Model structures for operads

Marcy Robinson

Let us begin with a symmetric monoidal category $(\mathcal{C}, \otimes, I)$. Define the category of operads $Op(\mathcal{C})$, whose objects are $P = \{P(n) | P(n) \in \mathcal{C}\}_{n \geq 0}$ and whose morphisms $f : P \to Q$ are given by $\{f_n : P(n) \to Q(n)\}_{n \geq 0}$ which commute with the structure morphisms. Similarly we can define the subcategory of reduced operads $Op^0(\mathcal{C})$ whose objects have $P(0) = I$ and whose morphisms have $f_0 = \mathrm{id} : P(0) \to Q(0)$.

Theorem (Berger-Moerdijk, '03)**.** *Let $\mathcal{C}$ be a* cofibrantly generated monoidal model *category with cofibrant unit such that*

1. *$\mathcal{C}$ has a symmetric manoidal fibrant replacement functor, and*
2. *$\mathcal{C}$ admits a* commutative Hopf interval*.*

Then we have a cofibrantly generated model category structure on $Op^0(\mathcal{C})$ in which a morphism $f : P \to Q$ is a weak equivalence (resp. fibration) iff $f_n : P(n) \to Q(n)$ is a weak equivalence (resp. fibration) in $\mathcal{C}$.

We will define the emphasized terms as we go.

Definition 1. A model category $\mathcal{C}$ is *cofibrantly generated* if $\mathcal{C}$ has all limits and colimits and there exist sets $I = \{\text{generating cofibrations}\}$ and $J = \{\text{generating trivial cofibrations}\}$ such that:

1. A map in $\mathcal{C}$ is a fibration (resp. trivial cofibration) iff it satisfies the RLP with respect to J (res. I).
2. The domains of I and J are small (in some appropriate sense).

We have the following **transfer principle**: Let $\mathcal{D}$ be a cofibrantly generated model category and $F : \mathcal{D} \leftrightarrows \mathcal{E} : G$ be an adjoint pair. Call a map $f \in \mathcal{E}$ a weak equivalence (resp. fibration) iff $G(f)$ is a weak equivalence (resp. fibration) in $\mathcal{D}$. If $\mathcal{E}$ satisfies the following requirements, then we have a cofibrantly generated model category structure on $\mathcal{E}$:

1. F preserves small objects (or G preserves all filtered colimits).
2. $\mathcal{E}$ has a fibrant replacement functor.
3. There is a functorial path object which factors the diagonal map $\Delta : X \to X \times X$ for all fibrant X: $X \xrightarrow{\sim} Path(X) \twoheadrightarrow X \times X$.

If we want to do this with operads, we've already seen the lift we want. Writing $\mathbb{S}$**-Mod** for symmetric sequences, then we have the adjunction $F : \mathbb{S}\textbf{-Mod} \leftrightarrows Op(\mathcal{C}) : U$, where F is the free functor and U is the forgetful functor.

It is a fact that $\mathbb{S}$**-Mod** has a cofibrantly generated model structure which it inherits from $\mathcal{C}$. Namely, $f : K \to L$ is a weak equivalence (resp. fibration) iff $f_n : K(n) \to L(n)$ is a weak equivalence (resp. fibration). So what remains is to check that $Op(\mathcal{C})$ satisfies the three requirements given above.

1. We'd like to know that F preserves small objects, but it's easier to check that U preserves all filtered colimits. But a colimit of operads is determined by the colimits in the underlying category, from which this follows immediately.
2. Given an operad $P = \{P(n)\}_{n \geq 0}$, we need a find a fibrant replacement (functorially). Let us add the assumption (which is already in our theorem) that $\mathcal{C}$ has a fibrant replacement functor $X \mapsto RX$. We can guess that $RP = \{RP(n)\}_{n \geq 0}$, but we want that $RP(k) \otimes RP(i_1) \otimes \cdots \otimes RP(i_k) = R(P(k) \otimes P(i_1) \otimes \cdots \otimes P(i_k))$ so we must additionally require (as we already have) that R is symmetric monoidal.
3. We need the following definition.

Definition 2. The category $\mathcal{C}$ admits a *commutative Hopf interval* if the fold map $I \sqcup I \to I$ can be factored into a cofibration and a trivial fibration: $I \sqcup I \rightarrowtail H \overset{\sim}{\twoheadrightarrow} I$. (This is just what we'd expect: an object with comultiplication and a commutative multiplication with unit and counit maps.)

Now if $H \in (\mathcal{C}, \otimes, I)$ is a Hopf object, then we can construct a *cooperad* which we call TH as follows. The objects of the cooperad are $TH(n) = H^{\otimes n}$, and the structure maps $H^{\otimes n} \to H^{\otimes k} \otimes H^{\otimes i_1} \otimes \cdots \otimes H^{\otimes i_k}$ (for $i_1 + \cdots + i_k = n$) are given by

$$H^{\otimes n} \overset{\Delta}{\to} H^{\otimes n} \otimes H^{\otimes n} \overset{p \otimes i}{\to} H^{\otimes k} \otimes H^{\otimes i_1} \otimes \cdots \otimes H^{\otimes i_k},$$

where $i : H^{\otimes n} \overset{\simeq}{\to} H^{\otimes i_1} \otimes \cdots \otimes H^{\otimes i_k}$ is the obvious deconcatenation and $p : H^{\otimes n} \to H^{\otimes i_1} \otimes \cdots \otimes H^{\otimes i_k} \overset{p_{i_1} \otimes \cdots \otimes p_{i_k}}{\to} H^{\otimes k}$.

