Triangulated categories

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1 Triangulated categories

1.1 Definitions

Definition 1. A *triangulated category* is an additive category $\mathcal{T}$ (i.e. a category with finite sums and finite products (in particular with initial and terminal objects) such that coproducts are isomorphic to products) equipped with an auto-equivalence $\Sigma : \mathcal{T} \to \mathcal{T}$ (called *suspension*) and a class of diagrams (called *distinguished triangles*) $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ (which for convenience we will often write as (f, g, h)) satisfying the axioms:

- *(T0)* the distinguished triangles are closed under isomorphisms;
- *(T1)* For all $X \in \mathcal{T}$, the triangle $0 \to X \xrightarrow{\mathrm{Id}} X \to 0$ is distinguished;
- *(T2)* If (f, g, h) is a distinguished triangle, then its *(forward) rotation* $(g, h, -\Sigma f)$ is also distinguished;
- *(T3)* Given the diagram whose rows are distinguished triangles and whose left square commutes

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Sigma\alpha} \\ A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & \xrightarrow[h']{} & \Sigma A' \end{array}$$

  there exists $\gamma : C \to C'$ such that the entire diagram commutes.

- *(T4, the octahedral axiom)* Given $f : A \to B$ and $f' : B \to D$ there is a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\ \| & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle x} & & \| \\ A & \xrightarrow{f'f} & D & \xrightarrow{g''} & E & \xrightarrow{h''} & \Sigma A \\ & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle \Sigma f} \\ & & F & = & F & \xrightarrow{h'} & \Sigma B \\ & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle (\Sigma g)h'} & & \\ & & \Sigma B & \xrightarrow[\Sigma g]{} & \Sigma C & & \end{array}$$

  in which all four triangles are distinguished.

(Note that in a morphism of triangles, we require that the map $\Sigma A \to \Sigma A'$ is the suspension of the map $A \to A'$.)

Remark 1. In the usual formulation, T2 is an iff and T4 says that 3 of the 4 triangles are given and then we get x and y. We will prove that these axioms are equivalent, which will be much more convenient for our future generalizations.

In contrast to many other sorts of structures in mathematics like groups or vector spaces, these axioms are not so aesthetically pleasing. What justifies these is the abundance of examples.

1.2 Examples

Exercise 1. Let k be a field, and let $\mathcal{T} = \mathbf{Vect}_k$. Define $\Sigma = \mathrm{Id}$, and call $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A$ distinguished whenever $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A \xrightarrow{f} B$ is exact. Prove that this satisfies the axioms.

Exercise 2. Let $\mathcal{F}(\mathbb{Z}/4)$ be the category of finitely generated free $\mathbb{Z}/4$-modules. Prove that there is a unique triangulated structure with $\Sigma = \mathrm{Id}$ such that $\mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4$ is distinguished.

Exercise 3. Let $\mathcal{A}$ be an additive category, and let $\mathcal{T} = K(A)$ be the homotopy category of chain complexes in $\mathcal{A}$. Show that this is triangulated with Σ any shift functor $C \mapsto C[1]$ (there are at least four different but isomorphic choices of conventions), and where distinguished triangles are mapping cone sequences $C \xrightarrow{f} D \to C(D) = D \oplus C[1] \to C[1]$, where the differential on $C(D)$ is

$$d = \begin{pmatrix} d_D & f \\ 0 & d_C \end{pmatrix}.$$

Example. Let $\mathcal{A}$ be an abelian category, and define the *derived category* by $\mathcal{D}(A) = K(A)[q\text{-}iso^{-1}]$. This is triangulated, with distinguished triangles the images of distinguished triangles in $K(A)$. (We will ignore the very real set-theoretic issues in these definitions.) For $M \in \mathcal{A}$ we can define the complex $M[n]$ which has M in degree n and 0 elsewhere. Then we have a natural isomorphism $\mathcal{D}(A)(M[0], N[n]) \cong \mathrm{Ext}^n_{\mathcal{A}}(M, N)$.

