Operadic Basics

Loday

1 Examples of associative algebras

Let $\mathbb{K}$ be a field, and let A be a vector space over $\mathbb{K}$ equipped with a multiplication $\mu : A \otimes A \to A$, which we will write as $\mu(x, y) = xy$. Associativity is the statement that $(xy)z = x(yz)$. Note that we are not assuming the existence of a unit.

Example. A *magmatic algebra* is an arbitrary algebra with no relations. These play an important role in the theory of operads.

Example. A *dendriform algebra* is a vector space A with two binary operations, $<, >$: $A \otimes A \to A$; we write $x < y$ and $x > y$. We then define a product by $x * y = x < y + x > y$. We impose the axioms

$$(x < y) < z = x < (y * z)$$
$$(x > y) < z = x > (y < z)$$
$$(x * y) > z = x > (y > z).$$

It is immediate that $*$ is associative. Moreover, if we think of the left and right products as A-actions, then this makes A into a bimodule over itself.

Example. An A_∞*-algebra* is a graded vector space $A = A_0 \oplus A_1 \oplus \cdots$ with maps $m_n : A^{\otimes n} \to A$ for all $n \geq 1$ (which we consider as $|m_n| = n - 2$). We require that for all $n \geq 1$,

$$\sum_{p+q+r=n} (-1)^{p+qr} m_{p+1+r}(\mathrm{id}, \dots, \mathrm{id}, m_q, \mathrm{id}, \dots, \mathrm{id}) = 0$$

(where the m_q is in the $(p + 1)^{st}$ slot, so there are p copies of id before and r copies of id after). For instance, the first three relations are

$$m_1(m_1) = 0$$
$$-m_2(\mathrm{id}, m_1) - m_2(m_1, \mathrm{id}) + m_1(m_2) = 0$$
$$m_3(\mathrm{id}, \mathrm{id}, m_1) + m_3(\mathrm{id}, m_1, \mathrm{id}) + m_3(m_1, \mathrm{id}, \mathrm{id}) - m_2(\mathrm{id}, m_2) + m_2(m_2, \mathrm{id}) + m_1(m_3) = 0.$$

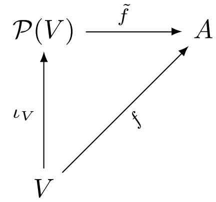
The first equation tells us that we can think of m_1 as a differential. The second equation tells us that m_1 is a derivation over the binary product m_2. The third equation tells us that the product is *almost* associative, but its associativity is obstructed by the higher order map m_3. Et cetera.

Note that the sorts of algebras we are considering have no symmetry in the operations, and in their relations the order of the factors is preserved. (This is implicit in the last example.) We chose these examples because we wish to work first with *non-symmetric operads*.

2 Nonsymmetric operads

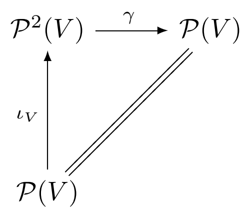
If we have some sort of algebra in mind that we call a $\mathcal{P}$-algebra, we have the forgetful functor $\mathcal{P}$**-Algebras** $\to$ **Vect**, which has a left adjoint $\mathcal{P}$: **Vect** $\to$ $\mathcal{P}$**-Algebras**. We call $\mathcal{P}(V)$ the *free* $\mathcal{P}$*-algebra over* V. For example, the free asociative algebra over V is $V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \cdots$ with product given by concatenation: $(v_1 \cdots v_n)(w_1 \cdots w_p) = v_1 \cdots v_n w_1 \cdots w_p$. We usually denote this $\overline{T}(V)$.

The free $\mathcal{P}$-algebra $\mathcal{P}(V)$ over V satisfies the following universal property: for any $\mathcal{P}$-algebra A, if we have a map of vector spaces $f : V \to A$, then there is a unique $\mathcal{P}$-algebra homomorphism $\tilde{f} : \mathcal{P}(V) \to A$ making the diagram

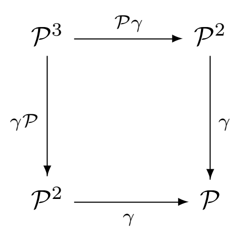


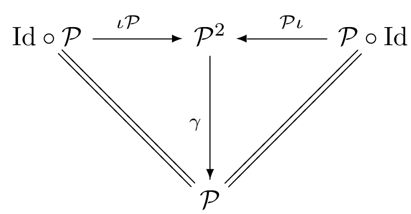
commute. Here, the morphism $\iota_V \in \mathbf{Vect}(V, \mathcal{P}(V))$ is the unit map, which corresponds to $\mathrm{id} \in \mathcal{P}\textbf{-Algebras}(\mathcal{P}(V), \mathcal{P}(V))$. (In fact, the unit maps assemble to a natural transformation $\iota : \mathrm{Id} \to \mathcal{P}$.)

Proposition. *There is a natural transformation of functors $\gamma : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ such that*



$\star\star\star$**is this the right diagram**??? *commutes. Moreover, this γ is associative and unital. Explicitly, associativity and unitality mean that the respective diagrams*





commute.

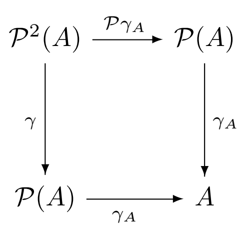
So it's a lot like we're looking at magmatic algebras, but we're considering endofunctors instead of products. To prove things about operads we will mimic much of the theory of associative algebras. This will be very fruitful, but there will always be some extra work to be done, for two reasons. First, the tensor product of two vector spaces is symmetric, but tensor product will be replaced by composition of functors and this need not be symmetric. Also, the tensor product of vector spaces is linear in each variable, whereas composition of functors is linear on the left but not on the right. This second issue will be the real meat of the work.

