Minimal Koszul models

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This is a simultaneous generalization of Koszul algebras and Sullivan models. We will quickly cover a bit of the theory, and then proceed to give example applications.

1 Theory

Let $\alpha : C \to P$ be a twisting morphism. This gives us the bar/cobar constructions as an adjunction $\mathrm{Hom}_{P\text{-}\mathbf{alg}}(\Omega_\alpha C, A) \cong Tw_\alpha(C, A) \cong \mathrm{Hom}_{C\text{-}\mathbf{coalg}}(C, B_\alpha A)$. Then, κ is a Koszul twisting morphism iff ϕ_κ and ψ_κ are quasi-isomorphisms.

Now for P a binary Koszul algebra, $A = P[V]/(R)$ for $R \subset P(2) \otimes_{\Sigma_2} V^{\otimes 2}$, and we get $A^! \cong P^!((sV)^*)/(R^\perp)$.

We set up our examples.

Example. We have $\alpha : (sCom)^* \to Lie$. We can formulate the relationship between Sullivan's and Quillen's approaches in this language. If $(\Lambda V, d)$ is a Sullivan model for some space X, then $\kappa : (s(\Lambda, d))^* \to \lambda(X)$, where $\lambda(X)$ is Quillen's DGL.

Example. We have $\alpha(sAs)^* \to As$, and this gives us $\kappa : C_*(X) \to X_*(\Omega X)$ (from a DGC to a DGA).

Example. Let us take $E_n \sim C_*(\mathcal{D}_n)$ and $G_n = H_*(\mathcal{D}_n)$. Then by a recent theorem of Benoit Fresse, we have $\alpha : (s^n E_n)^* \to E_n$. For $n = 1$, we recover the example above. Now, if X is X is an E_n-algebra, then $C_*(X)$ carries an E_n-coalgebra structure, and we have $\kappa : C_*(X) \to C_*(\Omega^n X)$ (from an E_n-coalgebra to an E_n-algebra).

Theorem. *If $\kappa : C \to A$ is a Koszul twisting morphism of dg objects, then the following are equivalent:*

1. *A and C are formal.*
2. *A is formal and $H_*(A)$ is a Koszul P-algebra.*
3. *C is formal and $H_*(C)$ is a Koszul C-algebra.*

Corollary. *An algebra A is Koszul iff $B_\alpha A$ is formal.*

2 Examples

2.1 Rational homotopy theory

In our first example we obtain the following result.

Theorem. *Suppose X is a 1-connected space of finite type. Then the following are equivalent:*

1. *X is formal (i.e. $\mathcal{A}^*_{PL}(X) \sim H^*(X;\mathbb{Q})$ as CDGAs) and X is coformal (i.e. $\lambda(X) \sim \pi_*(\Omega X) \otimes \mathbb{Q}$ as DGLs).*
2. *X is formal and $H^*(X;\mathbb{Q})$ is a Koszul commutative algebra.*
3. *X is coformal and $\pi_*(\Omega X) \otimes \mathbb{Q}$ is a Koszul Lie algebra.*

A space satisfying these conditions will be called a Koszul space. Moreover, in this situation, $\pi_*(\Omega X) \otimes \mathbb{Q} \cong H^*(X;\mathbb{Q})^{!\ Lie}$.

One might say that in this example, Eckmann-Hilton duality reduces to Koszul duality.

If X is n-connected, then we obtain that A_n-algebra $H_*(\Omega^n X;\mathbb{Q}) \cong H^*(X,\mathbb{Q})^{!\ G_n}$.

Example. Recall that $\mathcal{D}_n(k) \simeq F(\mathbb{R}^n, k) = \{(x_1, \dots, x_k) \in (\mathbb{R}^n)^k | x_i \neq x_j\}$. This is a Koszul space. It has cohomology

$$H^*(F(\mathbb{R}^n, k); \mathbb{Q}) = \frac{\Lambda(\{a_{pq} : 1 \leq p < q \leq k, |a_{pq}| = n-1\})}{(a_{pq}^2 = 0,\ a_{pq}a_{qr} + a_{qr}a_{rp} + a_{rp}a_{pq} = 0)}.$$

These are called the *Arnold relations.* On the other side,

$$\pi_*(\Omega F(\mathbb{R}^n, k)) \otimes \mathbb{Q} = \frac{\mathbb{L}(\{\alpha_{pq} : 1 \leq p < q \leq k, |\alpha_{pq}| = n-2\})}{([\alpha_{pq}, \alpha_{rs}] = 0,\ [\alpha_{pq}, \alpha_{pr} + \alpha_{qr}] = 0 \text{ for } \{p,q\} \cap \{r,s\} =)}.$$

These are called the *infinitesimal braid relations.*

But what if X is not formal or coformal? To analyze this situation, we begin with the following definition.

Definition 1. A *minimal Koszul model* for X (which is 1-connected and of finite type) is a CGDA (A, d) such that:

- $(A, d) \sim \mathcal{A}_{PL}^*(X)$ in **CDGA**.
- A is a Koszul commutative algebra.
- d strictly increases weight.

