Model Categories

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Model categories are a way of abstracting homotopy theory. Whenever one has abstractions, one must have examples.

1 Examples

Example (Classical homotopy theory (**Top**))**.** There are two kinds of equivalences: homotopy equivalences and weak homotopy equivalences (for which definition we require f_* to be an isomorphism for all basepoints in the domain). We then have the famous

Theorem (Whitehead)**.** *: A weak equivalence of cell complexes is actually a homotopy equivalence.*

Here, a cell complex is slightly more general than a CW-complex. A *relative cell complex* is as follows. Suppose $A \subset X$. Then (X, A) is a relative cell complex is there a sequence of spaces X_n such that $X_0 = A$ and

$$\begin{array}{ccc} \coprod S^{i_n-1} & \longrightarrow & X_{n_1} \\ \downarrow & & \downarrow \\ \coprod D^{i_n} & \longrightarrow & X_n \end{array}$$

is a pushout square. (The generalization here is that smaller cells can be glued to higher cells.) Then we require $X = \text{colim } X_n$.

An important fact is that every space is weakly equivalent to a cell complex. Moreover, let (B, A) be a relative cell complex and $f : X \to Y$ be a Serre fibration and weak equivalence. Then in every commutative diagram of the form

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & \overset{l}{\nearrow} & \downarrow f \\ B & \longrightarrow & Y \end{array}$$

there is a lift $l : B \to X$ making the two triangles commutative. (Recall that $f : X \to Y$ is a *Serre fibration* if we always have

$$\begin{array}{ccc} D^n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ D^n \times I & \longrightarrow & Y. \end{array}$$

These have long exact sequences in homotopy groups.)

We define the *homotopy category of topological spaces* $Ho(\mathbf{Top})$ to have $obj(Ho(\mathbf{Top})) = obj(\mathbf{Top})$ with morphisms the homotopy classes of maps: $Ho(\mathbf{Top})(X, Y) = [QX, QY]$, where QX is a cell complex weakly homotopy equivalent to X. (This definition does not depend on the choices of QX and QY, because it does not change the set of homotopy classes of maps.)

Example (Chain complexes of R-modules bounded below by 0: $Ch_{\geq 0}(R)$)**.** There are again two notions of equivalence: a chain map $f : X \to Y$ is a *chain homotopy equivalence* if there exists a chain map $g : Y \to X$ such that $fg \simeq id$ and $gf \simeq id$. Recall that $f, f' : X \to Y$ are *chain homotopic* (and we write $f \simeq f'$) if there exists a map $H : X_* \to Y_{*+1}$ such that $H\partial + \partial H = f - f'$.

We can write this differently. Let X and Y be chain complexes. Define $(X \otimes Y)_n = \oplus_{i+j=n} X_i \otimes Y_j$ with $\partial(x \otimes y) = \partial x \otimes y + (-1)^{|x|} x \otimes \partial y$. Then define the interval complex $R \to R \oplus R$ (in degrees 1 and 0) with differential given by $(1, -1)$. Then a chain map is just given by $\overline{H} : I \oplus X \to Y$.

The second kind of equivalence is called a *quasi-isomorphism*. $f : X \to Y$ is a quasi-isomorphism if it induces isomorphisms $f_* : H_n X \to H_n Y$. Obviously every chain homotopy equivalence is a quasi-isomorphism. Moreover, we have the following

Theorem. *If X, Y are complexes of projectives, then f is a quasi-isomorphism iff it is a chain homotopy equivalence.*

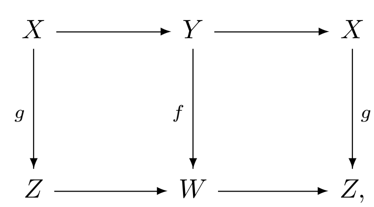
One can similarly define the homotopy category of chain complexes. This is often called the *derived category*. $\mathcal{D}^{\geq 0}(R) = Ho(Ch_{\geq 0}(R))$ has objects those of $Ch_{\geq 0}R$, and $D(R)(X, Y) = [QX, QY]$ where QX is a replacement of X by a module of projectives.

