$.

The idea is that S^TY is essentially just S^pY, where $p = |T| - 1$. This notation is also convenient because we don't need to put absolute values everywhere.

If $f : \{0, \ldots, p+q+1\} \to \{1, 2\}$ has $f^{-1}(1) = \{0, \ldots, p\}$ and $f^{-1}(2) = \{p+1, \ldots, p+q+1\}$ then we just recover $\langle f \rangle(x, y) = x \sqcup y$.

The family of $\langle f \rangle$ operations enjoys:

- a relation to the d^i and s^i;
- an appropriate sort of commutativity (given by permuting $\{1, 2\}$);
- an appropriate sort of associativity;
- an appropriate unitality

The reason we must allow $\emptyset \in \Delta_+$ is to obtain the correct version of unitality. Otherwise, we would just be symmetric monoidal with no unit.

Theorem (Theorem B)**.** *If $B^\bullet$ (or $X^\bullet$) has all $\langle f \rangle$ operations with the properties listed above, then this data induces an E_∞ structure on $N^*B^\bullet$ (or $Tot(X^\bullet)$).*

In particular, this gives us a new proof that the cochains on a topological space carry an E_∞ structure.

3.2 The E_n case for $1 < n < \infty$

For $n = 1$ we have partitions where the pieces are totally separated, whereas for $n = \infty$ we have partitions that are totally mixed. So to model $1 < n < \infty$, we will put restrictions on our f. More precisely, we will get a filtration of the operations $\langle f \rangle$ whose n^{th} level corresponds to an E_n structure.

If $T \in \Delta_+$ is a totally-ordered set, then $f : T \to \{1, 2\}$ can be thought of as a sequence of 1's and 2's. The *complexity* of such a sequence is the number of times it changes value. For example, 1221212211 has complexity 6. (This is fairly natural, because it is related to the skeletal filtration of $\mathbb{R}P^\infty$.)

For technical reasons, we must consider operations in many variables, $f : T \to \{1, \ldots, k\}$. It's clear how to define $\langle f \rangle : S^{f^{-1}(1)}Y \otimes \cdots \otimes S^{f^{-1}(k)}Y \to S^TY$, and these will have properties that are generalizations of the ones before. In this setting, we call the *complexity* of f to be the maximum of the complexities of the subsequences with only two different values. For example, 1232132 has 12-complexity 3, 13-complexity 3, and 23-complexity 4. So it has complexity 4. (This is less natural, but ends up working out.)

We will write $c(f)$ for the complexity of f. Of course, $c(f) = 1$ whenever f is constant; this corresponds to an iterated $\sqcup$ product. As expected, this gives us the symmetric E_1 operad.

Now, fix n. The family $\{\langle f \rangle | c(f) \leq n\}$ has properties analogous to the properties we've had before. This gives us the following.

Theorem (Theorem C)**.** *If $B^\bullet$ (or $X^\bullet$) has all $\langle f \rangle$ operations with $c(f) \leq n$ with these properties, then this data induces an E_n structure on $N^*B^\bullet$ (or $Tot(X^\bullet)$).*

Recall that an E_n operad is an operad that is weakly equivalent (or quasi-isomorphic) as operads to D_n (or S_*D_n), the (singular chains on the) little n-disks operad.

Corollary. *Deligne's conjecture.*

Proof. We just need to show that $C^\bullet A$ has all $\langle f \rangle$ operations with $c(f) \leq 2$. We employ a *proof by punctuation*. For example, let $f = 12221134431$. This has $c(f) = 2$, since things are nested (the 2's are nested inside the 1's, and the 3's are nested inside 1's, and the 4's are nested inside the 3's). This has $|T| = 11$, so we're looking at a 10-simplex. This is supposed to give us an operation $\langle f \rangle : C^3A \otimes C^2A \otimes C^1A \otimes C^1A \to C^{10}A$ (since there are 4 1's, 3 2's, etc.) Let us describe the action on $\varphi \otimes \psi \otimes \omega \otimes \chi$. Between the 11 numbers above we have 10 entries $a_1, \ldots, a_{10}$:

$$1_{a_1}2_{a_2}2_{a_3}2_{a_4}1_{a_5}1_{a_6}3_{a_7}4_{a_8}4_{a_9}3_{a_{10}}1.$$

The first occurence of 1 gives φ, the first occurrence of 2 gives ψ, etc. The subsequent occurences of these numbers give $\otimes$, and the last occurences of these numbers gives close-parentheses. So we get

$$\varphi([a_1\psi(a_2 \otimes a_3)a_4] \otimes [a_5] \otimes [a_6\omega(a_7\chi(a_8 \otimes a_9)a_{10})]).$$

(The square brackets delineate the three slots in φ.) If there is only one occurrence of a number, we simply put its corresponding cochain. One can check that this all works out and satisfies the right relation. $\square$

(Deligne's conjecture for ring spectra also follows.)

