# Rational homotopy theory

Kathryn Hess

## 1 The Sullivan model

### 1.1 Rational homotopy theory of spaces

We will restrict our attention to simply-connected spaces. Much of this goes through with nilpotent spaces, but this will keep things easier and less technical.

**Definition 1.** A 1-connected space $X$ is said to be *rational* if either of the following equivalent conditions holds:

1. $\pi_* X$ forms a graded $\mathbb{Q}$-vector space.
2. $\tilde{H}_* X$ forms a graded $\mathbb{Q}$-vector space.

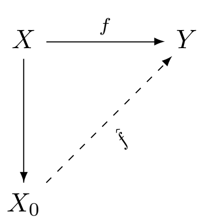
(If no mention of coefficients is made, we are using integral coefficients.)

**Remark 1.** The equivalence of the conditions can be proved first as follows. If $p$ is prime, we have $H_*(K(\mathbb{Q},1);\mathbb{F}_p) \cong H_*(pt;\mathbb{F}_p)$. An inductive spectral sequence argument shows that this holds for all $K(\mathbb{Q},n)$. Then for arbitrary $X$, we consider a Postnikov decomposition and argue inductively up the tower.

**Definition 2.** A *rationalization* of a 1-connected space $X$ is a map $\varphi : X \to X_0$ such that $X_0$ is rational and $\pi_*\varphi \otimes \mathbb{Q} : \pi_* X \otimes \mathbb{Q} \to \pi_* X_0 \otimes \mathbb{Q} \cong \pi_* X_0$ is an isomorphism.

Rationalizations always exist. For a CW-complex $X$, there is an explicit construction. It is easy to see how to rationalize a sphere by a telescope construction, and then we build $X_0$ out of rationalized spheres and discs corresponding to those in the decomposition of $X$.

**Theorem.** *For any 1-connected space $X$, there is a relative CW-complex $(X_0, X)$ with no 0- or 1-cells where $X_0$ is 1-connected and rational, such that the inclusion $j : X \hookrightarrow X_0$ is a rationalization. Moreover, this has the following universal property: if $Y$ is 1-connected and rational, then there is a map $\tilde{f} : X_0 \to Y$ which is unique up to homotopy that makes the diagram*




*commute up to homotopy. In particular, for any map $f : X \to Y$ we obtain a map $f_0 : X_0 \to Y_0$ on rationalizations.*

**Definition 3.** The *rational homotopy type* of a 1-connected space $X$ is the weak homotopy type of its rationalization $X_0$. (To say that two spaces $Y$ and $Z$ have the same weak homotopy type is to say that there is a zigzag of weak homotopy equivalences $Y \xleftarrow{\sim} \bullet \xrightarrow{\sim} \bullet \xleftarrow{\sim} \cdots \xleftarrow{\sim} \bullet \xrightarrow{\sim} Z$.)

**Definition 4.** A map of 1-connected spaces $f : X \to Y$ is a *rational homotopy equivalence* (RHE) if any of the following equivalent conditions hold:

1. $\pi_* f \otimes \mathbb{Q}$ is an isomorphism.
2. $H_*(f;\mathbb{Q})$ is an isomorphism.
3. $H^*(f;\mathbb{Q})$ is an isomorphism.

4. The induced map on rationalizations $f_0 : X_0 \to Y_0$ is a weak homotopy equivalence.

And now we can finally talk about the rational homotopy theory of spaces:

> **Rational homotopy theory of spaces is the study of rational homotopy types of spaces and of properties maps and spaces that are invariant under rational homotopy equivalence.**

Things ends up being much, much easier now that we've thrown out torsion. This is because there are very tidy algebraic models for spaces that allow us to make actual computations.
**Q.** Can you give us an example of a rational space?
**A.** Sure! $K(\mathbb{Q}, 5)$.