2 Model Categories

We now abstract the common features of these two examples to the notion of a model category. Let $\mathcal{C}$ be a category admitting all small limits and colimits. Then a *model structure* on $\mathcal{C}$ consists of three classes of morphisms: weak equivalences (in our examples, weak homotopy equivalences / quasi-isomorphisms), cofibrations (relative cell complexes / split inclusions (i.e. maps where the cokernel only consists of projectives)), and fibrations (Serre fibrations / degreewise epimorphism above degree 0). These must satisfy some axioms:

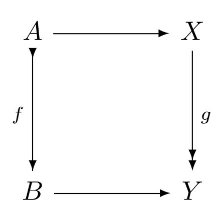
1. *(the 2 out of 3 property)* If $f : X \to Y$ and $g : Y \to Z$ are morphisms then if two of f, g, and gf are w.e.'s then the third is too.

2. *(retract axiom)* In the diagram of retracts



   if f is a weak equivalence / cofibration / fibration, then then same holds for g.

3. *(lifting axiom)* Consider the diagram



   where f is a cofibration and g is a fibration. Then there must be a lift $l : B \to X$ making the two triangles commute if either f or g is a weak equivalence.

4. *(factorization axiom)* Let $f : X \to Y$ be a map. Then there exist a factorization $p \circ i : X \to Z \to Y$ such that i is a cofibration and p is a fibration, and we can choose either i or p to be a weak equivalence.

We call a *trivial fibration* a fibration which is also a weak equivalence. Similarly, we call a *trivial cofibration* a cofibration which is also a weak equivalence.

We say that cofibrations have the *left lifting property* w/r/t all trivial fibrations, and dually all fibrations have the *right lifting property* w/r/t all trivial cofibrations.

This corresponds to the lifting theorem we wrote down in the case of topological spaces.

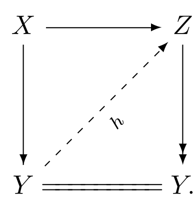
Example. Here are a few examples of model categories:

- topological spaces with weak homotopy equivalences, retracts of relative cell complexes, and Serre fibrations
- chain complexes $Ch_{\geq 0}(R)$ with qi's, maps that are injective and have projective cokernel in each degree, maps that are surjective in every degree above 0
- simplicial sets
- diagram categories
- algebras over topological operads
- ...

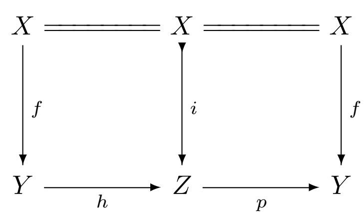
Remark. *We denote an initial object by $\emptyset$ and a terminal object by $*$. Then for any space X we can apply the factorization axiom to the unique maps: $\emptyset \to X$ gives $\emptyset \rightarrowtail QX \overset{\sim}{\twoheadrightarrow} X$. If $\emptyset \to Z$ is a cofibration, then Z is called cofibrant. Similarly, $X \to *$ factorizes to $X \overset{\sim}{\rightarrowtail} RX \twoheadrightarrow X$. Then QX is called the cofibration replacement of X and RX is called the fibrant replacement of X. [Note that in* **Top**, *all objects are fibrant.]*

Proposition. *Every map having the LLP with respect to all trivial fibrations is a cofibration. (Dually, every map having the RLP with respect to all trivial cofibrations is a fibration.)*

Proof. Let $f : X \to Y$ be a map with the LLP w/r/t trivial fibrations. Factorize $X \rightarrowtail Z \overset{\sim}{\twoheadrightarrow} Y$. Then we get the diagram



This gives us the diagram



using the retract axiom. Thus f is a cofibration. □

In a general model category we define a *cylinder object* for $B \in \mathcal{C}$ to be a factorization of $B \coprod B \to B$ as $B \coprod B \rightarrowtail Z \overset{\sim}{\to} B$. (One should of course think of this as a cylinder.) Note that we get two maps $i_0, i_1 : B \to Z$ by the universal property of the coproduct. Then, we define a *homotopy* between $f, g : B \to X$ to be a map $H : Z \to X$ such that $f = Hi_0$ and $g = Hi_1$. A *homotopy equivalence* is defined as usual.