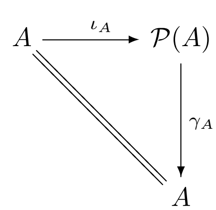
2.1 Monoidal definition

In all the above examples, we had $\mathcal{P}(V) = \bigoplus_{n \geq 0} (\mathcal{P}_n \otimes V^{\otimes n})$ for fixed vector spaces $\mathcal{P}_n$. This motivates our first definition of an operad.

Definition 1 (monoidal definition)**.** A *nonsymmetric operad* is a family of $\mathbb{K}$-vector spaces $\mathcal{P} = \{\mathcal{P}_n\}_{n \geq 0}$, considered as a functor $\mathcal{P} : \mathbf{Vect} \to \mathcal{P}\textbf{-Algebra}$ as above, equipped with natural transformations $\gamma : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ and $\iota : \mathrm{Id} \to \mathcal{P}$ such that $(\mathcal{P}, \gamma, \iota)$ is a monoid.

Definition 2. An *algebra* over a nonsymmetric operad $\mathcal{P}$ is a vector space A and a morphism $\gamma_A : \mathcal{P}(A) \to A$ such that the diagrams





commute. Note that γ_A has a priori nothing to do with γ. The diagrams give strong conditions on γ_A.

Thus we can get from a nonsymmetric operad $\mathcal{P}$ to a $\mathcal{P}$-algebra A: each vector in $\mathcal{P}_n$ gives us an n-fold multiplication on A.

2.2 Classical definition

It would be nice to somehow encode the action of γ in the spaces $\mathcal{P}_n$. To do this we need to know how to apply $\mathcal{P}$ to itself. So, we compute

$$(\mathcal{P} \circ \mathcal{P})_n = \bigoplus_{i_1+\ldots+i_k=n} \mathcal{P}_k \otimes \mathcal{P}_{i_1} \otimes \cdots \otimes \mathcal{P}_{i_k}.$$

Then, $\gamma : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ must be a morphism of *arity*-graded vector spaces, i.e. we need $(\mathcal{P} \circ \mathcal{P})_n \to \mathcal{P}_n$. This means we have maps

$$\gamma_{i_1\ldots i_k} : \mathcal{P}_k \otimes \mathcal{P}_{i_1} \otimes \cdots \otimes \mathcal{P}_{i_k} \to \mathcal{P}_n.$$

Definition 3 (classical definition)**.** A *nonsymmetric operad* is $\mathcal{P} = \{\mathcal{P}_n\}_{n\geq 0}$ with maps $\gamma_{i_1\ldots i_k}$ as above and a map $\iota : \mathbb{K} \to \mathcal{P}_1$, such that $\gamma : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ and $\iota : \mathrm{Id} \to \mathcal{P}$ make $(\mathcal{P}, \gamma, \iota)$ into a monoid. (Note that Id is represented by the operad with $\mathbb{K}$ in arity 1 and 0 elsewhere.)

Exercise 1. Write down the axioms for $\gamma_{i_1\ldots i_k}$.

2.3 Partial definition

It is very helpful to think of the spaces $\mathcal{P}_n$ as giving $\mathcal{P}_n \otimes A^{\otimes n} \to A$, $(\mu, a_1, \ldots, a_n) \mapsto \mu(a_1, \ldots, a_n)$. Thus we call $\mathcal{P}_n$ the space of *n-ary operations*. So using tree notation, what we are saying is that we can compose $\star\star\star$**DIAGRAM** to get $\star\star\star$**DIAGRAM**.

Definition 4 (partial definition)**.** A *nonsymmetric operad* is $\mathcal{P} = \{\mathcal{P}_n\}_{n\geq 0}$ with *partial composition* maps

$$\circ_i : \mathcal{P}_m \otimes \mathcal{P}_n \to \mathcal{P}_{m-1+n}$$

where in tree notation, $\mu \otimes \nu$ is given by plugging in ν to the i^{th} node of μ (for $1 \leq i \leq m$). There is a UNIT AXIOM. We impose the following *sequential composition axiom* for associating over partial compositions (DIAGRAM). If we want to plug both μ and ν into λ, there are two orders in which we can do this. Demanding this equality is the *parallel composition axiom.*

Exercise 2. Write out the sequential composition axiom in full.

It is a fact that these partial compositions generate all possible compositions.

2.4 Combinatorial definition

For the last definition we consider planar trees. Given a tree t, a vertex $v \in t$ and another tree s, we define the tree $t \circ_v s$ by substituting s at v into t. DIAGRAM (There is the obvious condition that the number of inputs at v needs to equal the number of leaves of s.) This comes with certain associativity properties: $t \circ_v (s \circ_w r) = (t \circ_v s) \circ_{\tilde{w}} r$, where $\tilde{w}$ is the image of w in $t \circ_v s$. And if $v_1, v_2 \in t$ are vertices, then $(t \circ_{v_1} s) \circ_{v_2} r = (t \circ_{v_2} r) \circ_{v_1} s$.

Now define $\mathcal{P} = \{\mathcal{P}_n\}_{n\geq 0}$ for $\mathbb{N}$-modules $\mathcal{P}_n$. $\star\star\star$??? We define $\mathbb{PT} : \mathbb{N}$-**Mod** $\to \mathbb{N}$-**Mod** by letting $\mathbb{PT}(M)_n$ be spanned by the planar trees decorated by M.