Exercise 4. Let R be a Frobenius ring, i.e. the class of projective right R-modules equals the class of injective **left/right/???** R-modules. Let $f, g : M \to N$ be two R-module maps. We say f is *homotopic* to g whenever $f - g : M \to N$ factors through a projective R-module. This gives us the *stable category* $\underline{\mathbf{mod}}$-R, whose whose objects are R-modules with $\underline{\mathbf{mod}}$-$R(M, N) = \mathbf{mod}$-$R(M, N)/\{$maps factoring through a projective$\}$. Show that this is triangulated.

Example (the stable homotopy category)**.** A *spectrum* X is a sequence of based topological spaces X_n for $n \geq 0$ and with structure maps $\sigma_n : \Sigma X_n = X_n \wedge S^1 \to X_{n+1}$. A morphism of spectra $f : X \to Y$ is a sequence of maps $f_n : X_n \to Y_n$ for $n \geq 0$ such that the diagrams

$$\begin{array}{ccc} X_n & \xrightarrow{\sigma_n} & X_{n+1} \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} \\ Y_n & \xrightarrow[\sigma_n]{} & Y_{n+1} \end{array}$$

are strictly commutative. The homotopy groups of a spectrum are defined by $\pi_k X = \mathrm{colim}_n \pi_{k+n}(X_n)$, where the colimit is taken over the maps

$$\pi_{k+n} X_n \xrightarrow{-\wedge S^1} \pi_{k+n+1}(X_n \wedge S^1) \xrightarrow{(\sigma_n)_*} \pi_{k+n+1} X_{n_1}.$$

A map $f : X \to Y$ of spectra is a π_*-*isomorphism* if $\pi_k f : \pi_k X \to \pi_k Y$ is an isomorphism for all $k \in \mathbb{Z}$. We then define the *stable homotopy category* by $\mathcal{SHC} = \mathbf{Spectra}[\pi_*\text{-}iso^{-1}]$. This is triangulated! The suspension functor is given by $(\Sigma X)_n = S^1 \wedge X_n$, wich structure maps $(\Sigma X)_n \wedge S^1 = S^1 \wedge X_n \wedge S^1 \xrightarrow{S^1 \wedge \sigma_n} S^1 \wedge X_{n+1}$.

1.3 Basic properties

Proposition (long exact sequences)**.** *Let $\mathcal{T}$ be a triangulated category and (f,g,h) be a distinguished triangle. The for all $X \in \mathcal{T}$, the sequences*

$$\mathcal{T}(\Sigma A, X) \xrightarrow{h^*} \mathcal{T}(C,X) \xrightarrow{g^*} \mathcal{T}(B,X) \xrightarrow{f^*} \mathcal{T}(A,X)$$
$$\mathcal{T}(X,A) \xrightarrow{f_*} \mathcal{T}(X,B) \xrightarrow{g_*} \mathcal{T}(X,C) \xrightarrow{h_*} \mathcal{T}(X,\Sigma A)$$

are exact. (By rotating, we can actually extend these indefinitely in either direction.)

Proof. First, consider the diagram

$$\begin{array}{ccccccc}
A & = & A & \longrightarrow & 0 & \longrightarrow & \Sigma A \\
\| & & \downarrow{\scriptstyle f} & & \downarrow & & \| \\
A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & C & \xrightarrow[h]{} & \Sigma A.
\end{array}$$

The existence of the morphism $0 \to C$ implies that $gf = 0$.

Now we can show exactness of $\mathcal{T}(C,X) \xrightarrow{g^*} \mathcal{T}(B,X) \xrightarrow{f^*} \mathcal{T}(A,X)$. Since $gf = 0$, then certainly $\mathrm{im}(g^*) \subseteq \ker(f^*)$. On the other hand, suppose $\psi : B \to X$ is in $\ker(f^*)$. This gives us an extension problem

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\
\downarrow & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi} & & \downarrow \\
0 & \longrightarrow & X & = & X & \longrightarrow & 0.
\end{array}$$

(The top row is distinguished by assumption, and the bottom row is distinguished by T1.) By T3 there is $\varphi : C \to X$ such that $g^*\varphi = \varphi g = \mathrm{Id}_X f = f$. Thus $\mathrm{im}(g^*) \supseteq \ker(f^*)$. To check exactness at $\mathcal{T}(C,X)$ we can simply rotate our triangle.