Theorem. *Let (A, d) be a minimal Koszul model for X. Then there is a minimal L_∞-algebra structure $l = \{l_n\}_{n \geq 2}$ on $A^{!\ Lie}$ and an L_∞ quasi-isomorphism*

$$(A^!\ Lie, l) \overset{\sim}{\rightsquigarrow} \lambda(X).$$

In particular, there is an isomorphism of graded Lie algebras $\pi_*(\Omega X) \otimes \mathbb{Q} \cong (A^{!\ Lie}, l_2)$. Thus in a sense, we are lifting the previous theorem to the not-necessarily-formal case. Here, l_2 depends only on the quadratic part d_2 of d.

Remark. *Any minimal Sullivan model is a Koszul model. Indeed, given $(\Lambda V, d)$, ΛV is a Koszul commutative algebra, and $\Lambda V^{!\ Lie} = (sV)^*$ (with abelian Lie structure). We recover the fact that $\pi_n(X) \otimes \mathbb{Q} \cong (V^n)^*$ from the theorem, since $\pi_*(\Omega X) \otimes \mathbb{Q} \cong (sV)^*$.*

This theorem gives us new information, too.

Example. Let X be a formal space with $H^*(X; \mathbb{Q}) = \Lambda(u_3, v_3, w_3)/(uvw)$. This cohomology is not quadratic so it cannot be a Koszul algebra, so X cannot be coformal.

Let us try to construct a Sullivan model. We resolve $H^*(X; \mathbb{Q})$ into a free algebra as follows. We need $y_8 \mapsto uvw$ to kill the relation, but then we get nontrivial cycles uy, vy, and wy. So we introduce $a_{10} \mapsto uy$, $b_{10} \mapsto vy$, $c_{10} \mapsto wy$. But this gives us the new cycles $va + ub$, $wa + uc$, $wb + vc$. The killing never stops! This is unpleasant. To continue this the killing would need to go on forever.

So instead, let us construct a Koszul model. The idea is that we stop resolving when we have Koszul relations. This gives us $\Lambda(u, v, w, y)/(uy, vy, wy, y^2)$, with only nonzero differential $dy = uvw$. Thus, $\pi_*(X) \otimes \mathbb{Q} \cong (A^{!\ Lie}, l_2) \cong \mathbb{L}[\alpha, \beta, \gamma, \xi]/([\alpha, \beta], [\alpha, \gamma], [\beta, \gamma])$. (This is finitely generated, despite the fact that it is infinite-dimensional.

If a space admits finitely-generated Koszul model, then **???**.

2.2 Hochschild cochains

Let $P = As$, and let R be an associative ring.

Theorem L (e). *t (A, d) be a minimal Koszul model for R. Then there is an A_∞-algebra structure $m = \{m_n\}_{n \geq 1}$ on $A^! \hat{\otimes} A$ and an A_∞-quasi-isomorphism $\mathbb{C}^\bullet(R, R) \overset{\sim}{\rightsquigarrow} (A^! \hat{\otimes} A, m)$. If $d = 0$ (or $R = A$ then m reduces to $m_1 = \kappa, -]$, $m_a m_2$ is the standard multiplication, and $m_1 = 0$ for $n > 2$.*

Corollary. *If A is Koszul, then $C^\bullet(A, A) C^\bullet(A^!, A^!)$ as A_∞ algebras.*

Example. Let $R = \mathbb{Z}[x]/(x^{n+1})$ with $|x| = -2$, $|y| = -2n-1$. Then we have the Koszul model $(\Lambda_{\mathbb{Z}}(x, y), dy = x^{n+1}) \overset{\sim}{\to} R$, and $\Lambda(x, y)^! \cong \Lambda(\alpha, \beta)$ with $|\alpha| = 1$ and $\beta = 2n$. The A_∞-algebraic model for $C^\bullet(R, R)$ looks like $(\Lambda(x, y) \hat{\otimes} \Lambda(\alpha, \beta), m)$ with

$$m_1(y) = x^{n+1},\ m_1(\alpha) = (n+1)\beta x^n$$
$$m_r(\alpha, \dots, \alpha) = \binom{n+1}{r} \beta x^{n+1-r} \text{ for } 1 \leq r \leq n+1$$

(and m_r is linear in x, y, β for $r \geq 2$. The generating cyclies are $x, \beta, \zeta = (n+1)\beta y - \alpha x$ with relations $x^{n+1} = m_1(y)$, $(n+1)\beta x^n = m_1(\alpha)$, $3x^n = m_1(\alpha y)$. This all gives that

$$\mathbb{H}_*(L\mathbb{C}P^n) = HH^*(R,R) \cong \frac{\Lambda(x_{-2}, \beta_{2n}, \zeta_{-1}}{(x^{n+1}, 3x^n, (n+1)\beta x^n)}.$$