We claim that the notion of substitution of trees gives us a natural transformation of functors $\Gamma : \mathbb{PT} \circ \mathbb{PT} \to \mathbb{PT}$.

To see how this works, we look at $\mathbb{PT}(\mathbb{PT}(M))_n \to \mathbb{PT}(M)_n$. In the domain, the decorations themselves are now trees decorated by M. We get $\Gamma(M)_n$ by substitution of trees. Associativity follows readily from the associativity of the substitution process described above.

Definition 5 (combinatorial definition)**.** We have the monad $(\mathbb{PT}, \Gamma, \iota)$, a monoid on endofunctors of $\mathbb{N}^+$-**Mod** $\star\star\star$**when did we switch to** $\mathbb{N}^+$???. An algebra over $(\mathbb{PT}, \Gamma, \iota)$ is an $\mathbb{N}^+$-module $\mathcal{P}$ compatible with Γ and ι.

2.5 Equivalence of definitions

Theorem. *These four definitions are equivalent.*

3 Further exercises

Exercise 3. Recall that $\{\mathcal{P}_n\}_{n\geq 0}$ gives $\mathcal{P} : \mathbf{Vect} \to \mathbf{Vect}$ by $\mathcal{P}(V) = \bigoplus_n(\mathcal{P}_n \otimes V^{\otimes n})$. Let $\{\mathcal{Q}_n\}_{n\geq 0}$ give $\mathcal{Q}$. Find $(\mathcal{P} \circ \mathcal{Q})_n$ giving $\mathcal{P} \circ \mathcal{Q}$. Make clear the properties of $\otimes$ that you use.

Exercise 4. Write $Mag(V)$ for the free magmatic algebra on V, $Mag(V) = \bigoplus_n(Mag_n \otimes V^{\otimes n})$. What is Mag_n? What is $\circ_i : Mag_m \otimes Mag_n \to Mag_{m-1+n}$?

Exercise 5. Let $\mathcal{P}$ be a nonsymmetric operad such that $\mathcal{P}_0 = 0$ and $\mathcal{P}_1 = \mathbb{K} \cdot \mathrm{id}$. Define

$$G = \{\underline{\mu} = (\mathrm{id}, \mu_2, \mu_3, \ldots)|\mu_n \in \mathcal{P}_n\},$$

and define an operation $\cdot : G \times G \to G$ by

$$\underline{\mu} \cdot \underline{\nu} = \left(\mathrm{id}, \mu_2 + \nu_2, \ldots, \sum_{i_1+\ldots+i_k=n} \gamma(\mu_k, \nu_{i_1}, \ldots, \nu_{i_k}), \ldots\right),$$

where the indicated sum is in arity n. Show that $(G, \cdot)$ is a group.

Exercise 6. Let V be a vector space, and define $\mathrm{End}(V)_n = \mathrm{Hom}(V^{\otimes n}, V)$. Show that there is a nonsymmetric operad structure on $\{\mathrm{End}(V)_n\}_{n\geq 1}$.

4 Nonsymmetric operads, cont.

Recall the endomorphism operad, which for a vector space V has the space of n-ary operations given by $\mathrm{End}(V)_n = \mathrm{Hom}(V^{\otimes n}, V)$, with partial composition defined by

$$(f \circ_i g)(v_1, \ldots, v_{m-1+n}) = f(v_1, \ldots, g(v_i, \ldots, v_{i+n}), \ldots, v_{m-1+n}).$$

Claim. *The data of a $\mathcal{P}$-algebra A is equivalent to a morphism $\mathcal{P} \to \mathrm{End}(A)$ of operads.*

$$\mathcal{P}(A) \to A \quad \mathcal{P}_n \otimes A^{\otimes n} \to A \quad \mathcal{P}_n \to \mathrm{Hom}(A^{\otimes n}, A) \star\star\star\, \textbf{what were all these???}$$

This nonsymmetric operad $\mathcal{P}$ is completely determined by $\mathcal{P}(\mathbb{K}_x)$, where $\mathbb{K}_x$ is the free $\mathcal{P}$-algebra on one element x. DIAGRAM??? $\mathcal{P}(\mathbb{K}) = \oplus_n \mathcal{P}_n$.

We needed our category **Vect** to be symmetric monoidal for our theory to go through. But there are many other such categories:

$$\begin{array}{ccc} & & (\mathbf{Top}, \times) \\ & & \downarrow \\ (\mathbf{Sets}, \times) & \longrightarrow & (\mathbf{SimpSets}, \times) \\ \downarrow & & \downarrow \\ (\mathbb{K}\textbf{-mod}, \otimes) & \longrightarrow & (\mathbf{dg}\ \mathbb{K}\textbf{-mod}, \otimes) \end{array}$$

So really the category we care about most is **dg $\mathbb{K}$-mod**, which is the same as chain complexes.

Recall that an A_∞-algebra is a $\mathbb{K}$-module A with maps $m_n : A^{\otimes n} \to A$ for $n \geq 1$ with $m_1^2 = 0$ and higher relations. We consider $m_1 = d$, so the data is equivalent to having $((A, d), \{m_n\}_{n\geq 2})$, a chain complex with higher multiplication maps. Recall that

$$\partial(m_n) = -\sum_{p+q+r=n,\ p+1+r=k} (-1)^{p+qr} m_k \circ_{p+1} m_q$$

for $n \geq 2$.

5 Examples of commutative algebras

By commutative, we mean that the relations no longer respect the order of the factors.

Example. Standard commutative algebras are commutative in our sense.