Recall that if A is a k-algebra and C is a k-coalgebra, we can construct a k-algebra $A^C = \mathrm{Hom}_k(C, A)$, called the *convolution* of C and A, whose multiplication is given by $f * g : C \overset{\Delta}{\to} C \otimes C \overset{f \otimes g}{\to} A \otimes A \overset{\mu}{\to} A$. We can do the same construction on our operads. Given an operad P and a cooperad Q, we construct the operad P^Q by $P^Q(n) = \mathrm{Hom}(Q(n), P(n))$. The morphisms are given in the same way, as

$$Q(n) \overset{\Delta}{\to} Q(k) \otimes Q(i_1) \otimes \cdots \otimes Q(i_k) \to P(k) \otimes P(i_1) \otimes \cdots \otimes P(i_k) \overset{\gamma}{\to} P(n).$$

Claim. *For a commutative Hopf interval $H \in (\mathcal{C}, \otimes, I)$, P^{TH} is a functorial path object for any fibrant operad P:*

$$P = P^{TI} \overset{\sim}{\to} P^{TH} \twoheadrightarrow P^{T(I \sqcup I)} = P \times P.$$

Unraveling this slightly, we need $\mathrm{Hom}(H^{\otimes n}, P(n)) \to \mathrm{Hom}((I \sqcup I)^{\otimes n}, P(n))$ to be a fibration, which we get if we know that $(I \sqcup I)^{\otimes n} \to H^{\otimes n}$ is a cofibration.

Definition 3. Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category and $\mathcal{C}$ be a model category. If for all $i : A \hookrightarrow B$ and $j : K \hookrightarrow L$ we take the pushout

$$\begin{array}{ccc} A \otimes K & \xrightarrow{\mathrm{id} \otimes j} & A \otimes L \\ \downarrow{\scriptstyle i \otimes \mathrm{id}} & & \downarrow \\ B \otimes L & \longrightarrow & A \otimes L \coprod_{A \otimes K} B \otimes K \end{array}$$

and obtain the map $p : A \otimes L \coprod_{A \otimes K} B \otimes K \to B \otimes L$ and it is a cofibration, and if additionally whenever i or j is a weak equivalence then p is also a weak equivalence, then $\mathcal{C}$ is called a *monoidal model category.*

Exercise 1. If I is cofibrant, then $(I \sqcup I)^{\otimes n} \to H^{\otimes n}$ is a cofibration.

Remark 1. If $\mathcal{C}$ is cartesian closed and the terminal object is cofibrant, then the theorem extends from $Op^0(\mathcal{C})$ to $Op(\mathcal{C})$.

Remark 2. If everything is fibrant (e.g. **Top**, $Ch(R)$) then we can use Id as our symmetric monoidal fibrant replacement. So the only difficult category for the second axiom is **sSet**.

Example. Recall that **Top** is a symmetric monoidal category that is cartesian closed. We've seen its model structure, which has weak equivalences the weak homotopy equivalences. Moreover, everything is fibrant. The interval $I = [0, 1]$ is our Hopf interval.

Example. Recall that $Ch(R)$ is the category of $\mathbb{Z}$-graded chain complexes of R-modules. This is a symmetric monoidal category with graded tensor product, but it is not cartesian closed. We have the *normalized chain functor* $N_*^R : \mathbf{sSet} \to Ch(R)$ (coming from the Dold-Kan correspondence), and $N_*^R(\Delta[1])$ plays the role of a Hopf object. So the category of *reduced* operads over $Ch(R)$ admits a model structure. (This was already discovered by Hinich outside of this framework.)

Example. Recal that **sSets** is a cartesian closed symmetric monoidal model category. Its Hopf interval is $\Delta[1]$. We can take the Kan functor Ex^∞ as our symmetric monoidal fibrant replacement functor, or we can take $S_\bullet|-|$.

And here is a non-example:

Theorem (Lewis)**.** *There does not exist a model structure on the category Sp^Σ of symmetric spectra which has both a cofibrant unit and a symmetric monoidal fibrant replacement functor.*

Let us say just a word about algebras.

Theorem (Berger-Moerdijk)**.** *For $(\mathcal{C}, \otimes, I)$ as above, if we are given that P is cofibrant (or even just locally cofibrant) and that H is cocommutative, then the category of P-algebras admits a cofibrantly generated model structure.*

This is done much the same way: we look at the forgetful-free adjunction $F : \mathcal{C} \leftrightarrows Alg(P) : U$ and follow the same process as before.