Proposition (generalization of Whitehead). *Let $f : X \to Y$ be a map between cofibrant-fibrant objects. Then f is a homotopy equivalence iff it is a weak equivalence.*

Predefinition: Define the *homotopy category* $Ho(\mathcal{C})$ by $obj(Ho(\mathcal{C})) = obj(\mathcal{C})$ and $Ho(\mathcal{C})(X, Y) = [RQX, RQY] = \mathcal{C}(RQX, RQY)/\sim$. (One can check that RQX is still cofibrant.) We must check 3 things to make sure this makes sense. For cofibrant-fibrant objects:

1. homotopy equivalence is an equivalence relation;
2. $[RQX, RQY]$ is independent of choices;
3. composition is well defined (i.e. if we have homotopies of two composable maps, we get a homotopy of their composition).

Theorem. *Let $f, g : B \to X$. Then:*

1. *If $f \sim g$ and $h : X \to Y$, then $hf \sim hg$.*

2. *If X is fibrant, $h : A \to B$, then $fh \sim gh$.*

3. *If B is cofibrant, is an equivalence relation.*

Assuming the replacements R and Q are actually functors, then the functor $\mathcal{C} \to Ho\,\mathcal{C}$ given by $X \mapsto RQX$ is the "minimal category" where all weak equivalences are sent to isomorphisms.

3 A taste of the proof that Top has a model category structure

Recall from before that we discussed a model structure on **Top**. We have that $p : E \to B$ is a trivial fibration iff it has a RLP with respect to $\{S^{n-1} \to D^n\}_{n\geq 0}$. The heart of today's lecture is the following proposition.

Proposition. *Any map $f : X \to Y$ can be factored as $f = pi$, where i is a cofibration and p is a trivial fibration.*

There is a very general method for proving such statements, called *Quillen's small object argument.*

Proof. We will construct inductively the following factorizations:

$$GAAAAAH$$

We begin with $Z_{-1} = X$. Now suppose we have constructed our diagram through Z_{n-1}. Consider the set $\mathcal{D}$ of all possible commutative diagrams of the form

$$\begin{array}{ccc} S^{m-1} & \longrightarrow & Z_{n-1} \\ \downarrow & & \downarrow \\ D^m & \longrightarrow & Y. \end{array}$$

We then form the pushout

$$\begin{array}{ccc} \coprod_\infty S^{m_\alpha - 1} & \longrightarrow & Z_{n-1} \\ \downarrow & & \downarrow \\ \coprod_\infty D^{m_\alpha} & \longrightarrow & Z_n. \end{array}$$

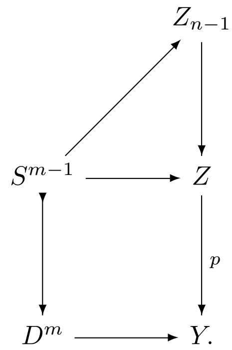
The universal property of pushouts then gives us a map $Z_n \to Y$. Moreover, note that the map $\coprod S^{m_\alpha-1} \to \coprod D^{m_\alpha}$ is a cofibration, so since the pushout of a cofibration is a cofibration then so is the map $Z_{n-1} \to Z_n$. So we define $Z = \mathrm{colim}_n Z_n$. Then we get $X \to Z \to Y$ factorizing f. We know that $X \to Z$ is a cofibration, and so it only remains to show that $Z \to Y$ is a fibration.

Now, recall that showing that $p : Z \to Y$ is a trivial fibration is equivalent to showing that it has the RLP with respect to $\{S^{m-1} \to D^m\}_{m\geq 0}$. So suppose we have a diagram

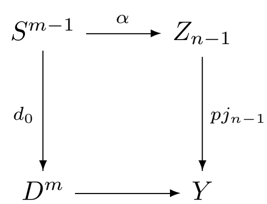
$$\begin{array}{ccc} S^{m-1} & \longrightarrow & Z \\ \downarrow & & \downarrow p \\ D^m & \longrightarrow & Y. \end{array}$$

There is the basic point-set topological fact, coming from the fact that S^{m-1} is compact, that any map $S^{m-1} \to Z$

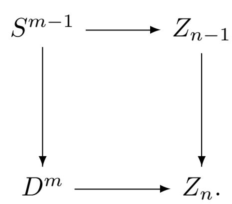
factors through some Z_{n-1}. So this gives us



So now we get the commutative diagram



which is an element of $\mathcal{D}$. This gives us



Thus there exists a lift $D^m \to Z_n$, which when composed with $Z_n \to Z$ gives us our desired lift $D^m \to Z$. □

4 Derived functors & model categories

Let R be any ring and suppose $M \in \mathbf{Mod}\text{-}R$ and $N \in R\text{-}\mathbf{Mod}$. Then we get the *Tor groups* $\mathrm{Tor}_n^R(M, N) = H_n(M \otimes_R P_*)$, where $P_* \to N \to 0$ is a projective resolution of N. We will reformulate this definition in the language of model categories.

