Koszul duality for operads

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(This is in Chapters 6 and 7 of Loday-Vallette; we'll do exercises 1,7,8,9,10.)

1 Motivation

We are interested in an arbitrary of P-algebras. We would like to understand the homotopy category of this category, and we would like to carry through the HTT. However, in general these categories do not have nice homotopy properties. The problem is that the category of P-algebras may be represented by an operad P which is not cofibrant. Thus, we replace P with P_∞, a cofibrant replacement, and we consider the category of P_∞-algebras. This category does have nice homotopy properties. We obtain the following diagram:

$$\begin{array}{ccc} P & \xleftarrow{\sim} & P_\infty \\ \downarrow & & \downarrow \\ P\text{-}\mathbf{alg} & \rightarrowtail & P_\infty\text{-}\mathbf{alg}. \end{array}$$

So we can still hope to understand P-algebras by understanding P_∞-algebras. Thus, we will look for explicit quasi-free resolutions of operads.

2 Operadic homological algebra

Let C be a cooperad and P be an operad. We can form the convolution operad $P^C = \mathrm{Hom}(C, P)$, in which we can look at the Maurer-Cartan equation.

Lemma. *For any operad Q, we can put an operation $\star$ on $\prod_{n\geq 0} Q(n)$ as follows. If $\mu \in Q(m)$ and $\nu \in Q(n)$, then $\mu \star \nu = \sum_{i=1}^{n} \mu \circ_i \nu$.*

Exercise 1. Check that this is a preLie algebra.

We apply this lemma to $Q = \mathrm{Hom}(C, P)$.

Proposition. *If we endow $\mathrm{Hom}(C, P) = \prod_{n\geq 0} \mathrm{Hom}(C(n), P(n))$ with a bracket defined by $[\mu, \nu] = \mu \star \nu - (-1)^{|\mu||\nu|}\nu \star \mu$, then we obtain a Lie algebra.*

Unfortunately we have not yet taken the quotient with respect to the symmetric action. However, the bracket does indeed descend to the quotient. On the convolution $\mathrm{Hom}(C, P)$ operad we define $\partial(f) = d_P f - (-1)^{|f|} f d_C$, and this becomes a dg Lie algebra.

Definition 1. A *twisting morphism* is an element $\alpha \in \mathrm{Hom}_{\mathbb{S}}(C, P)_{-1}$ satisfying the Maurer-Cartan equation $\partial\alpha + \frac{1}{2}[\alpha, \alpha] = 0$.

Let us assume our operads are augmented and coaugmented: $C = \overline{C} \oplus I$ and $P = \overline{P} \oplus I$, and we require that our Maurer-Cartan solutions satisfy $aug_P \circ \alpha = \alpha \circ coaug_C = 0$.

We would like to represent and corepresent functor giving the set of twisting morphisms. We first do this in the categories of operads and cooperads for the convolution, as follows:

$$\begin{array}{ccccc} \mathrm{Hom}_{op}(\mathcal{T}(s^{-1}C), P) & \cong & \mathrm{Hom}_{\mathbb{S}}(C, P)_{-1} & \cong & \mathrm{Hom}_{coop}(C, \mathcal{T}^c(s\overline{P})) \\ & & \uparrow & & \\ \mathrm{Hom}_{dg\ op}(?, P) & \cong & Tw(C, P) & \cong & \mathrm{Hom}_{dg\ coop}(C, ?). \end{array}$$

Here, $\mathcal{T}C$ is the free operad on C, which functor is left adjoint to the forgetful functor from operads to $\mathbb{S}$-modules. For an $\mathbb{S}$-module E, this satisfies

$$\begin{array}{ccc} E & \longrightarrow & \mathcal{T}(E) \\ & \searrow^{f} & \vdots\ \exists! \tilde{f} \\ & & P \end{array}$$

for any operad P. Explicitly, we define $\mathcal{T}(E)$ to be the direct sum over rooted trees whose interior vertices are labelled by $E(n)$, where n is the number of edges extending upwards. The partial compositions are given by the grafting of trees. Note that this has a natural weight grading, given by the number of vertices. Denote by $\mathcal{T}(E)^{(k)}$ those trees with k vertices.