4 Proofs

We will first do everything topologically, and then comment on how the arguments must be modified. We begin with the proof of Theorem A.

Proof sketch of Theorem A. Recall that we must show that $X^\bullet$ with a cup product determines an A_∞ structure on $Tot(X^\bullet)$.

The first step is to show that there is a monoidal structure $\square$ on the category $\mathbf{Top}^\Delta$ of cosimplicial spaces. (This construction works on the category of all functors between two monoidal categories.) We set $X^\bullet \square Y^\bullet$ to have in degree m the object

$$\coprod_{p+q=m} (X^p \times Y^q)/\langle (x, d^0 y) \sim (d^{p+1}x, y)\rangle.$$

The coface maps and codegeneracy maps are given by

$$\begin{aligned} d^i(x,y) &= \begin{cases} (d^i x, y), & i \leq p \\ (x, d^{i-p}y), & i > p \end{cases} \\ s^i(x,y) &= \begin{cases} (s^i x, y), & i < p \\ (x, s^{i-p}y), & i \geq p. \end{cases} \end{aligned}$$

A key fact is that $X^\bullet$ has a cup product iff $X^\bullet$ is a $\square$-monoid.

Then, similar to Problem 2, we let $A(k) = \mathrm{Hom}_\Delta(\Delta^\bullet, (\Delta^\bullet)^{\square k})$, and this ends up giving a nonsymmetric operad A. Now, if $X^\bullet$ is a $\square$-monoid, then A acts on $Tot(X^\bullet) = \mathrm{Hom}_\Delta(\Delta^\bullet, X^\bullet)$. Indeed, we map

$$A(k) \times Tot(X^\bullet)^k = \mathrm{Hom}_\Delta(\Delta^\bullet, (\Delta^\bullet)^{\square k}) \times \mathrm{Hom}_\Delta(\Delta^\bullet, X^\bullet)^{\times k} \overset{\text{composition}}{\longrightarrow} \mathrm{Hom}_\Delta(\Delta^\bullet, (X^\bullet)^{\square k}) \to \mathrm{Hom}_\Delta(\Delta^\bullet, X^\bullet) = Tot(X^\bullet).$$

(The second arrow uses the fact that $X^\bullet$ is a $\square$-monoid.)

So it only remains to show that A is an A_∞ operad. For this, we compute,

$$A(k) = \mathrm{Hom}_\Delta(\Delta^\bullet, (\Delta^\bullet)^{\square k}) \cong \mathrm{Hom}_\Delta(\Delta^\bullet, \Delta^\bullet) \simeq *,$$

where the isomorphism comes from Problem 7 and the homotopy equivalence comes from Problem 6 (which is solved using a straight-line homotopy). □

Proof sketch of Theorem B. Recall that we must show that if $X^\bullet$ has all $\langle f\rangle$ operations then $Tot(X^\bullet)$ has an E_∞ structure.

The main ingredient of the proof is a symmetric monoidal structure $\boxtimes$ on $\mathbf{Top}^\Delta$ such that $X^\bullet$ has all $\langle f\rangle$ operations. This is equivalent to the fact that $X^\bullet$ is a commutative $\boxtimes$-monoid.

So, for $S \in \Delta_+$, an element of $(X^\bullet \boxtimes Y^\bullet)^S$ should admit all $\langle f\rangle$ operations. We need the cosimplicial operators, so if $\phi : T \to S$ is a morphism in Δ_+ then we need to be able to apply ϕ_* to $\langle f\rangle(x,y) \in (X^\bullet \boxtimes Y^\bullet)^T$. If $S = \{1,2\}$, $x \in X^{f^{-1}(1)}$, $y \in Y^{f^{-1}(2)}$, then we quotient by the relation in Problem 5. (This is actually a Kan extension.)

From here on out, we follow the same proof structure. From the $\langle f\rangle$ operations we can create an operad $\mathcal{E}$ with $\mathcal{E}(k) = \mathrm{Hom}_\Delta(\Delta^\bullet, (\Delta^\bullet)^{\boxtimes k})$ for which $X^\bullet$ is an algebra. This is our E_∞ operad.

□

Proof sketch of Theorem C. Recall that we must show that if $X^\bullet$ has $\langle f\rangle$ operations with complexity $c(f) \leq n$, then $Tot(X^\bullet)$ has an E_n-structure.

For this, we essentially just filter $\boxtimes$ appropriately using the complexity filtration. The question is, what's the structure between a monoidal structure and a symmetric monoidal structure? There are many answers, and the one that McClure-Smith arrived at they call a "functor operad" structure.

□

For the algebraic results, we must replace $\Delta^\bullet$ everywhere with $C_*\Delta^\bullet$, its normalized simplicial chains. This was the reason for Problem 3: if we know that the conormalization can be represented in this way (which the alternating sum of coface maps cannot), then the proof goes through identically.