We proceed to the covariant sequence. To check exactness at $\mathcal{T}(X,A) \xrightarrow{f_*} \mathcal{T}(X,B) \xrightarrow{g_*} \mathcal{T}(X,A)$, again we know that $\mathrm{im}(f_*) \subseteq \ker(g_*)$ since $gf = 0$. Now, suppose $\psi : X \to B$ is in $\ker(g_*)$, i.e. $g\psi = 0$. We now get the extension diagram

$$\begin{array}{ccccccc}
X & \longrightarrow & 0 & \longrightarrow & \Sigma X & \xrightarrow{-\mathrm{Id}_{\Sigma X}} & \Sigma X \\
\downarrow{\scriptstyle \psi} & & \downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \Sigma f} \\
B & \xrightarrow[g]{} & C & \xrightarrow[h]{} & \Sigma A & \xrightarrow[-\Sigma f]{} & \Sigma B.
\end{array}$$

(The first square commutes by the assumption that $\psi \in \ker(g_*)$.) Now, since Σ is a full functor, then $\gamma = \Sigma\varphi$, and $\Sigma\psi = \Sigma f \circ \Sigma\varphi = \Sigma(f\varphi)$. On the other hand, since Σ is also faithful, then $\psi = f\varphi$, i.e. $f \in \mathrm{im}(f_*)$. Again we can rotate to obtain exactness at $\mathcal{T}(X,C)$. □

Proposition (2-out-of-3 property)**.** *Let (a,b,c) be a morphism between distinguished triangles. If two out of the three of these morphisms are isomorphisms, then so is the third.*

Proof. Consider the diagram

$$\begin{array}{ccccccccc}
\mathcal{T}(X,A) & \xrightarrow{f_*} & \mathcal{T}(X,B) & \xrightarrow{g_*} & \mathcal{T}(X,C) & \xrightarrow{h_*} & \mathcal{T}(X,\Sigma A) & \xrightarrow{(-\Sigma f)_*} & \mathcal{T}(X,\Sigma B) \\
\downarrow{\scriptstyle a_*} & & \downarrow{\scriptstyle b_*} & & \downarrow{\scriptstyle c_*} & & \downarrow{\scriptstyle (\Sigma a)_*} & & \downarrow{\scriptstyle (\Sigma b)_*} \\
\mathcal{T}(X,A') & \xrightarrow{f'_*} & \mathcal{T}(X,B') & \xrightarrow{g'_*} & \mathcal{T}(X,C') & \xrightarrow{h'_*} & \mathcal{T}(X,\Sigma A') & \xrightarrow{(-\Sigma f')_*} & \mathcal{T}(X,\Sigma B').
\end{array}$$

The rows are exact by the previous proposition. Suppose first that a and b are isomorphisms. Then so are a_*, b_*, $(\Sigma a)_*$ and $(\Sigma b)_*$. So c_* is an isomorphism by the 5-lemma. Since X is arbitrary, then by the Yoneda lemma c is an isomorphism.

The other cases are similar. □

Proposition (backwards rotation)**.** *Let (f, g, h) be a triangle such that the rotated triangle $(g, h, -\Sigma f)$ is distinguished. Then (f, g, h) is also distinguished.*

Proof. Choose a distinguished triangle $A \xrightarrow{f} B \xrightarrow{\overline{g}} \overline{C} \xrightarrow{\overline{h}} \Sigma A$ (by axiom T4 using f and Id_B). Rotating this three times gives us that $(-\Sigma f, -\Sigma\overline{g}, -\Sigma\overline{h})$ is distinguished, and rotating $(g, h, -\Sigma f)$ two times gives that $(-\Sigma f, -\Sigma g, -\Sigma h)$ is distinguished. Then we have the diagram of distinguished triangles

$$\begin{array}{ccccccc}
\Sigma A & \xrightarrow{-\Sigma f} & \Sigma B & \xrightarrow{-\Sigma g} & \Sigma C & \xrightarrow{-\Sigma h} & \Sigma A \\
\| & & \| & & \downarrow & & \| \\
\Sigma A & \xrightarrow[-\Sigma f]{} & \Sigma B & \xrightarrow[-\Sigma \overline{g}]{} & \Sigma \overline{C} & \xrightarrow[-\Sigma \overline{h}]{} & \Sigma A
\end{array}$$