## 1.2 From spaces to CDGA's

**Definition 5.** If $V = \bigoplus_{k\geq 1} V^k$ is a positively graded $\mathbb{Q}$-vector space, by $\Lambda V$ we will mean the *free commutative graded algebra* on $V$. As a vector space, $\Lambda V = \mathbb{Q} \oplus V \oplus \Lambda^2 V \oplus \Lambda^3 V \oplus \cdots$. Here, $\Lambda^n V$ can be thought of as the words of length $n$ in $V$ up to symmetric action: $\Lambda^n V = (V^{\otimes n})_{\Sigma_n}$. Thus, $\Lambda V = TV/(v \otimes w - (-1)^{|v||w|} w \otimes v) = \mathbb{Q}[V^{even}] \otimes E(V^{odd})$.

**Remark 2.** For any commutative graded $\mathbb{Q}$-algebra $A$, a $\mathbb{Q}$-linear degree-0 map $V \to A$ has a unique extension to $\Lambda V \to A$.

**Remark 3.** For any $\mathbb{Q}$-linear degree-1 map $\delta : V \to \Lambda V$, there is a unique extension $\hat{\delta} : \Lambda V \to \Lambda V$ to a derivation: $\hat{\delta}(ab) = \hat{\delta}(a)b + (-1)^{|a|} a\hat{\delta}(b)$. Thus, to define a differential on $\Lambda V$ it is enough to define it on generators.

**Definition 6.** A *commutative differential graded algebra* (CDGA) $(A, d)$ is a commutative graded algebra $A$ with differential $d : A^n \to A^{n+1}$ which acts by derivations. These form a category **CDGA** with the obvious morphisms.

To get from spaces to CDGAs, we first pass to simplicial sets. Intermediately, we look at simplicial CDGAs. Let $\mathcal{A}^*_\bullet \in \mathbf{sCDGA}$ be given by

$$\mathcal{A}^*_n = \Lambda(t_0, \ldots, t_n; y_0, \ldots, y_n) \Bigg/ \left\{ \sum_{i=0}^{n} t_i - 1, \sum_{i=0}^{n} y_i \right\},$$

where we consider $|t_i| = 0$ and $|y_i| = 1$ and we set $dt_i = y_i$.

This gives us the following two important functors. First, $\mathcal{A}^* : \mathbf{sSet}^{op} \to \mathbf{CDGA}$ is given by $K \mapsto \mathbf{sSet}(K, \mathcal{A}^*_\bullet)$. We also have $\mathcal{K}_\bullet : \mathbf{CDGA}^{op} \to \mathbf{sSet}$ given by $A \mapsto \mathbf{CDGA}(A, \mathcal{A}^*_\bullet)$.

**Theorem** (polynomial Stokes-de Rham)**.** *As graded algebras, $H^*(\mathcal{A}^*(K)) \cong H^*(K; \mathbb{Q})$.*

**Theorem** (Bousfield-Gugenheim, 1976)**.** *There is a model category structure on* **CDGA** *with respect to which $\mathcal{A}^* : \mathbf{sSet}^{op} \leftrightarrows \mathbf{CDGA} : \mathcal{K}_\bullet$ is a Quillen pair, which gives rise to an equivalence*

$$[RHE]^{-1}\mathbf{sSet}_1^{fin} \simeq [QI]^{-1}\mathbf{CDGA}_1^{fin}.$$

(These are localized categories: on the left side we have inverted rational homotopy equivalences; on the right side we have inverted quasi-isomorphisms. The $fin$s refer to finiteness conditions and the 1s refer to 1-connectedness: on the left side we require that $\pi_0 X = *$, $\pi_1 X = 0$, and $\dim H_n(X; \mathbb{Q}) < \infty$ for all $n$; on the right side we require that $A^0 = \mathbb{Q}$, $A^1 = 0$, and $\dim H^n(A) < \infty$ for all $n$.)

**Remark 4.** One can replace **sSet** by **Top** using the adjunction $|-| : \mathbf{sSet} \leftrightarrows \mathbf{Top} : S_\bullet$.