Now, consider the cobar construction. Let d_2 be the unique derivation which extends

$$s^{-1}\overline{C} \overset{\Delta_{(1)}}{\to} s^{-1}\mathcal{T}(\overline{C})^{(2)} \overset{s^{-1}}{\to} \mathcal{T}(s^{-1}\overline{C})^{(2)} \hookrightarrow \mathcal{T}(s^{-1}\overline{C})$$

where $\Delta_{(1)}$ is the partial decomposition product. So d_2 applied to a graph is the sum over vertices of decomposing the edges coming in according to the cooperad structure: this is what Kontsevich calls the "vertex expansion".

We would like to construct d_1, the unique derivation which extends $s^{-1}\overline{C} \overset{d_C}{\to} s^{-1}\overline{C} \hookrightarrow \mathcal{T}(s^{-1}C)$.

Exercise 2. Find d_1.

Definition 2. We define the *cobar construction* by $\Omega C = (\mathcal{T}(s^{-1}\overline{C}), d = d_1 + d_2)$.

And this is the object that corepresents: $Tw(C, P) \cong \mathrm{Hom}_{dg\ op}(\Omega C, P)$.

Exercise 3. Dualize the story to obtain $Tw(C, P) \cong \mathrm{Hom}_{dg\ coop}(C, BP)$.

3 Method

Given a dg operad P, we would like to find a quasi-free resolution. The cobar construction is indeed quasi-free, so given P we would like to find C (the "generators" of "syzygies" of P) along with a map $f_\alpha \in \mathrm{Hom}_{dg\ op}(\Omega C, P)$ corresponding to $\alpha \in Tw(C, P)$.

But how can we check that this is a quasi-isomorphism? We would like to recognize the quasi-isomorphisms among all of $\mathrm{Hom}_{dg\ op}(\Omega C, P) \cong Tw(C, P) \cong \mathrm{Hom}_{dg\ coop}(C, BP)$.

Definition 3. The *twisted composite product*, which is the correct analog of the tensor product, is defined by $C \circ_\alpha P = (C \circ P, d_\alpha = d^r_\alpha + d_{C \circ P})$. This twist is defined by

$$d^r_\alpha : C \circ P \overset{\Delta_{(1)} \circ P}{\to} \mathcal{T}(C)^{(2)} \circ P \to \cdots \to C \circ P.$$

Exercise 4. If $\alpha \in Tw(C, P)$, then $d^2_\alpha = 0$. (This is always equal to the expression in the Maurer-Cartan equation.)

Definition 4. If $C \circ_\alpha P$ is acyclic, then α is called a *Koszul morphism*.

And finally we obtain $Kos(C, P) \subset Tw(C, P)$, and these correspond to the quasi-isomorphisms.

4 Koszul duality

Suppose we are given quadratic data, i.e. a pair (E, R) of an $\mathbb{S}$-module E and a subspace $R \subset \mathcal{T}(E)^{(2)}$ of quadratic relations.

Example. The toy model (in the nonsymmetric case) to keep in mind is E is a single binary operation xy and $R(x, y, z) = (xy)z - x(yz)$.

As we have seen before, the quadratic operad is defined by $P(E, R) = \mathcal{T}(E)/(R)$. (In our toy example, we recover the operad As.)

Alternatively, we can dualize the story to get a quadratic cooperad $C(E, R)$. (We will not explicitly define this, but it exists.)

Definition 5. The *Koszul dual cooperad* is $P^{¡} = C(sE, s^2R)$. There is a natural choice of twisting morphisms $\kappa \in Tw(P^{¡}, P)$ given by $\kappa : P^i \twoheadrightarrow sE \overset{s^{-1}}{\to} E \hookrightarrow P$. Then, $P^{¡} \circ_\kappa P$ is called the *Koszul complex*, and this is acyclic iff $\Omega P^i \overset{\sim}{\to} P$.

5 Applied Koszul duality theory

Definition 6. The *Koszul dual operad* is defined by

$$P^! = \mathrm{End}_{\mathbb{K}s^{-1}} \otimes_H (P^{¡})^*.$$

Here the first factor is the suspension operad, and we define $(P \otimes_H Q)(n) = P(n) \otimes Q(n)$.

Proposition. *Suppose* $\dim E < \infty$ *and* E *is binary, i.e. it has nothing outside of arity 2. Then* $P^! = P(E^* \otimes sgn_{S_2}, R^\perp)$ *is quadratic, and* $P^{!^!} = P$.