Example. Let $(A, [\cdot,\cdot])$ be a Lie algebra. Besides satisfying antisymmetry, the bracket must satisfy the Jacobi identity, which we will write as $[[x,y],z] = [[x,z],y] + [x,[y,z]]$. (This is equivalent to the usual formulation by antisymmetry.)

Example. Let $(A, [\cdot,\cdot])$ be an algebra with a bracket satisfying the above Jacobi identity. If we consider $[\cdot, z]$ as a right action of A on itself, then the Jacobi identity is the statement that this action is a derivation. We call this a *Leibniz algebra*.

Example. Recall our dendriform algebra A with the left and right operations. A *commutative* dendriform algebra has the additional axiom that $x < y = y > x$. These then satisfy, e.g., that $(x < y) < z = x < (y < z + z < y)$. Commutative dendriform algebras are also known as *Zinbiel algebras*. Note that if we make the definition $xy = x < y + y < x$, our product is automatically commutative, and from the dendriform axioms it is associative too. So a Zinbiel algebra gives a commutative algebra. Also, if we define $x^{<n} = x < (x < (\cdots < x))$, it follows that $x^n = n! x^{<n}$. This is very convenient, because then regardless of whether we restrict ourselves to characteristic 0 we can always define $\exp(x) = 1 + x + x^{<2} + x^{<3} + \cdots$.

Our examples are related by the following diagram:

$$\begin{array}{ccccc}
\textbf{Zinbiel-alg} & \longrightarrow & \textbf{Dendr-alg} & \longrightarrow & \textbf{preLie-alg} \\
\downarrow & & \downarrow & & \downarrow \\
\textbf{Comm-alg} & \longrightarrow & \textbf{As-alg} & \longrightarrow & \textbf{Lie-alg}.
\end{array}$$

A *preLie algebra* A is an algebra A with a bracket $\{x,y\}$ satisfying the axiom $\{\{x,y\},z\} - \{x,\{y,z\}\} = \{\{x,z\},y\} - \{x,\{z,y\}\}$. From this we can define a Lie bracket $[x,y] = \{x,y\} - \{y,x\}$.

6 Symmetric operads

If we are thinking of algebras of type $\mathcal{P}$, the free $\mathcal{P}$-algebra on V is now given by $\mathcal{P}(V) = \bigoplus_n (\mathcal{P}(n) \otimes_{S_n} V^{\otimes n})$, where now each $\mathcal{P}(n)$ is an S_n-module. Thus we call the family $\{\mathcal{P}(n)\}_{n \geq 0}$ an $\mathbb{S}$-*module*. We call $\mathcal{P}$ a *Schur functor*.

As before, we must see what happens when we compose two Schur functors. So suppose $\{Q(n)\}_{n\geq 0}$ is another $\mathbb{S}$-module giving the Schur functor $\mathcal{Q}$. One can compute that $\mathcal{P} \circ \mathcal{Q}$ is indeed again a Schur functor, and

$$(\mathcal{P} \circ \mathcal{Q})(n) = \bigoplus_k \mathcal{P}(k) \otimes_{S_k} \left(\bigoplus_{i_1 + \cdots + i_k = n} \operatorname{Ind}^{S_n}_{S_{i_1} \times \cdots \times S_{i_k}} (Q(i_1) \otimes \cdots \otimes Q(i_k)) \right).$$

Note that $S_{i_1} \times \cdots \times S_{i_k} \subset S_n$ in a natural way, and in general, if we have a subgroup $G \subset \Gamma$ and a G-module M then we can define an *induced* Γ-*module* by $\operatorname{Ind}^{\Gamma}_G M = M \otimes_{\mathbb{K}[G]} \mathbb{K}[\Gamma]$.

6.1 Monoidal definition

Definition 6 (monoidal definition)**.** A *symmetric operad* is an $\mathbb{S}$-module $\{\mathcal{P}(n)\}_{n\geq 0}$ with natural transformations $\gamma : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ and $\iota : \mathrm{Id} \to \mathcal{P}$ (coming from a map of $\mathbb{S}$-modules) such that $(\mathcal{P}, \gamma, \iota)$ is a monoid. An *algebra* over a symmetric operad is an algebra A with a morphism $\gamma_A : \mathcal{P}(A) \to A$ satisfying the same axioms as before.

6.2 Comparing nonsymmetric and symmetric operads

The associative nonsymmetric operad has $\mathcal{P}(V) = \bigoplus_n (\mathcal{P}_n \otimes V^{\otimes n}) = \bigoplus_n (\mathcal{P}_n \otimes \mathbb{K}[S_n]) \otimes_{S_n} V^{\otimes n}$, where $\mathbb{K}[S_n]$ is the regular representation of S_n. We write As for the nonsymmetric operad and Ass for the symmetric operad: $Ass(n) = \mathbb{K}[S_n]$.

Exercise 7. Make γ explicit in this case.

⋆⋆⋆**what did i miss here???**

To determine an operad, we of course need to define $\mathcal{P}(n)$, and then the usual route is to define the operad via its partial operations $\circ_i$.

Example. $Com(n) = \mathbb{K}\mu_n$, since $Com(V) = \overline{S}(V) = V \oplus (V^{\otimes 2})_{S_2} \oplus \cdots \oplus (V^{\otimes n})_{S_n} \oplus \cdots$. (If W is a G-module, then $W_G = W/\{w - g \cdot w\}$ are the *coinvariants.*)

Example. $Lie(V) \subset \overline{T}(V)$ is generated by V under $[\cdot, \cdot]$.