## 1.3 Minimal models

**Definition 7.** A CDGA $(\Lambda V, d)$, with $V^1 = 0$ is called *minimal* if $d = d_2 + d_3 + \cdots$, where $d_k : V \to \Lambda^k V$ is the projection of $d : V \to \Lambda V$ to the summand $\Lambda^k V \subset \Lambda V$ (i.e., $d$ has no linear part). We then call $(\Lambda V, d)$ a *minimal Sullivan algebra.*

**Theorem.** *Any $(A, d) \in \mathbf{CDGA}_1^{fin}$ is modeled by a minimal Sullivan algebra $(\Lambda V, d)$ (i.e. there is a quasi-isomorphism of algebras $(\Lambda V, d) \xrightarrow{\sim} (A, d)$), and this is unique up to isomorphism. This is called a* minimal Sullivan model *for $(A, d)$.*

**Definition 8.** For a space $X$, we define $\mathcal{A}_{PL}(X) = \mathcal{A}^* \circ S_\bullet(X)$, the *piecewise-linear de Rham algebra* on $X$. The *minimal Sullivan model* of $X$ is by definition the minimal Sullivan model of $\mathcal{A}_{PL}(X)$. If $(\Lambda V, d) \xrightarrow{\sim} \mathcal{A}_{PL}(X)$ but $(\Lambda V, d)$ is not minimal, then this is just called a *Sullivan model*.

The minimal Sullivan model enjoys the following properties:

1. $H^*(\Lambda V, d) \cong H^*(X; \mathbb{Q})$ as graded commutative algebras.
2. For $n \geq 2$, $\mathrm{Hom}_{\mathbb{Q}}(V^n, \mathbb{Q}) \cong \pi_n X \otimes \mathbb{Q}$.

**Q.** Can you give us an example of a non-minimal Sullivan model?
**A.** Sure! Take $(\Lambda(v_2, w_3), d)$ with $dv = w$.
**Q.** So, minimal Sullivan models are cofibrant in the Bousfield-Gugenheim model structure. Is this true of all Sullivan models?
**A.** Yes, assuming you're careful about your constructions with well-ordered bases (which we avoided when we assumed $V^1 = 0$).

### 1.4 A tool for computing Sullivan models

Here is the slogan: "The model of a pullback is the pushout of the models." More precisely, we have the following.

**Theorem.** *Suppose we have a Serre fibration $p : E \to B$ and a pullback diagram*

$$\begin{array}{ccc} E \times_B X & \longrightarrow & E \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & B \end{array}$$

*where $B$ and $X$ are 1-connected and $E$ is path-connected, and suppose we have*

$$\begin{array}{ccccc} (\Lambda V', d') & \xleftarrow{\varphi} & (\Lambda V, d) & \hookrightarrow & (\Lambda(V \oplus W), D) \\ \big\downarrow{\scriptstyle \simeq} & & \big\downarrow{\scriptstyle \simeq} & & \big\downarrow{\scriptstyle \simeq} \\ \mathcal{A}_{PL}(X) & \xleftarrow[\mathcal{A}_{PL}(f)]{} & \mathcal{A}_{PL}(B) & \xrightarrow[\mathcal{A}_{PL}(p)]{} & \mathcal{A}_{PL}(E) \end{array}$$

*with Sullivan models along the top of the diagram. (The second map along the top is a quasi-free extension: as an algebra it's a free extension, and $(\Lambda V, d) \subset (\Lambda(V \oplus W), D)$ sits as a sub-CDGA. Think of a relative CW-complex, whose inclusion of the base is a cofibration. But since $\mathcal{A}_{PL}$ is contravariant, this models fibrations.) Then the induced map*

$$(\Lambda V', d') \otimes_{(\Lambda V, d)} (\Lambda(V \oplus W), D) \to \mathcal{A}_{PL}(E \times_B X)$$

*is a quasi-isomorphism of CDGAs.*

**Remark 5.** Explicitly, this pushout is $(\Lambda V', d') \otimes_{(\Lambda V, d)} (\Lambda(V \oplus W), D) \cong (\Lambda(V' \oplus W), D')$, where $D'v' = d'v'$. We have $(\Lambda V', d')$ sitting as a sub-DGA; note that the map $\Lambda(V \oplus W) \cong \Lambda V \otimes \Lambda W \xrightarrow{\varphi \otimes \mathrm{Id}} \Lambda V' \otimes \Lambda W \cong \Lambda(V' \oplus W)$ satisfies $D'w = (\varphi \otimes \mathrm{Id}) \circ Dw$.

