Derived A_∞-algebras

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We would like to study dga's by studying A_∞-algebra structures on their homology. There are many nice constructions here, but they often depend on rather strong projectivity assumptions. Thus we must *derive* our theory so that we can replace non-projective algebras with projective resolutions.

1 A_∞-algebras

In everything that follows, $\mathbb{K}$ is a commutative ring. We will assume it has no 2-torsion (to simplify our lives, otherwise we'd have to work with preLie structures instead of just Lie structures). Let $A = \sum_{n\in\mathbb{Z}} A^n$ be a (cohomologically) graded $\mathbb{K}$-module. We take the convention that a morphism of degree i raises degree by i.

Definition 1. An A_∞-structure on A is a collection of multiplication maps $m_n : A^{\otimes n} \to A$ of degree $2-n$ such that

$$\sum_{r+s+t=n} (-1)^{rs+t} m_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0$$

for all n.

We write an A_∞-structure as a formal sum $m = m_1 + m_2 + \ldots$; this is finite in each degree.

Theorem (Kadeishvili)**.** *If (C, ∂, μ) is a dga such that $H^*(C)$ is degreewise projective over $\mathbb{K}$, then there is an A_∞-structure m on $H^*(C)$ such that:*

- $m_1 = 0$ *(we call this case* minimal*);*
- m_2 *is induced by* μ*;*
- $(H^*(C), m)$ *is quasi-isomorphic to* $(C, \partial + \mu)$ *(i.e. the* f_1 *part of the map* f *is a quasi-isomorphism of the underlying dgas).*

Moreover, there is a bijection

$$\frac{\{\text{dgas } A \text{ with } H^A \cong H^*C \text{ (as associative algebras)}\}}{\text{quasi-isomorphism}} \cong \frac{\{A_\infty\text{-structures on } H^*C \text{ such that } m_1 = 0 \text{ and } m_2 = \mu\}}{\text{quasi-isomorphism}}.$$

2 Hochschild cohomology

Let A be a graded $\mathbb{K}$-module. This gives us a bigraded $\mathbb{K}$-module $C^{n,m}(A, A) = \mathrm{Hom}_{\mathbb{K}}((A^{\otimes n})^*, A^{*+m})$. For example, we have $m_2 \in C^{i,2-i}(A, A)$. We give this a Lie algebra structure as follows. If $f \in C^{n,k}(A, A)$ and $g \in C^{m,l}(A, A)$, we set

$$\begin{aligned}[f, g] \quad = \quad & \sum_{i=0}^{n-1} (-1)^{(n-1)(m-1)+(n-1)l+i(m-1)} f(1^{\otimes i} \otimes g \otimes 1^{\otimes(n-i)}) \\ & -(-1)^{(n+k-1)(m+l-1)} \sum_{i=0}^{m-1} (-1)^{(m-1)(n-1)+(m-1)k+i(n-1)} g(1^{\otimes i} \otimes f \otimes 1^{\otimes(m-i-1)}).\end{aligned}$$

This is an element of $C^{n+m-1,k+l}(A, A)$.

The sign here comes from the shift $S(A)$ defined by $(S(A))^k = A^{k+1}$. If $f \in C^{n,k}(A,A)$, we define $\sigma(f)$ by

$$\begin{array}{ccc} (SA)^{\otimes n} & \xrightarrow{\sigma(f)} & SA \\ \uparrow {\scriptstyle S^{\otimes n}} & & \downarrow {\scriptstyle S^{-1}} \\ A^{\otimes n} & \xrightarrow[(-1)^{k+n-1} f]{} & A. \end{array}$$

Then we just get that $[f,g] = \sigma^{-1}(\{\sigma(f), \sigma(g)\})$, where $\{\ ,\ \}$ is the "bracket without signs". It is now clear that m is an A_∞-structure iff $[m,m] = 0$.