Example. There is a generalized BCH formula for more than two variables:

$$e^{x_1} \cdots e^{x_n} = e^{x_1 + \cdots + x_n + \cdots + H_n(x_1, \ldots, x_n) + \cdots}.$$

The power series in the exponent on the right side has all degrees, but we are particularly interested in degree n. We denote $h_n(x_1, \ldots, x_n)$ for the monomial in which each x_i shows up exactly once. Then $h_n = e_n^1(x_1 \cdots x_n) \in \mathbb{Q}[S_n]$.

Theorem. *e_n^1 is an idempotent (in fact it is Eulerian), and as a representation of S_n it is exactly $Lie(n)$.*

6.3 Other definitions

All the other definitions go through the same as before, replacing $\mathbb{N}$-modules with S_n-modules.

For the classical definition, recall that we had to make our associativity precise for $\mathcal{P} \circ \mathcal{P} \circ \mathcal{P}$. Even $\mathcal{P} \circ \mathcal{P}$ was terrible to write down, but if anybody outside of topology asks, tell them nobody uses this one.

For the partial definition, recall that we had $\circ_i : \mathcal{P}(m) \otimes \mathcal{P}(n) \to \mathcal{P}(m - 1 + n)$. Then there is an ugly but straightforward formula for the symmetric action on the source and target.

For the combinatorial definition, we no longer have planar trees but arbitrary trees. We give our leaves an ordering. This gets messy too, but André Joyal came up with a new way of indexing with his theory of *species*. Instead of thinking of finite sets themselves, we look at functors off the category of finite sets and bijections. This category admits a skeleton by the objects $\underline{n} = \{1, \ldots, n\}$.

7 Topological operads

A *topological operad* is one in which the operations are themselves topological spaces.

Example (The small discs operad)**.** An n-ary operation is given by a disc with n small discs inside labelled 1 through n. The partial composition $d_1 \circ_i d_2$ is given by fitting disc d_2 onto the i^{th} small disc of d_1 and numbering all its discs in order before continuing to the rest of the little discs in d_1.

Example (Planar algebras)**.** Vaughn Jones invented something called "planar algebras", which are not really algebras at all, but are operads. Take a big disc and punch out a bunch of discs, and put a bunch of vertices on the boundary (which consists of multiple circles). Connect these vertices with nonintersecting edges, and shade the regions of the swiss cheese black and white. There are various other axioms that must be satisfied, but this gives an operation in a planar algebra. Planar algebras are used to construct knot invariants and C^*-algebra invariants.

Example. We define an operad as a sort of algebra in a different framework. First note that we have the table

algebras	**Vect**	$\otimes$	$\mathbb{K}$
n.s. operads	$\mathbb{N}$-**Mod**	$\circ$	$Id = (0, \mathbb{K}, 0, 0, \ldots)$
sym. operads	$\mathbb{S}$-**Mod**	$\circ$	$Id = (0, \mathbb{K}, 0, 0, \ldots)$

Dual to the notion of algebra we have the notion of *coalgebra*. A coalgebra is a vector space C with a comultiplication map $\Delta : C \to C \otimes C$ which is coassociative and counital. So in analogy we can define *cooperads*, which have natural transformations $\Delta : \mathcal{C} \to \mathcal{C} \circ \mathcal{C}$.

8 Credits

The mathematicians chiefly responsible for the development of the theory of operads are from the Chicago school: Mac Lane, May, Boardman, Vogt, and Stasheff. Also important are the Australians Kelly and Street.

9 Further exercises

Exercise 8. Suppose we have an algebra with two generating operations, a symmetric one and an antisymmetric one: $xy = yx$, $[x, y] = -[y, x]$. Impose the relations $[xy, z] = x[y, z] + [x, z]y$ and $(xy)z - x(yz) = [y, [x, z]]$. What is this type of algebra? (*Hint: You know it.*)

Exercise 9. Suppose we have an algebra with a single symmetric binary operation $xy = yx$. Denote the category of these by **ComMag** (for "commutative magmas"). What is $\dim ComMag(n)$?

Exercise 10. Consider $Lie(V) \subset \overline{T}(V)$, the free Lie algebra on V considered as a subalgebra of the free tensor algebra. Try to make sense of this sentence operadically: "Every relation satisfied by the product comes from the anticommutativity of the bracket and the Jacobi identity".

Exercise 11. Let A be an associative algebra with a Lie bracket. Define $x \circ y = xy + yx$. This is known as a *Jordan algebra*. Find and write out the *Jordan identity* satisfied by this product. Do we have the same property as for *Lie* hinted at above?

Exercise 12. Let A be a preLie algebra, i.e. it has a binary operation $\{x, y\}$ whose associator is right symmetric. Check that $[x, y] = \{x, y\} - \{y, x\}$ is a Lie bracket. Do the same for $x \cdot y = \{x, y\} + \{y, x\}$. What relations do these satisfy? (Make some computations and guess what the result should be.)

Exercise 13. Consider the monomial $x_1 \cdots x_n$, and let $\sigma \in S_n$ act by $x_1 \cdots x_n \mapsto x_{\sigma(1)} \cdots x_{\sigma(n)}$. Is this a left action or a right action?

Exercise 14. Let Ass be the symmetric associative operad. Recall that $Ass(n) = \mathbb{K}[S_n]$. An element $\sigma \in S_n$ operates on $x_1 \cdots x_n$ on the left as $\sigma(x_1 \cdots x_n) = x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(n)}$. For $\sigma \in S_m$ and $\omega \in S_n$, write explicitly the operation $\sigma \circ_i \omega \in S_{m-1+n}$.