(whose left square clearly commutes) which gives us a map $c' : \Sigma C \to \Sigma\overline{C}$ by axiom T3. But by the 2-out-of-3 property, c' must be an isomorphism. Since Σ is an equivalence, we can obtain a map $c : C \to \overline{C}$ which is also an isomorphism. Now $(\mathrm{Id}_A, \mathrm{Id}_B, c) : (f, g, h) \to (f, \overline{g}, \overline{h})$ is an isomorphism of triangles, which by axiom T0 means that (f, g, h) is distinguished. □

2 Stable cofibration categories

Cofibration categories are substantially more general than triangulated categories, but they include almost all known examples of triangulated categories. The work here is due to Brown, Waldhausen, Baues, Anderson, et al.

Definition 2. A *cofibration category* is a category $\mathcal{C}$ equipped with cofibrations and weak equivalences such that:

- *(C1)* isomorphisms are both cofibrations and weak equivalences, cofibrations are closed under composition, and there is an initial object;
- *(C2)* 2-out-of-3 for weak equivalences;
- *(C3)* cofibrations admit cobase change:

$$\begin{array}{ccc}
A & \overset{i}{\rightarrowtail} & B \\
{\scriptstyle f}\downarrow & & \downarrow \\
C & \underset{j}{\rightarrowtail} & P,
\end{array}$$

  and if i is a weak equivalence then so is j;.

Definition 3. The *homotopy category* of $\mathcal{C}$ is the localization $\gamma : \mathcal{C} \to Ho(\mathcal{C})$ at the class of weak equivalences.

Example. One can take the cofibrant objects in a model category.

Example. If $\mathcal{A}$ is an additive category, then on $Ch(\mathcal{A})$ (or one of its bounded cousins) we can take weak equivalences to be the chain homotopy equivalences and cofibrations to be dimensionwise split monomorphisms. Then, the homotopy category of this cofibration category is the same as if we consider $Ch(\mathcal{A})$ as a triangulated category.

Remark 2. The homotopy category $Ho(\mathcal{C})$ may not have small Hom sets. One can fix this by assuming the existence of enough fibrant objects (as before, an object is fibrant if it has the right lifting property with respect to all acyclic cofibrations). Thus we can write $Ho(\mathcal{C})(X, Y) = \mathcal{C}(X, Y^{fib})/\sim$, where we quotient out by homotopy equivalences. We will ignore these issues since they are completely orthogonal to everything we want to do.

The homotopy category $Ho(\mathcal{C})$ has a "calculus of fractions". This yields the following consequences.

- Every morphism in $Ho(\mathcal{C})$ is of the form $\gamma(s)^{-1} \circ \gamma(a)$ for $\mathcal{C}$-morphisms $A \xrightarrow{a} B' \overset{s}{\underset{\sim}{\leftarrowtail}} B$.
- Let $f, g : A \to B$ be $\mathcal{C}$-morphisms such that $\gamma(f) = \gamma(g)$. These may not be homotopic, but there must be an acyclic cofibration $s : B \overset{\sim}{\rightarrowtail} B$ such that $sf \simeq sg$.
- $\gamma : \mathcal{C} \to Ho(\mathcal{C})$ preserves coproducts.
- Just because $\gamma(f)$ is an isomorphism in $Ho(\mathcal{C})$ does not mean that f is a weak equivalence. However, we can modify our cofibration category to extend weak equivalences to all maps that become weak equivalences in $Ho(\mathcal{C})$, and this will again be a cofibration category and it will have the same homotopy category as before.