Example. Let us consider the nonsymmetric operad As. Then $\mathcal{T}(E(2))$ looks like $(xy)z \oplus x(yz)$, and $R \subset \mathcal{T}(E(2))$. If a basis for E is given by $\{e_i\}_{i=1,\dots,e}$ with dual basis for E^* given by $\{e_i^*\}_{i=1,\dots,e}$, we get a scalar product $\langle\ ,\ \rangle$ on $\mathcal{T}(E)^{(2)} \otimes \mathcal{T}(E^*)^{(2)}$ which on two left-com trees gives 1, on two right-com trees gives -1, and on trees of different shapes gives 0. Now, the Koszul dual has that $((xy)z - x(yz))^\perp = (xy)z - x(yz)$; thus $As^! = As$.

Theorem. $Com^! = Lie$.

To prove that this is Koszul, we must compute that $Lie \circ_\kappa Com^*$ is acyclic. But this was already done in the rational homotopy theory lecture!

Lastly, we consider the little discs operad D_2. This is a topological operad, so when we pass to chains we get a dg operad. Thus $H_*(D_2)$ is the free operad $\mathcal{G}$ for *Gerstenbaber algebras*.

Exercise 5. Compute the generators of $H_*(D_2)$. To begin, the generators of $D_2(2)$ are configurations of two discs inside the disc, and this is homotopy equivalent to S^1. This gives us $H_0(S^1) = \bullet \oplus \langle\ ,\ \rangle$ in degrees 0 and 1, and we claim that these give generators for the Gerstenbaber algebras.

Now, consider framings of one point on each disc. This gives the *framed* little discs operad fD_2. Then for instance $H_0(fD_2(1))$ has a framed disc inside the disc, and these give generators for the *Batalin-Vilkovisky algebra*. The relations here are no longer homogeneous, though, so our framework does not carry through as it stands.

Remark. $C(L) = \Lambda^c(L)$ *is a BV-algebra.*

6 Homotopy theory for algebras

This is in Chapter 9.

Our goals for today are:

- HTT
- $Ho(As - alg) \cong \infty - As - alg/\sim$

Everything will be dg-graded today.

Here is our **Rosetta Stone**: Let P be a Koszul operad and let A be a chain complex. We would like to understand $\{P_\infty$**-alg** structure on $A\}$. We have the string of natural isomorphisms

$$\{P_\infty\text{-}\mathbf{alg} \text{ structure on } A\} \cong Tw(P^{¡}, \mathrm{End}_A) \cong \mathrm{Hom}_{dg\ coop}(P^{¡}, B\mathrm{End}_A) \cong Codiff(P^{¡}(A)).$$

The first can be taken as a definition, and this is very nice because Tw is a deformation theoretic object. $\mathrm{Hom}_{dg\ coop}$ takes us to the language of the HTT, and $Codiff$ brings us to $\infty-$morphisms.

Let us revisit the bar-cobar adjunction. For any twisting morphism $\alpha \in Tw(C, P)$, we have B_α : P**-alg** $\leftrightarrows$ C**-coalg** : Ω_α. For any P-algebra A, we define $B_\alpha A = (C \circ_\alpha P) \circ_\rho A$ as the coequalizer

$$(C \circ_\alpha P) \circ P \circ A \overset{C\circ\gamma_\rho\circ A}{\rightarrow} \rightarrow C \circ P \circ \gamma_A(C \circ P) \circ A \twoheadrightarrow (C \circ_\alpha P) \circ_P A.$$

Explicitly, this gives us $(C(A), d)$, where $C(A)$ is the cofree C-coalgebra on A, and d is given by **DIAGRAM**.

Example. Let $P = As$, $C = As^{¡}$, and $\alpha = \kappa$. Then we have B : *assoc alg* $\leftrightarrows$ *coassoc coalg* : Ω, the classical bar-cobar adjunction.

Example. Let $P = Lie$, $P^{¡} = Lie^{¡} = (\mathcal{S})Com^*$, and $\alpha = \kappa$. Then we have C_* : *Lie alg* :$\leftrightarrows$ *cocom coalg* : L_*. The counit of the adjunction is $(P \circ_\alpha C)(A) = \Omega_\alpha B_\alpha A \to A$. This is a quasi-isomorphism iff $\alpha \in Kos(C, P)$. This gives us a functorial cofibrant replacement for P-algebras, which gives us derived functors, etc.