The functor $M \otimes_R - : R\text{-}\mathbf{Mod} \to \mathbb{Z}\text{-}\mathbf{Mod}$ induces a functor $M \otimes_R - : Ch_{\geq 0}(R) \to Ch_{\geq 0}(\mathbb{Z})$. We also have the homology functors $H_n : Ch_{\geq 0}(\mathbb{Z}) \to \mathbf{AbGrps}$, and we define the composite functor $F_n^{\otimes} = H_n \circ (M \otimes_R -)$. Thus $\mathrm{Tor}_n^R(M, N) = F_n^{\otimes}(P_*)$.

In the language of model categories, $P_* \to N$ is a projective resolution iff $P_* \to N$ is a trivial fibration and P_* is cofibrant. This motivates the following definition.

Definition 1. Let $F : \mathcal{M} \to \mathcal{A}$ be a functor off a model category $\mathcal{M}$ which sends weak equivalences between cofibrant objects to isomorphisms. Then define the *left derived functor* of F to be the functor $LF : Ho(\mathcal{M}) \to \mathcal{A}$ which takes X to $F(QX)$ for any cofibrant replacement $QX \overset{\sim}{\rightarrowtail} X$.

Our functor $F_n^{\otimes} : Ch(R) \to \mathbf{AbGrps}$ satisfies the hypothesis of the definition, and so admits a left derived functor $LF_n^{\otimes} : \mathcal{D}_{\geq 0}(R) \to \mathbf{AbGrps}$, and now we have that $LF_n^{\otimes}(N) = \mathrm{Tor}_n^R(M, N)$. So $\mathrm{Tor}_n^R(M, -)$ is the composition $R\text{-}\mathbf{Mod} \to Ch_{\geq 0}(R) \to \mathcal{D}_{\geq 0}(R) \to \mathbf{AbGrps}$. So the derived functor $LF_n^{\otimes}$ contains much more information that the Tor groups themselves.

We will indicate why this is independent of cofibrant replacement. This follows from the following lemma.

Lemma (comparison lemma)**.** *Suppose $f : X \to Y$ is any map and that QX and QY are any cofibrant resolutions.*

Then there is a unique $\tilde{f} : QX \to QY$ *up to left homotopy making the diagram*

$$\begin{array}{ccc} QX & \xrightarrow[\sim]{p_X} & X \\ {\scriptstyle \tilde{f}} \downarrow & & \downarrow {\scriptstyle f} \\ QY & \xrightarrow[\sim]{p_Y} & Y. \end{array}$$

commute, and the left homotopy class of $\tilde{f}$ *depends only on the left homotopy class of* f*.*

The easiest thing here is to construct $\tilde{f}$, which we do via the diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & QY \\ \downarrow & \overset{\tilde{f}}{\nearrow} & \downarrow {\scriptstyle \sim\, p_Y} \\ QX & \xrightarrow[fp_X]{} & Y. \end{array}$$

Thus $f \simeq g$ implies that $F(f) = F(g)$.

Definition 2. Suppose we have an adjunction $F : \mathcal{M} \leftrightarrows \mathcal{N} : G$. This is called a *Quillen adjunction* if F preserves cofibrations and trivial cofibrations (or equivalently if G preserves fibrations and trivial fibrations). In this case we call F a *left Quillen functor* and G a *right Quillen functor*.