Definition 4. Assume $\mathcal{C}$ is pointed, i.e. every initial object is also terminal (so these are zero objects, which we will denote $*$). A *cone* of $A \in \mathcal{C}$ is a cofibration $i : A \rightarrowtail CA$ such that $CA \simeq *$. Note that C need not be a functor in A on the level of the cofibration category. A *suspension* is any pushout

$$\begin{array}{ccc} A & \overset{i}{\rightarrowtail} & CA \\ \downarrow & & \downarrow p \\ * & \longrightarrow & \Sigma A. \end{array}$$

Proposition. Σ *passes to a functor* $\Sigma : Ho(\mathcal{C}) \to Ho(\mathcal{C})$ *(regardless of our choices on* $\mathcal{C}$*).*

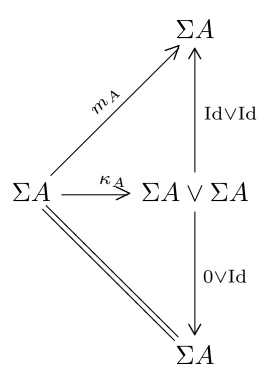
Definition 5. Let $j : A \rightarrowtail B$ be a cofibration. Then the *connecting morphism* $\delta(j) : B/A \to \Sigma A$ in $Ho(\mathcal{C})$ is defined as the composite $B/A \overset{\sim}{\leftarrow} CA \cup_j B \to CA/A = \Sigma A$, where the first map collapses the cone CA and the second map collapses B.

Definition 6. A triangle in $Ho(\mathcal{C})$ is *distinguished* if it is isomorphic to $A \xrightarrow{\gamma(j)} B \xrightarrow{\gamma(q)} B/A \xrightarrow{\delta(i)} \Sigma A$ for some cofibration $j : A \rightarrowtail B$.

Theorem. *If* $\mathcal{C}$ *is stable (i.e.* $\Sigma : Ho(\mathcal{C}) \to Ho(\mathcal{C})$ *is an equivalence) then* $Ho(\mathcal{C})$ *is triangulated.*

Before we prove this, we first discuss additivity. Define the *collapse map* $\kappa_A : \Sigma A \to \Sigma A \vee \Sigma A$ in $Ho(\mathcal{C})$ by $\Sigma A = CA/A \underset{0 \cup p}{\overset{\sim}{\longleftarrow}} CA \cup_A CA \xrightarrow{p \cup p} \Sigma A \vee \Sigma A$. Note that this is *not* the pinch map.

Proposition. *In* $Ho(\mathcal{C})$*, the diagram*



commutes,and $m_a = (\mathrm{Id} \vee \mathrm{Id}) \circ \kappa_A : \Sigma A \to \Sigma A$ *satisfies* $m_A^2 = \mathrm{Id}$.

Proof. First, in $\mathcal{C}$ the diagram

$$\begin{array}{ccc} * \simeq CA & \xrightarrow{p} & \Sigma A \\ \uparrow \scriptstyle{\mathrm{Id} \cup \mathrm{Id}} & & \uparrow \scriptstyle{\mathrm{Id} \vee \mathrm{Id}} \\ CA \cup_A CA & \xrightarrow[p \cup p]{} & \Sigma A \vee \Sigma A \end{array}$$

commutes. In $Ho(\mathcal{C})$ we have $\kappa_A = (p \cup p) \circ (0 \cup p)^{-1}$, so