**Corollary.** *With $p : E \to B$ as above and with Sullivan model*

$$\begin{array}{ccccc} (\mathbb{Q}, 0) & \xleftarrow{\varepsilon} & (\Lambda V, d) & \hookrightarrow & (\Lambda(V \otimes W), D) \\ \big\downarrow{\scriptstyle \simeq} & & \big\downarrow{\scriptstyle \simeq} & & \big\downarrow{\scriptstyle \simeq} \\ \mathcal{A}_{PL}(*) & \longleftarrow & \mathcal{A}_{PL}(B) & \longrightarrow & \mathcal{A}_{PL}(E) \end{array}$$

*then if $F$ is the fiber of $p$, then $(\mathbb{Q}, 0) \otimes_{(\Lambda V, d)} (\Lambda(V \otimes W), D) \cong (\Lambda W, \overline{D})$ is a model of $\mathcal{A}_{PL}(F)$, where $\overline{D}w = (\varepsilon \otimes \mathrm{Id}_{\Lambda W}) \circ Dw$.*

# 2 The Quillen model

This model is Eckmann-Hilton/Koszul dual (depending on your perspective) to the Sullivan model. It involves differential graded Lie algebras as opposed to just differential graded algebras.

## 2.1 Homotopy Lie algebras

We will first explain how Lie algebras show up in rational homotopy theory.

### 2.1.1 The case of spaces

Given a 1-connected topological space $X$, $H_*(\Omega X;\mathbb{Q})$ is naturally a graded Hopf algebra. The multiplication comes from the Eilenberg-Zilber map

$$C_*\Omega X \otimes C_*\Omega X \overset{\simeq \text{ by E-Z}}{\longrightarrow} C_*(\Omega X \times \Omega X) \overset{C_*\mu}{\longrightarrow} C_*\Omega X.$$

The comultiplication comes from the Alexander-Whitney map

$$C_*\Omega X \overset{C_*\Delta}{\longrightarrow} C_*(\Omega X \times \Omega X) \overset{A-W}{\longrightarrow} C_*\Omega X \otimes C_*\Omega X.$$

Meanwhile, $\pi_*\Omega X \otimes \mathbb{Q}$ is naturally a graded Lie algebra with the *Samelson bracket*. To define this, suppose we are given pointed maps $\alpha : S^p \to \Omega X$ and $\beta : S^q \to \Omega X$. Then we can take the composite

$$S^p \vee S^q \to S^p \times S^q \overset{\alpha\times\beta}{\to} \Omega X \otimes \Omega X \overset{\gamma}{\to} \Omega X$$

where $\gamma$ is conjugation: $(\lambda,\mu) \mapsto \lambda\mu\overline{\lambda}\overline{\mu}$, and this is homotopically trivial. So there is an induced map

$$(S^p \times S^q)/(S^p \vee S^q) \cong S^{p+q} \overset{\alpha\wedge\beta}{\to} \Omega X,$$

and we set $[\overline{\alpha},\overline{\beta}] = [\alpha \wedge \beta]$. And thus we obtain $\mathcal{L}_X = (\pi_*\Omega X \otimes \mathbb{Q}, [-,-])$, the *homotopy Lie algebra* of $X$.

**Theorem** (Milnor-Moore, 1965)**.** *The rational Hurewicz homomorphism $h : \pi_*\Omega X \otimes \mathbb{Q} \to H_*(\Omega X;\mathbb{Q})$ induces two equivalent isomorphisms:*

$$\pi_*\Omega X \otimes \mathbb{Q} \overset{\cong}{\to} \mathrm{Prim}(H_*(\Omega X;\mathbb{Q})) \text{ (as graded Lie algebras)}$$
$$\mathcal{U}(\pi_*\Omega X \otimes \mathbb{Q}) \overset{\cong}{\to} H_*(\Omega X;\mathbb{Q}) \text{ (as graded Hopf algebras).}$$

**Remark 6.** If $L$ is a Lie algebra, we define $\mathcal{U}L = TL/\langle x \otimes y - (-1)^{|x||y|} y \otimes x - [x,y]\rangle$, the *universal enveloping algebra.*

**Remark 7.** Taking the primitives is the dual construction to taking the indecomposables $QA = A/A^2$ in an algebra. The primitives are exactly the elements $c$ such that $\Delta(c) = c \otimes 1 + 1 \otimes c$. It is not hard to show that the primitive elements of a Hopf algebra form a Lie algebra.