5 The framed little disks operad

The framed little disks operad fD_2 has little circles that each come with a marked point. So its spaces admit torus actions, since we can rotate the little circles. In particular, we have a single little pointed circle in $fD_2(1)$, so $S^1 \subset fD_2(1)$, so any fD_2 space has an S^1 action (and the fD_2 action is "generated" by the S^1 action and the D_2 action; cf. a paper of Ralph Kaufmann).

So we will need to use cosimplicial spaces whose totalizations admit circle actions, which if you know about this stuff immediately means that you need to work with cyclic cosimplicial spaces.

A digression on the cyclic category Λ: Consider the Hochschild chain complex $C_pA = A^{\otimes p+1}$ (thought of as a simplicial object). The face maps come from multiplication, including cycling the last goat around to the front. The degeneracy maps come from inserting unit anywhere besides in the first slot (since we're secretly thinking about this one as a bimodule). But now we also have cyclic permutations. This is a faithful representation of Λ^{op}, whose morphisms are generated by d^i, s^i, and t, subject to certain relations. (There's a different $t = t_p$ in each degree, but we usually ignore that.) So $\Delta^{op} \subset \Lambda^{op}$, and hence $\Delta \subset \Lambda$. As before, *cyclic object* is a functor in $\mathcal{C}^{\Lambda^{op}}$, and a *cocyclic object* is a functor in $\mathcal{C}^{\Lambda}$. More precisely, a *cocyclic space* is a covariant functor $\Lambda \to$ **Top**. For example, the coHochschild complex of a dg coalgebra C is given in degree p by $C^{\otimes p+1}$, with cofaces given by comultiplication, codegeneracies given by the counit, and cyclic permutations; thus, this is a cocyclic chain complex.

Definition 7. Let $X^\bullet$ be a cocyclic space. A *compatible* $\sqcup$ on $X^\bullet$ is the same as before but with the additional requirement that if $|x| = p$, then $t^{p+1}(x \sqcup y) = y \sqcup x$.

Example. This is related to Problem 8. If A is a Frobenius algebra over k, then the coHochschild complex of A has a compatible $\sqcup$. The formula is a little messy, but what's easier is to write down the corresponding $\smile$ (guaranteed by Problem 5): set $(a_0 \otimes \cdots \otimes a_p) \smile (b_0 \otimes \cdots \otimes b_q) = (a_0 b_0 \otimes a_1 \otimes \cdots \otimes a_p \otimes b_1 \otimes \cdots \otimes b_q)$.

Theorem (Theorem D)**.** *A compatible $\sqcup$ on a cocyclic object $B^\bullet$ (or $X^\bullet$) induces an fE_2 on $N^*B^\bullet$ (or $Tot(X^\bullet)$).*

(In analogy with the above, fE_2 is quasi-isomorphic to fD_2.)

The proof is similar to the proofs of Theorems A, B, and C, but to show that the operad we get is equivalent (as an operad) to fD_2 is difficult. To show this, we use Ralph Kaufmann's proof that the cactus operad is equivalent to fD_2. The difference here is that if we have an $\langle f \rangle$ operation of degree one, say, 1112222, we might get an operation of degree two by a cyclic permutation that takes us to 2211122.

Corollary (cyclic Deligne conjecture, by Kaufmann)**.** *The Hochschild cochain complex of a Frobenius algebra has an fE_2 structure.*

6 Connections with string topology

Recall that H_*LM has a BV structure. Here, LM is the free loopspace on a compact oriented manifold M. Ordinarily a BV structure is induced by an fD_2 action, and so we have a Deligne-like question: **Is there a chain-level fD_2 structure inducing this BV structure?** The answer, which was worked out by Costello, probably Lurie (in some form), and McClure, all individually, is yes. (McClure's work applies to PL manifolds, and the others may even apply to Poincaré duality spaces.)

The idea is that, assuming M is simply-connected, then S_*LM is equivalent to the coHochschild complex of S_*M. (When $\pi_1(M) \neq 0$, a twisted version still holds. Even more generally, there is a categorical version phrased in the language of presheaves.) So we just need a compatible $\sqcup$ on S_*M. To do this, we just modify the ordinary formula. But we need to be able to multiply chains on a manifold. The well-known way to do this is the intersection pairing. But we can only use the transverse subcomplex $G_2 \hookrightarrow S_*M \otimes S_*M$, and so (McClure 2006) proves that this inclusion is a quasi-isomorphism. (This gives us what is known as a *Leinster partial structure*.) This gives us a partial multiplication on S_*M. And then, Scott Wilson's recent work shows that we can always rectify a Leinster partial algebra structure. Over $\mathbb{Q}$, this can even be rectified to a CDGA; McClure conjectures that this is a rational model for the homotopy type of M. Also, there is a spectral sequence of BV algebras from the coHochschild complex on S_*M converging to H_*LM.