10 Associative unital algebras

The free tensor algebra is $T(V) = \mathbb{K} \oplus V \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$, and dually we have the *cofree algebra* $T^c(V) = \mathbb{K} \oplus V \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$ with comultiplication $\Delta : T^c(V) \to T^c(V) \otimes T^c(V)$ given by *deconcatenation*: if $v_1 \cdots v_n \in V^{\otimes n}$, then

$$\Delta(v_1 \cdots v_n) = \sum_{i=0}^{n} v_1 \cdots v_i \otimes v_{i+1} \cdots v_n.$$

Now suppose $\alpha, \beta : C \to A$ are linear maps from a coalgebra C from an algebra A. Then we can dfine the *convolution* $\alpha \star \beta : C \to A$ via $\alpha \otimes \beta(\Delta(x))$. **DIAGRAMMMMM**

Exercise 15. Show that $\star$ is associative and unital.

$\star\star\star$**WHAT IS THIS???** Now we have the composition $C \to \mathbb{K} \to A$, and $d_\alpha : C \otimes A \to C \otimes A$ via the diagram **DIAGRAMMMM**: $d_\alpha(c \otimes a) = (1 \otimes \mu) \circ (\Delta \otimes 1)(c \otimes a)$.

Lemma. $d_{\alpha \star \beta} = d_\alpha \circ d_\beta$.

Proof. **DIAGRAM INVOLVING COASSOCIATIVITY** □

Now suppose C and A additionally have dga structure. Then we define $\partial(\alpha) = d^A \alpha - (-1)^{|\alpha|} \alpha d^C$, and we also redefine $d_\alpha = d^C \otimes 1_A + 1_C \otimes d^A + d^r_\alpha$, where d^r_α is the "right" differential that we have defined earlier.

Lemma. *If* $\partial(\alpha) + \alpha \star \alpha = 0$, *then* $(d_\alpha)^2 = 0$.

If we compute the bracket with respect to $\star$, then $[\alpha, \alpha] = 2\alpha \star \alpha$. Then our condition becomes $\partial\alpha + \frac{1}{2}[\alpha, \alpha] = 0$, the *Maurer-Cartan equation*. We will call such an α a *twisting morphism*, and the set of these is denoted $Tw(C, A)$.

We have $Tw(C, A) \subset \mathrm{Hom}^{-1}(C, A)$. Considered as a functor in C or A, this ends up being representable or corepresentable (resp.), i.e.

$$\mathrm{Hom}_{dga\ alg}(\Omega C, A) \cong Tw(C, A) \cong \mathrm{Hom}_{dga\ coalg}(C, BA).$$

We call ΩC the *cobar construction* on C, and we call BA the *bar construction* on A. For the moment, all our algebras are augmented (so $\iota : \mathbb{K} \to A$ is the inclusion of a direct summand) and all our coalgebras are coaugmented (so $\epsilon : C \to \mathbb{K}$ splits).

To make these bar and cobar constructions, we begin with the following. FOr a coalgebra C, we can extend Δ to $T(C)$ by for example on $C^{\otimes 2}$ using $\Delta \otimes 1_C + 1_C \otimes \Delta$. We don't want our $\mathbb{K}$ showing up over and over, so we modify to $T(\overline{C})$. But also we need to shift our degrees, to get $\overline{C}$ in degree 0, so we take s to be the shift-by-1 map (so $|s^{-1}(x)| = |x| - 1$) and we take instead $T(s^{-1}\overline{C})$. This has a differential that *does* square to zero. And so finally we get define $(T(s^{-1}\overline{C}), d)$. When we add up the "internal differential" to make a bicomplex and then take the total complex, we get the right definition of ΩC.

Theorem. $\mathrm{Hom}_{dga\ alg}(\Omega C, A) \cong Tw(C, A)$.

Exercise 16. Dualize this story.

Suppose now that $d_\alpha : C \otimes A \to C \otimes A$ gives an acyclic complex. (Here, C and A are both graded so $C \otimes A$ is bigraded, and $|d| = (-1, +1)$.) In this case we call α a *Koszul morphism*. So now we have $Kos(C, A) \subset Tw(C, A) \subset \mathrm{Hom}^{-1}(C, A)$, and we can ask the obvious question: Which are the elements of $\mathrm{Hom}_{dga\ alg}(\Omega C, A)$ that correspond to Koszul morphisms? The answer is simple: the quasi-isomorphisms are exactly those elements. In other words,

$$\mathrm{Hom}^{-1}(C, A)$$

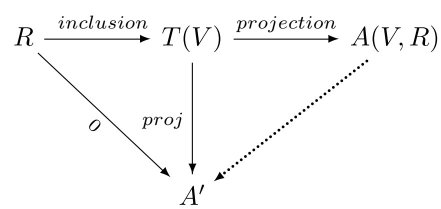
$$\mathrm{Hom}_{dga\ alg}(\Omega C, A) \quad \cong \quad Tw(C, A) \quad \cong \quad \mathrm{Hom}_{dga\ coalg}(C, BA)$$

$$q - isom_{dga\ alg}(\Omega C, A) \cong \quad Kos(C, A) \quad \cong q - isom_{dga\ coalg}(C, BA).$$

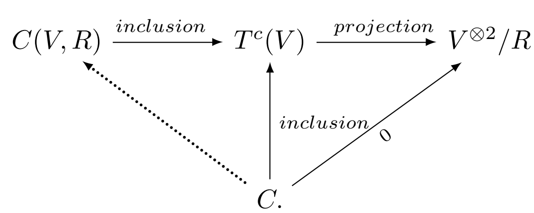
11 Quadratic algebras

Any algebra A can be be thought of as $A = A(V, R) = T(V)/(R)$ for some vector space V and a two-sided ideal (R) generated by R. A is a *quadratic algebra* if $R \subset V^{\otimes 2}$. For example we can take $R = 0$, or $R = V^{\otimes 2}$ (so $A = \mathbb{K} \oplus V$), or $V = \mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n$ and $R = \{x_i \otimes x_j - x_j \otimes x_i$ so that $A = S(V) = \mathbb{K}[x_1, \ldots, x_n]$.