6.1 ∞-morphisms

We will apply our bar-cobar adjunction to $B_\kappa : P\textbf{-alg} \leftrightarrows P^{\text{¡}}\textbf{-coalg} : \Omega_\kappa$. Note that sitting inside $P^{\text{¡}}\textbf{-coalg}$ we have the quasi-free $P^{\text{¡}}\textbf{-coalg}$.

Definition 7. A *quasi-free* $P^{\text{¡}}$*-coalgebra* is $(P^{\text{¡}}(A), d)$, where $P^{\text{¡}}(A)$ is a free P-coalgebra and d is a *codifferential* which is a coderivation satisfying $d^2 = 0$.

Proposition. *The data of a codifferential on* $P^{\text{¡}}(A)$ *is equivalent to a* P_∞*-algebra structure on* A.

Proof. A coderivation is completely characterized by $P^{\text{¡}}(A) \to P(A) \to A$, which is equivalent to $P^{\text{¡}} \to \text{End}_A$. Since $d^2 = 0$ this is equivalent to an element of $Tw(P^{\text{¡}}, \text{End}_A)$. ☐

Example. Let $P = As$, $As^{\text{¡}}(-) = \mathcal{S}^* \otimes_H As^*$, $P^{\text{¡}}(A) = \overline{T}^c(sA)$. Now, $d : \overline{T}^c(sA) \to sA$ (which has degree -1 is completely characterized by $\{(sA)^{\otimes n} \overset{\tilde{\mu_n}}{\to} sA\}_{n\geq 2}$, where $|\tilde{\mu}_n| = -1$. This is in turn equivalent to $\{A^{\otimes n} \overset{\mu_n}{\to} A\}_{n\geq 1}$, where $|\mu_n| = n-2$. That $d^2 = 0$ is equivalent to the fact that the sum of all n-ary partial compositions of the μ_i is 0.

Definition 8. An ∞*-morphism* between two P_∞-algebras is a dg morphism of dg $P^{\text{¡}}$-coalgebras $(P^{\text{¡}}(A), d) \to (P^{\text{¡}}(B), d)$. Such data is completely characterized by $P^{\text{¡}}(A) \to B$ (which must satisfy certain conditions). This is equivalent to a map of $\mathbb{S}$-modules $P^{\text{¡}} \to \text{End}^A_B = \{\text{Hom}(A^{\otimes n}, B)\}_{n\geq 2}$.

This gives us an equivalence $\tilde{B}_\iota : \infty - P_\infty\textbf{-alg} \overset{\cong}{\to} P^{\text{¡}}\textbf{-coalg}$. (Note that ∞-morphisms are made up of many maps.)

Definition 9. An ∞-morphism is an ∞*-quasi-isomorphism* if $A \cong I(A) \hookrightarrow P^{\text{¡}}(A) \to B$ is a quasi-isomorphism. We write $A \rightsquigarrow B$.

6.2 HTT

Definition 10. A *homotopy retract* is $h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\leftrightarrows}} (H, d_H)$ such that $pi = \text{Id}_H$ and $ip - \text{Id}_A = hd_A + d_A h$, such that i is a quasi-isomorphism.

Theorem. *Any* P_∞*-algebra structure* μ *on* A *transfers to a* P_∞*-algebra structure* N *on* H *such that* i *extends to an* ∞*-quasi-isomorphism with* **???**.

Proof. Use the Rosetta Stone. A P_∞-algebra structure on A is a map in

$$\text{Hom}_{dg\ op}(\Omega P^{\text{¡}}, \text{End}_A) \cong Tw(P^{\text{¡}}, \text{End}_A) \cong \text{Hom}_{dg\ coop}(P^{\text{¡}} B\text{End}_A),$$

and we consider $\mu \in Tw(P^{\text{¡}}, \text{End}_A)$. We want to map down to the same diagram for H. We have a map $\Phi : \text{Hom}_{dg\ coop}(P^{\text{¡}}, B\text{End}_A) \to \text{Hom}_{dg\ coop}(P^{\text{¡}}, B\text{End}_H)$. We consider μ as associated to g_μ, and then we should have $g_\nu = \Phi \circ g_\mu$. We need the following result.