$$\begin{aligned}
m_A^2 &= (\Sigma A \xrightarrow{\kappa_A} \Sigma A \vee \Sigma A \xrightarrow{\mathrm{Id}\vee\mathrm{Id}} \Sigma A \xrightarrow{\kappa_A} \Sigma A \vee \Sigma A \xrightarrow{\mathrm{Id}\vee\mathrm{Id}} \Sigma A) \\
&= (\Sigma A = CA/A \xleftarrow[0\cup p]{\sim} CA \cup_A CA \xrightarrow{p\cup p} \Sigma A \vee \Sigma A \xrightarrow{\mathrm{Id}\vee\mathrm{Id}} \Sigma A = CA/A \xleftarrow[0\cup p]{\sim} CA \cup_A CA \xrightarrow{p\cup p} \Sigma A \vee \Sigma A \xrightarrow{\mathrm{Id}\vee\mathrm{Id}} \Sigma A) \\
&= (\Sigma A = CA/A \xleftarrow[0\cup p]{\sim} CA \cup_A CA \xrightarrow{\mathrm{Id}\cup\mathrm{Id}} CA \xrightarrow{p} \Sigma A = CA/A \xleftarrow[0\cup p]{\sim} CA \cup_A CA \xrightarrow{\mathrm{Id}\cup\mathrm{Id}} CA \xrightarrow{p} \Sigma A) \\
&= (\Sigma A = CA/A \xleftarrow[0\cup p]{\sim} CA \cup_A CA \xrightarrow{p\cup p} \Sigma A = CA/A \xleftarrow[0\cup p]{\sim} CA \cup_A CA \xrightarrow{p\cup p} \Sigma A) \\
&= \textbf{MAGIC} \\
&= (\Sigma A \xrightarrow{\mathrm{Id}} \Sigma A).
\end{aligned}$$

Moreover, the commutative diagram

$$\begin{array}{ccccc}
\Sigma A & \xleftarrow[\sim]{0\cup p} & CA \cup_A CA & \xrightarrow{p\cup p} & \Sigma A \vee \Sigma A \\
 & & & \searrow^{0\cup p} & \downarrow{\scriptstyle 0\vee\mathrm{Id}} \\
 & & & & \Sigma A
\end{array}$$

has the identity along the bottom. $\square$

Proposition. *Consider $[B, X \vee Y] \to [B, X] \times [B, Y]$ given by $\varphi \mapsto (p_X\varphi, p_Y\varphi)$.*

1. *If B is a suspension, then the map is surjective.*
2. *Let $\varphi, \psi : B \to X \vee Y$ such that $p_X\varphi = p_X\psi$ and $p_Y\varphi = p_Y\psi$. Then $\Sigma\varphi = \Sigma\psi$.*
3. *If additionally Σ is an equivalence, then $Ho(\mathcal{C})$ is additive and $m_A = -\mathrm{Id}_{\Sigma A}$.*

Proof. For 1, take $B = \Sigma A$, and let $\alpha : B \to X$ and $\beta : B \to Y$. Then note that in the diagram

$$\begin{array}{ccccc}
\Sigma A & \xrightarrow{\kappa_A} & \Sigma A \vee \Sigma A & \xrightarrow{(\alpha\circ m_a)\vee\beta} & X \vee Y \\
 & \searrow^{m_A} & \downarrow{\scriptstyle \mathrm{Id}\vee 0} & & \downarrow{\scriptstyle p_X} \\
 & & \Sigma A & \xrightarrow{\alpha\circ m_A} & X,
\end{array}$$

following along the bottom is $(\alpha \circ m_A) \circ m_A = \alpha : \Sigma A \to X$. Then the top row is our desired preimage of (α, β).

We will skip 2.

For 3, we know that from 1 and 2 we get that the map is always bijective in this case. In $Ho(\mathcal{C})$, coproducts are products. Thus $[B, Z]$ has a natural abelian monoid structure, as follows. Let $f, g \in [B, Z]$. Then there is a unique map $f \perp g : B \to Z \vee Z$ such that $(1 + 0)(f \perp g) = f$ and $(0 + 1)(f \perp g) = g$. Then we set $f + g = (1 + 1)(f \perp g)$. Now we have that $\kappa_A : \Sigma A \to \Sigma A \vee \Sigma A$, and $(1 + 0)\kappa_A = m_A$ and $(0 + 1)\kappa_A = \mathrm{Id}$, so $\kappa_A = m_A \perp \mathrm{Id}_{\Sigma A}$. Thus $m_A + \mathrm{Id}_{\Sigma A} = (1 + 1)\kappa_A = 0$. $\square$