### 2.1.2 The case of CGDAs

Let $(\Lambda V, d)$ be a 1-connected minimal Sullivan algebra. Then write $d = d_2 + d_3 + d_4 + \cdots$, where $d_k : V \to \Lambda^k V$. Assume for simplicity that $\Lambda V$ is of finite type (which is equivalent to demanding that $\dim V^i < \infty$ for all $n$).

**Micro-exercise 1.** If $d^2 = 0$, then $d_2^2 = 0$.

Let $L_*$ be a graded $\mathbb{Q}$-vector space defined by $L_n = \mathrm{Hom}_{\mathbb{Q}}(V^{n+1},\mathbb{Q})$, i.e. $L_* = s^{-1}\mathrm{Hom}_{\mathbb{Q}}(V^*,\mathbb{Q})$ (where $s^{-1}$ is the desuspension).

**Definition 9.** Let $(v_i | i \in I)$ be a basis of $V$, with dual basis $(\hat{v}_i | i \in I)$. Then, define $[-,-] : L_* \otimes L_* \to L_*$ by dualizing $d_2 : V \to \Lambda^2 V$ to $\mathrm{Hom}_{\mathbb{Q}}(\Lambda^2 V,\mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Q}}(V,\mathbb{Q})$. The Jacobi identity then ends up being equivalent to the condition that $d_2^2 = 0$, and the anticommutativity comes from the fact that multiplication in $\Lambda V$ is commutative. This gives us another homotopy Lie algebra $\mathcal{L}_{(\Lambda V,d)}$ depending only on the $d_2$ part of the differential.

**Theorem.** *If $(\Lambda V, d)$ is a Sullivan minimal model of $X$, then $\mathcal{L}_X \cong \mathcal{L}_{(\Lambda V,d)}$.*

## 2.2 From spaces to DGLs

There are two apparently different ways to take a space and get a differential graded Lie algebra, although they end up being equivalent (though this was a difficult theorem).

The first is due to Quillen in 1969, which runs through a series of Quillen pairs:

$$\mathbf{Top}_1 \xrightarrow{\overline{S}_\bullet} \mathbf{sSet}_1 \xrightarrow{\mathbb{G}} \mathbf{sGr}_0 \xrightarrow{\hat{\mathbb{Q}}[-]} \mathbf{sHA}_0^{comp} \xrightarrow{\mathrm{Prim}} \mathbf{sLie}_0 \xrightarrow{N_*} \mathbf{DGL}_0.$$

(Here, $\overline{S}_\bullet(X) = S_\bullet(X)/\{$0- and 1-simplices$\}$, $\mathbb{G}$ the Kan loop group functor, $\hat{\mathbb{Q}}[-]$ is the completed group ring functor, Prim is the functor of primitives, and $N_*$ is the normalized chain functor.) Quillen called the entire composite functor $\lambda$, and he proved that $\lambda$ induces an equivalence

$$[RHE]^{-1}\mathbf{Top}_1 \simeq [QI]^{-1}\mathbf{DGL}_0.$$

The second approach is much more concrete, which we will call the "Sullivan" approach (since it is not really due to Sullivan). We begin with a Sullivan model and work out way to a Quillen model. Our key tool here is the adjunction $\mathcal{L}_* : \mathbf{DGCC} \leftrightarrows \mathbf{DGL}_0 : \mathcal{C}_*$ ($\mathbf{DGCC}$ is the category of differential graded cocommutative coalgebra (i.e. coalgebras over the cocommutative cooperad)). First, $\mathcal{C}_*(L,\delta) = (\Lambda_{co}(sL), d)$, where $sL$ is the suspension of $L$, given by $(sL)_n = L_{n-1}$, and $\Lambda_{co}W = \mathbb{Q} \oplus \bigoplus_{n\geq 1} \Lambda^n_{co}W$, where $\Lambda^n_{co}W = (T^nW)^{\Sigma_n}$. This has $d = d_1 + d_2$, where