Lemma. *The following map induces a morphism of dg cooperads:*

$$B\text{End}_A = T^c(s\text{End}_A) \to s\text{End}_H.$$

Proof. On the left side we have labelled trees, and we will map this to the same tree but using $i : H \to A$ at the leaves and p at the root, and using the homotopy h on the internal edges. ☐

☐

We can obtain an explicit construction for ν:

$$N : P^{\text{¡}} \overset{\Delta_{iterated}}{\to} T(P^{\text{¡}}) \overset{T(s\mu)}{\to} T(s\text{End}_A) \overset{s^{-1}\varphi}{\to} \text{End}_H.$$

7 The category $Ho(P\textbf{-alg})$

Theorem. *For any ∞-quasi-isomorphism $A \overset{\sim}{\rightsquigarrow} B$, there exists an ∞-quasi-isomorphism $B \overset{\sim}{\rightsquigarrow} A$.*

Note that these are very different from our usual qi's.

Proof. An ∞-isomorphism is an invertible map $A \overset{\cong}{\rightsquigarrow} B$. We now have the sequence

$$B \overset{p^*}{\rightsquigarrow} H_*(B) \overset{H_*(F)^{-1}}{\rightsquigarrow} H_*(A) \overset{i_0}{\rightsquigarrow} A\,.$$

□

Thus, being ∞-quasi-isomorphic defines an equivalencerelations, which we call *homotopy equivalence*. For example, we write $(H_*(A), 0) \overset{\sim}{h.e.} (A, d_A)$ for a P_∞-algebra A.

Theorem (rectification)**.** *For any P_∞-algebra A, $A \overset{\sim}{\rightsquigarrow} \Omega_k \tilde{B}_L A$; the target is a strict P-algebra.*

Corollary. *Let A and B be P-algebras. Then there is a zigzag of quasi-isomorphisms $A \overset{\sim}{\leftarrow} \cdot \overset{\sim}{\rightarrow} \cdots\cdot \overset{\sim}{\leftarrow} \cdot \overset{\sim}{\rightarrow} B$ iff there is an ∞-quasi-isomorphism $A \overset{\sim}{\rightsquigarrow} B$.*

Thus we begin to develop the following theorem.

Theorem. $Ho(P\textbf{-alg}) \cong Ho(\infty P\textbf{-alg}) \cong \cdots \cong Ho(\infty P_\infty\textbf{-alg})$.

Theorem. *THere exists a model category structure on P^{i}**-coalg** such that:*

- *weak equivalences are maps $f : C \to D$ such that $\Omega_\kappa(f)$ is a quasi-isomorphism;*
- *B_κ, Ω_κ is a Quillen equivalence;*
- *the fibrant-cofibrant objects are the quasi-free P^{i}-coalgebras;*
- *$\sim_h$, and a good cylinder is given by the Laurence-Sullivan model for the interval $\hat{Lie}(0, 1, I)$ with*

$$\partial I = \mathrm{ad}_I(1) + \sum_{n\geq -1}^{\infty} \frac{B_n}{n!} \mathrm{ad}_I^n(1, 0).$$

This last point is what opened the door to deformation theory.

8 Deformation theory of algebras

8.1 The convolution dg Lie algebra

Recall our second definition: $\mathfrak{g} = \mathrm{Hom}_{\mathbb{S}}(p^{\text{i}}, \mathrm{End}_A), [\ ,\ ], \partial)$ a dg Lie algebra, $MC(\mathfrak{g}) = \{P_\infty$**-alg** structures on $A\}$.

Example. Let $P = As$. Then $\mathfrak{g} = \mathrm{Hom}_\emptyset(As^{\text{i}}, \mathrm{End}_A)$ (where $\emptyset$ denotes that we are looking at the nonsymmetric case). Then, this is $\prod_{n\geq 1} \mathrm{Hom}(As^{\text{i}}(n), \mathrm{Hom}(A^{\otimes n}, A)) = \prod_{n\geq 1} \mathrm{Hom}(A^{\otimes n}, A)$. This looks a lot like the Hochschild cochain complex. Suppose $\partial\alpha + \frac{1}{2}[\alpha, \alpha] = 0$. Then $P^{\text{i}} = C(sE, s^2R) = I \oplus sE \oplus s^2R \oplus \cdots$, which is weight-graded and injects into $\mathcal{T}^c(sE)$ by using the number of vertices. This takes us to $\mathfrak{g}^{(n)} = \mathrm{Hom}(P^{(n)}, \mathrm{End}_A)$, a weight-graded dg Lie algebra.