When we have this *quadratic data*, i.e. such a vector space V and ideal R, then we can also immediately form a *coalgebra*. First, for the algebra case, we have the following universal property:



So for the coalgebra case, we dualize the diagram:



Explicitly, we have

$$\begin{aligned} A(V, R) &= \mathbb{K} \oplus V \oplus V^{\otimes 2}/R \oplus V^{\otimes 3}/(R \otimes V + V \otimes R) \oplus \cdots \\ C(V, R) &= \mathbb{K} \oplus V \oplus R \oplus (R \otimes V \cap V \otimes R) \oplus \cdots . \end{aligned}$$

The comultiplication is defined by, for $r = \sum v \otimes w$, setting $\Delta(r) = 1 \otimes r + r \otimes 1 + \sum r \otimes w$. (The tensor factors are now considered as living in the two copies of C.)

We immediately have the following twisting morphism. Define $\kappa : C(V, R) \to A(V, R)$ by the composition $C(V, R) \to^{projection} V \to^{inclusion} A(V, R)$.

Lemma. $\kappa \star \kappa = 0$.

Proof. The only place this is not completely obvious is on $R \subset C(V, R)$. This has $R \to V^{\otimes 2} \to V^{\otimes 2}/R$. □

Given quadratic data, we can define the composition $\kappa : A^{upsidedownexclamationpoint} = C(sV, s^2 R) \to sV \to^{congruent} V \to^{inclusion} A(V, R)$. This anti-shriek coalgebra is called the *Koszul dual coalgebra of the quadratic algebra A*.

Definition 7. A is *Koszul* whenever κ is a Koszul morphism.

We use this as follows. Let A be a Koszul algebra. From the theorem involving the big diagram, our twisting morphism gives us a quasi-isomorphism $\Omega A^{antishriek} \to A$. But the source here is just $(T(s^{-1}\overline{A^{antishriek}}), d)$ which is free as an algebra, so we have constructed a *quasi-free resolution* of A (in fact it is a *minimal* resolution). One major aim of this whole theory is to construct quasi-free resolutions and minimal models.

12 Homotopy transfer theorem

Here is the problem at hand. Suppose we have a deformation retract pair (X, A) (i.e. a subspace $i : A \to X$ with projection $p : X \to A$ so that $p \circ i = \mathrm{id}_A$ and $i \circ p \simeq \mathrm{id}_X$). In homological algebra we may have a similar situation with $((C, d), (V, d))$. Explicitly, $p : C \to V$ and $i : V \to C$ must be so that $p \circ i = \mathrm{id}_V$ and $i \circ p - \mathrm{id}_C = hd + dh$ for some degree-1 $h : C \to C$. This all gives that $H_*(C) \cong H_*(A)$. Now suppose these are complexes of A-modules. Then the homologies naturally have A-module structures. But does the A-module structure on $H_*(V)$ come from that on $H_*(C)$? Of course this is usually not true. However, we have the following theorem.

Theorem. *If A is a Koszul algebra, then the A-module structure on (C, d) transfers a $\Omega A^{antishriek}$-module structure on (V, d). Thus we get an $\Omega A^{antishriek}$-structure on $H_*(V)$, and this is a much more refined structure than the original A-module structure on $H_*(V) \cong H_*(C)$.*

There are examples where indeed there is some extra structure that comes out.

13 Further exercises

Exercise 17. Describe the free nonsymmetric operad over $(0, 0, \mathbb{K}\mu, 0, 0, \ldots)$ (supported in arity 2). The forgetful functor ns operad $\to$ $\mathbb{N}$-**Mod** has an adjoint; it will be the analog of the tensor algebra, but in the operad case instead of in the vector space case. We can generalize to $(0, 0, M_2, 0, \ldots)$, to $(0, 0, M_2, M_3, \ldots)$, or even to $(0, M_1, M_2, \ldots)$.

Exercise 18. Do the previous exercise in the symmetric case (replacing $\mathbb{N}$-modules with $\mathbb{S}$-modules).

Exercise 19. Describe the cofree nonsymmetric cooperad. Again this is adjoint to the forgetful functor ns cooperad $\to$ $\mathbb{N}$-**Mod**. (*Hint: The underliying module will be the same as for the free object, but the coproduct will be more subtle. This will generalize our concatenation-deconcatenation constructions in the algebra cases.*)

14 HTT, cont.

Given an algebra A, then in some cases (e.g. quadratic and Koszul) we constructed a differential graded associative algebra $\Omega A^{antishriek}$ with a quasi-isomorphic map to A, which is quasi-free and minimal. We say $\Omega A^{antishriek}$ is a *model* for A. (By minimal we mean that in $(T(V), d)$, $d|_V : V \to \sum_{n \geq 2} V^{\otimes n}$.)

Here we are working in the category **As-alg**, whereas rational homotopy theory takes place in the category **Com-alg**. In that case, the role of T is played by Λ. But algebras and coalgebras no longer work: we must replace coalgebras by Lie coalgebras. This will make sense when we start talking about Koszul duality for operads.

15 Applications of HTT

15.1 Spectral sequences

The simplest possible case is to take the quadratic algebra $A = \mathbb{K}[\varepsilon]/(\varepsilon^2)$ (coming from $V = \mathbb{K}\varepsilon$ and $R = V^{\otimes 2} = \mathbb{K}\varepsilon \otimes \mathbb{K}\varepsilon = \mathbb{K}\varepsilon^2$). This will lead us to discover spectral sequences!