Definition 2. Let (A,m) be an A_∞-algebra. We define the *tangent cohomology* (or just *the Hochschild cohomology of an A_∞-algebra*) by

$$HH^*(A,A) = H^*\left(\prod_i C^{i,*-i}(A,A), [m,-]\right).$$

Remark. *The Jacobi identity implies that $[m,[m,-]] = 0$. If $m = m_2$, we recover the usual Hochschild cohomology of an algebra and the degree splits as $HH^{**}_{alg}(A,A)$.*

Theorem (Kadeishvili)**.** *Let C be a dga with H^*C degreewise projective. Then if $HH^{n,2-n}_{alg}(H^*C, H^*C) = 0$ for $n \geq 3$, then C is intrinsically formal, i.e. C is the only dga up to (quasi-isomorphism of dgas) that realizes H^*C.*

3 Derived A_∞-algebras

The previous theorem is great, but we would like to get rid of the projectivity assumption. Thus, we must derive everything. So we consider $(\mathbb{N}, \mathbb{Z})$-bigraded $\mathbb{K}$-modules $A = \sum_{i \in \mathbb{N},\ j \in \mathbb{Z}} A_i^j$. By convention, a morphism of bidegree (s,t) takes the form $A_*^* \to B_{*-s}^{*+t}$.

Definition 3 (Sagave)**.** A *derived A_∞-structure* on A consists of a family m of multiplication maps $m_{ij} : A^{\otimes j} \to A$ of bidegree $(i, 2-i-j)$ (for $i \geq 0$ and $j \geq 1$) such that

$$\sum_{u=i+p,\ v=j+q-1,\ j=1+r+t} (-1)^{rq+t+pj} m_{ij}(1^{\otimes r} \otimes m_{pq} \otimes 1^{\otimes t}) = 0$$

for all (u,v). The definitions of morphisms are obvious, and we define *E_2-equivalences* to be those maps which induce quasi-isomorphisms on the E_2-pages of the modules (which are obtained by taking homology first with respect to m_{01} and then with respect to m_{11}).

Example. Given a bicomplex, we can set $m = m_{01} + m_{11}$ to obtain a derived A_∞-structure.

Example. A "classical" A_∞-algebra has a derived A_∞-algebra structure concentrated in lower-degree 0.

Example. A bidga, which is by definition a monoid in the category of bicomplexes, is has a derived A_∞-structure with $m = m_{01} + m_{11} + m_{02}$.

Theorem (Sagave)**.** *Let C be a dga. Then there is a derived A_∞-algebra (E,m) such that:*

- $m_{01} = 0$ *(we say m is* minimal*);*
- *we have an E_2-equivalence $E \to C$;*
- *the derived A_∞-algebra $(E, m_{11} + m_{02})$ is a termwise projective resolution of H^*C.*

4 Uniqueness

This is good, but we would still like to remove the assumption of termwise projective resolution. We consider trigraded complexes $C_k^{n,i}(A,A) = \mathrm{Hom}_{\mathbb{K}}((A^{\otimes n})_*^*, A_{*-k}^{*+i})$ with Lie algebra structure almost the same as before. Given $f \in C_k^{n,i}(A,A)$, we define $f^\# = (-1)^k f$.

Lemma. *m is a derived A_∞-structure iff $[m, m^\#] = 0$.*

Unfortunately we cannot always define Hochschild cohomology in this setting. However, for "nice" derived A_∞-algebras, we can set $HH^*(A,A) = H^*\left(\prod_{i,j} C_j^{i,*-i-j}(A,A), D\right)$, where D is built from [,]. When A is a bidga, we can split $HH^*(A,A) = \prod_s H_{bidga}^{s,*-s}(A,A)$.

Theorem (R-Whitehouse)**.** *Let C is a dga and let (E,m) be its minimal model. Write $\tilde{E} = (E, m_{11} + m_{02})$ for the underlying bidga. If $HH_{bidga}^{n,2-n}(\tilde{E},\tilde{E}) = 0$ for $n \geq 3$, then C is intrinsically formal, i.e. it is the only dga (up to quasi-isomorphism of dgas) realizing H^*C.*

5 Work in progress

This is joint with S. Whitehouse and M. Livenet. We would like to understand the foregoing material in an operadic context.

We work in the category $\mathcal{C}$ of $\mathbb{K}$-bicomplexes with $m_{11} = 0$, which is equivalent to the category of $\mathbb{N}$-graded chain complexes. We have the operad $dAs = P(\mathbb{K}m_{11} \oplus \mathbb{K}m_{02}; \{\text{relations}\})$ which encodes bidgas. We must check that:

- dAs is Koszul, from which it follows that $\Omega(dAs^{¡}$ is a minimal model of dAs;
- derived A_∞-algebras are $\Omega(dAs^{¡})$-algebras.