The equivalence of the definitions follows from the fact that commutative diagrams of the form

$$\begin{array}{ccc} FA & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ FB & \longrightarrow & Y \end{array}$$

are in bijective correspondence to commutative diagrams of the form

$$\begin{array}{ccc} A & \longrightarrow & GX \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & GY. \end{array}$$

Example. There is a self-adjunction $F : Ch(\mathbb{K}) \leftrightarrows Ch(\mathbb{K}) : G$ given by $F = C \otimes_{\mathbb{K}} -$ and $G = \mathrm{Hom}_{\mathbb{K}}(C, -)$ for a fixed complex C. If C happens to be a complex of projectives, then this adjunction is a Quillen adjunction. For example, $\mathrm{Hom}_{\mathbb{K}}(C, -)$ preserves degreewise surjections.

Example. Given any model category $\mathcal{M}$, we have the category $\mathcal{M}^{\omega}$ of sequences $X_0 \to X_1 \to \cdots$ in $\mathcal{M}$, and there is an adjunction $\mathrm{colim} : \mathcal{M}^{\omega} \leftrightarrows \mathcal{M} : \mathrm{const}$ (i.e. $\mathrm{const}(Y) = Y \to Y \to \cdots$). This is a Quillen adjunction.

Example. There is a Quillen adjunction between $|-| : \mathbf{sSet} \leftrightarrows \mathbf{Top} : S(-)$.

We now have the following central theorem.

Theorem (Quillen's total derived functor theorem)**.** *Suppose that we have a Quillen adjunction* $F : \mathcal{M} \leftrightarrows \mathcal{N} : G$*. We can define the composite functors*

$$\mathcal{M} \xrightarrow{F} \mathcal{N} \xrightarrow{\gamma_{\mathcal{N}}} Ho(\mathcal{N})$$

$$\mathcal{N} \xrightarrow{G} \mathcal{M} \xrightarrow{\gamma_{\mathcal{M}}} Ho(\mathcal{M}).$$

These allow us to construct the derived functors $LF : Ho(\mathcal{M}) \to Ho(\mathcal{N})$ *and* $RG : Ho(\mathcal{N}) \to Ho(\mathcal{M})$*. These functors are also adjoint.*

Proof sketch. Suppose (without loss of generality up to isomorphism) that $X \in \mathcal{M}$ and $Y \in \mathcal{N}$ are both cofibrant-fibrant. Then

$$\mathrm{Hom}_{Ho(\mathcal{M})}(X, RG(Y)) \cong \mathrm{Hom}_{Ho(\mathcal{M})}(X, G(Y)) = [X, G(Y)] \cong [F(X), Y],$$

where the last isomorphism comes from the fact that Quillen functors respect homotopies (since they respect cylinder objects). Thus the above is

$$= \mathrm{Hom}_{Ho(\mathcal{N})}(F(X), Y) = \mathrm{Hom}_{Ho(\mathcal{N})}(LF(X), Y).$$

□

Let us return to our examples. From the self-adjunction on $Ch(\mathbb{K})$ we get derived tensor product and derived hom. But this is uninteresting, because these are already their own derived functors! More interesting is the Quillen adjunction colim : $\mathbf{Top}^{\omega} \leftrightarrows \mathbf{Top}$: const. We obtain $L(\mathrm{colim})$, the *homotopy colimit*, which we can compute (up to homotopy). Take

$$X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots.$$

Then $L\mathrm{colim}\ X \simeq Tel_n X_n$ is the *mapping telescope*, defined by

$$Tel_n X_n = \coprod_n X_n \times [0,1]/(x,1)\ (f_n(x), 0) \text{ for } x \in X_n.$$

This has that $\pi_k(Tel_n X_n) \cong \mathrm{colim}_n \pi_k X_n$, so the telescope really is the right homotopical notion of the colimit.

Lastly, we will define the correct notion of equivalence.

Definition 3. Suppose we have a Quillen adjunction $F : \mathcal{M} \leftrightarrows \mathcal{N} : G$. This induces an adjunction $LF : Ho(\mathcal{M}) \leftrightarrows Ho(\mathcal{N}) : RG$. We say that (F, G) is a *Quillen equivalence* if (LF, RG) is an adjoint equivalence of categories (i.e. the units and counits of the adjunction are isomorphisms).

This essentially tells us that the homotopy theories of $\mathcal{M}$ and $\mathcal{N}$ are somehow "the same thing".

The most important example is that $|-| : \mathbf{sSet} \leftrightarrows \mathbf{Top} : S(-)$ is a Quillen equivalence.