First we compute $A^{antishriek} = \mathbb{K} \oplus \mathbb{K}s\varepsilon \oplus \cdots \oplus \mathbb{K}(s\varepsilon)^n \oplus \cdots$, where $|s\varepsilon| = 1$. This is a coalgebra under deconcatenation: $\Delta((s\varepsilon)^n) = \sum_{i+j=n} (s\varepsilon)^i \otimes (s\varepsilon)^j$. So now $\Omega A^{antishriek} = T(s^{-1}\overline{A^{antishriek}})$. The overbar just means get rid of the unit, so if we write $t_n = s^{-1}(s\varepsilon)^n$ then $s^{-1}A^{antishriek} = \mathbb{K}t_1 \oplus \cdots \oplus \mathbb{K}t_n \oplus \cdots$, where $|t_n| = n-1$, and hence $\Omega A^{antishriek}$ is a (noncommutative) polynomial algebra over $t_1, t_2, t_3, \ldots$.

Now we compute the differential ∂. It ends up that $\partial t_n = -\sum_{i+j=n, i\geq 1, j\geq 1}(-1)^i t_i t_j$.

Let us apply the HTT to a chain complex (C, d). Suppose we have the inclusion of a deformation retract $i : (V, d) \to (C, d)$ as before and (C, d) is a $\mathbb{K}[\varepsilon]/(\varepsilon^2)$-module. Here $\varepsilon : C \to C$, $\varepsilon d = d\varepsilon$, $\varepsilon^2 = d^2 = 0$. Then by the theorem, (V, d) is a module over $\Omega(\mathbb{K}[\varepsilon]/(\varepsilon)^2)^{antishriek}$. On this side, $t_i : V \to V$ for $i \geq 1$ and by definition $\partial t_i = dt_i + (-1)^{i-1}t_i d$. Then, $\partial t_n = -\sum(-1)^i t_i t_j$. If we write $d = t_0$ then we get $\sum_{i+j=n, i\geq 0, j\geq 0}(-1)^i t_i t_j = 0$.

We take the example of a bicomplex $C_{\bullet,\bullet}$ of vector spaces on a grid with downward vertical differentials d^v and leftward horizontals d^h satisfying $(d^h)^2 = (d^r)^2 = d^r d^h + d^h d^r = 0$. Then we can sum along the rows to get a complex $C_n = \oplus_p C_{p,n}$ with $d = d^v$ and $\varepsilon = (-1)^n d^h$. So any bicomplex gives a chain complex of $\mathbb{K}[\varepsilon]/(\varepsilon^2)$-modules. If we are working over a field $\mathbb{K}$, then we get a deformation retract of the inclusion $(H_*(C), 0) \to (C, d)$. Since the differential on $H_*(C)$ is zero, we get that $\sum_{i+j=n, i\geq 1, j\geq 1}(-1)^i t_i t_j = 0$. The first nontrivial implication is that $t_1 t_1 = 0$. Writing $E^1 = C$, this gives us $t_1 : E^1 \to E^1$. Then define $E^2 = H_*(E^1, t_1)$. Next we have $-t_1 t_2 + t_2 t_1 = 0$, meaning we get $t_2 : E^2 \to E^2$. Now $-t_1 t_3 + t_2 t_2 - t_3 t_1 = 0$, so t_2 is a differential on E^2, and we can define $E^3 = H_*(E^2, t_2)$. Et cetera. When all is said and done, we get from the HTT the spectral sequence associated to the bicomplex! In fact, there are generalizations of spectral sequences from this perspective.

Exercise 20. Give the formula for t_i in terms of ε and h.

15.2 Hochschild homology and cyclic homology

Let A be an associative unital algebra. The *Hochschild complex* is given by $\cdots A^{\otimes 3} \to A^{\otimes 2} \to A$. (For example, the first Hochschild boundary map $b : A^{\otimes 2} \to A$ is given by $b(a_1 \otimes a_2) = a_1 a_2 - a_2 a_1$.) Let us align these vertically and make a similar complex just to the right but with differentials $-b'$ given by throwing out the last summand in b, and then repeat the Hochschild complex to the right of that. Underlying these are periodic maps between the two complexes giving a bicomplex, where the map $A^{\otimes n} \to A^{\otimes n}$ is given first by $1 - t$ and then by N; $t(a_0, \ldots, a_n) = (-1)^n(a_n, a_0, \ldots, a_{n-1})$ and $N = 1 + t + \cdots + t^n$, and these obvious compose to zero (in either order). Alain Connes discovered that these squares anticommute, and so we obtain the cyclic bicomplex $HC_*(A) := H_*(\mathrm{Tot}C_{\bullet,\bullet}(A))$.

One often likes to kill acyclic subcomplexes (either by quotienting out an acyclic subcomplex or by taking a subcomplex the quotient by which is acyclic). We might try to do this to our columns here, but neither works. So instead Loday-Quillen found an ad hoc method which turned out to use something that had already been discovered: the *Connes boundary map* $B : C_{p,q} \to C_{p-1,q+1}$. Explicitly, $B(a_0, \ldots, a_n) = \sum_i \pm(1, a_i, \ldots, a_n, a_0, \ldots a_{i-1}) + \cdots$.

If we make the same bicomplex but with all the horizontal differentials zero, the obvious map of this bicomplex into the original bicomplex is a deformation retract. This gives us right back that $t_2 = B$ and $t_3 = t_4 = \cdots = 0$.