$$d_1(sx_1 \wedge \cdots \wedge sx_n) = \sum_i \pm sx_1 \wedge \cdots \wedge s\delta x_i \wedge \cdots \wedge sx_n$$
$$d_2(sx_1 \wedge \cdots \wedge sx_n) = \sum_{i<j} \pm s[x_i,x_j] \wedge sx_1 \wedge \cdots \wedge sx_{i-1} \wedge sx_{i+1} \wedge \cdots \wedge sx_{j-1} \wedge sx_j \wedge \cdots sx_n.$$

There is a natural quasi-isomorphism $\mathcal{C}_*(L,\delta) \xrightarrow{\simeq} B\mathcal{U}(L,\delta)$. On the other side, we define $\mathcal{L}_*(C,d) = (\mathbb{L}(s^{-1}C_{>0},\delta)$, where $\mathbb{L}$ stands for the free Lie algebra and

$$\delta(s^{-1}c) = \pm s^{-1}(dc) + \frac{1}{2}\sum_i \pm[s^{-1}c_i, s^{-1}c^i]$$

where $\Delta(c) = c \otimes 1 + 1 \otimes c + \sum_i c_i \otimes c^i$.

**Remark 8.** $\mathcal{L}_*(C,d) \cong \mathrm{Prim}(\Omega(C,d))$.

**Theorem.** *$(\mathcal{L}_*, \mathcal{C}_*)$ are adjoint, and moreover $\eta : (C,d) \xrightarrow{\simeq} \mathcal{C}_*\mathcal{L}_*(C,d)$ and $\varepsilon : \mathcal{L}_*\mathcal{C}_*(L,\delta) \xrightarrow{\simeq} (L,\delta)$ for all $(C,d)$ and all $(L,\delta)$.*

To make the connection with DGCA, we use $\mathcal{D} = \mathrm{Hom}_{\mathbb{Q}}(-,\mathbb{Q})$. This gives us an adjunction

$$\mathcal{L}^* = \mathcal{L}_* \circ \mathcal{D} : \mathbf{CDGA}_1^{fin} \leftrightarrows \mathbf{DGL}_0 : \mathcal{D} \circ \mathcal{C}_* = \mathcal{C}^*,$$

giving $(\Lambda V, d) \xrightarrow{\simeq} \mathcal{A}_{PL}(X)$. Thus $\mathcal{L}^*(\Lambda V, d) \in \mathbf{DGL}_0$.

## 2.3 Minimal models

**Definition 10.** The *minimal model* of the DGL $(L,\delta)$ is a DGL $(\mathbb{L}V,\delta)$ together with a quasi-isomorphism of DGLs $(\mathbb{L}V,\delta) \xrightarrow{\simeq} (L,\delta)$ such that $\delta(V) \subseteq \mathbb{L}^{\geq 2}(V)$.

**Theorem** (Majewski, 2000)**.** *The minimal models of $\lambda(X)$ and of $\mathcal{L}^*(\Lambda V, d)$ (for $(\Lambda V, d)$ the Sullivan model of $X$) are isomorphic; we call either the* Quillen model *of $X$.*

**Theorem.** *Minimal models exist and are unique.*

The Quillen model enjoys the following properties:

1. $(\mathbb{L}V,\delta) \xrightarrow{\simeq} \lambda X$
2. $sV \cong \tilde{H}_*(X;\mathbb{Q})$
3. $H_*(\mathbb{L}V,\delta) \cong \mathcal{L}_X$

## 3 References


1. Felix, Halperin, Thom: *Rational Homotopy Theory*.
2. Hess: *A brief intro* (arXiv).
3. Griffiths and Morgan (geometric perspective)
4. Bousfield and Gugenheim (model categorical perspective)