Proposition. *MC elements concentrated in weight 1 correspond to to P-algebra structures.*

8.2 Deformation theory

Given $\alpha \in MC(\mathfrak{g})$, we can *deform* this by $\varphi \in \mathfrak{g}_{-1}$ whenever $\alpha + \varphi \in MC(\mathfrak{g})$. In general, we can put for φ the maximal ideal $\mathfrak{m}$ of our artinian ring.

Definition 11. A *twisted differential* ∂_α is given by $\partial_\alpha = \partial + [\alpha, -]$. (This squares to zero, and is a derivation with respect to the bracket.) A *Twisted dg Lie algebra* has the same underlying space and bracket but has a new differential: $\mathfrak{g}^\alpha = (\mathfrak{g}, [\ ,\ ], \partial_\alpha)$. Then, $(\mathfrak{g}, \partial_\alpha)$ is called the *deformation complex*, and $H_*(\mathfrak{g}, \partial_\alpha)$ is called the *tangent homology.* The bracket $[\ ,\ ]$ on the tangent homology is called the *intrinsic Lie bracket.*

Example. Take $P = As$. Write $\alpha(\text{Y-graph}) = \mu$, and for $f \in \mathrm{Hom}(A^{\otimes n}, A)$ we have $\partial_\alpha(f) = \alpha \star f \pm f \star \alpha$. We must apply this to corollas, and (with the proper labellings) we get $\partial_\alpha(f) = ((xxx)x) \pm (x(xxx)) \pm \sum_i (x \ldots (xx)x)$ in $\mathrm{Hom}(A^{\otimes n+1}, A)$. This is just the Hochschild differential. Moreover, the bracket is just the Gestenhaber bracket.

Example. Take $P = Lie$, and set $\mathfrak{g}_\alpha = (\coprod_n (\Lambda^n A, A), [\ ,\ ], \partial_\alpha)$; the first and third components here were found by Chevalley-Eilenberg, and the bracket is the Nijenhuis-Richardson bracket.

Proposition. *Consider the deformations* $Def(\mathbb{K}[\epsilon]/(\epsilon^2))$*. This is* $H_{-1}(\mathfrak{g}^\alpha)$*, and is also* $MC(\mathfrak{g}^\alpha \otimes \epsilon)/\sim$*, the infinitesimal deformations.*

Proof. Note that $\alpha + \varphi \in MC(\mathfrak{g})$ iff $\varphi \in MC(\mathfrak{g}^\alpha)$. Moreover, $\alpha + a\varphi \in MC(\mathfrak{g})$ iff $\varphi \in MC(\mathfrak{g}^\alpha \otimes^a \mathfrak{m})$. □

The picture to keep in mind is a quadric surface cut out by $MC(\mathfrak{g})$, and the tangent space at α is $T_\alpha MC(\mathfrak{g}) = H_{-1}(\mathfrak{g}^\alpha)$.

8.3 Hot topics in deformation theory

8.3.1 Deligne conjecture

We have $CH^*(A, A)$ endowed with $[\ ,\ ]$ and $\smile$, which descends to the Gestenhaber algebra $HH^*(A, A)$ with bracket and commutative product. Deligne asked: can we lift this to an action of the operad G_∞ on $CH^*(A, A)$? This is basically the opposite of the HTT.

Then, the BRST complex acts on the topological vertex operator algebra. Taking homology gives us a BV-algebra. The question is: can we lift this to a BV_∞ algebra structure? The answer is yes.

8.3.2 Deformation quantization of Poisson manifolds

Given a Poisson manifold M, we have $(C^\infty(M), \cdot, [\ ,\ ])$. The question is: Can we extend to a $\star$-associative algebra? This would be $f \star g = f \cdot g + B_1(f, g)t + B_2(f, g)t^2 + \cdots$ on $C^\infty(M)[[t]]$ Konsevich directly constructed an L_∞-isomorphism. A few month later, Tamarkin proved it beautifully using very different methods.