Proposition. *The connecting morphism is natural in $\mathcal{C}$.*

Proof. We have the diagram

$$\begin{array}{ccc}
A & \overset{j}{\rightarrowtail} & B \\
{\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\
A' & \underset{j'}{\rightarrowtail} & B'
\end{array}$$

which gives us that in $Ho(\mathcal{C})$, the diagram

$$\begin{array}{ccc} B/A & \xrightarrow{\delta(j)} & \Sigma A \\ {\scriptstyle j(\beta/\alpha)}\downarrow & & \downarrow{\scriptstyle \Sigma\gamma(a)} \\ B'/A' & \xrightarrow[\delta(j')]{} & \Sigma A' \end{array}$$

commutes. $\square$

We can now prove the axioms T1-T4 for $Ho(\mathcal{C})$.

- *(T1)* Suppose $* = \emptyset \rightarrowtail X$ is a cofibration. Then $0 \to X \to X \to 0$ is an elementary distinguished triangle since $X = X/0$.

- *(T2)* It suffices to consider $A \overset{\gamma(j)}{\to} B \overset{\gamma(q)}{\to} B/A \overset{\delta(j)}{\to} \Sigma A$ since these are the elementary distinguished triangles. Then the diagram in $\mathcal{C}$

$$\begin{array}{ccccc} A & \overset{j}{\rightarrowtail} & B & \longrightarrow & * \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle k} & & \downarrow \\ CA & \longrightarrow & CA \cup_j B & \xrightarrow[p\cup 0]{} & \Sigma A \end{array}$$

has that all three rectangles are pushouts (by the 2-out-of-3 property for cofibrations). Thus in $Ho(\mathcal{C})$, the diagram

$$\begin{array}{ccccccc} B & \xrightarrow{\gamma(k)} & CA \cup_j B & \xrightarrow{\gamma(p\cup_j 0)} & \Sigma A & \xrightarrow{\delta(k)} & \Sigma B \\ \| & & \cong \downarrow {\scriptstyle \gamma(0\cup q)} & & \| & & \| \\ B & \xrightarrow[\gamma(q)]{} & B/A & \xrightarrow[\delta(j)]{} & \Sigma A & \xrightarrow[\Sigma\gamma(j)\circ\delta(i_A)]{} & \Sigma B \end{array}$$

has top row a distinguished triangle since it comes from a cofibration. (The second square commutes by definition of δ, and the third square commutes by its naturality.) Note that

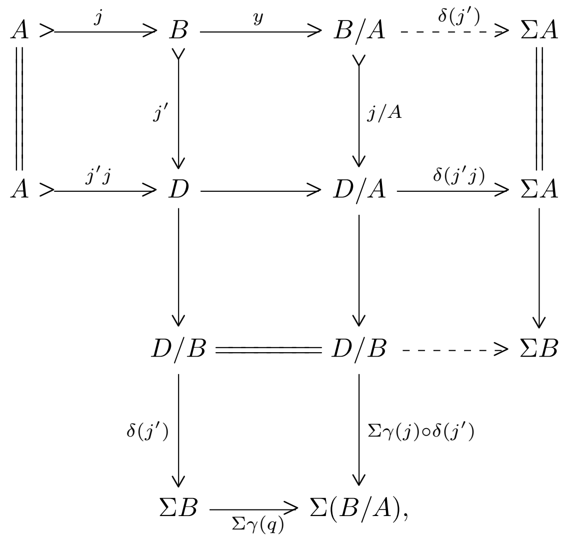
$$\delta(i_A) = (\Sigma A \underset{0\cup p}{\overset{\sim}{\longleftarrow}} CA \cup_A CA \underset{p\cup 0}{\overset{\sim}{\longrightarrow}} \Sigma A) = m_A = -\mathrm{Id}_{\Sigma A}$$

and so in fact $(\Sigma\gamma(j)) \circ \delta(i_A) = -\Sigma\gamma(j)$.

- *(T3)* We will skip this one.

- *(T4)* We first reduce to an easier case. Suppose we have $A \overset{f}{\to} B \overset{f'}{\to} D$. Then T4 says we have four distinguished triangles in the given diagram. First, we can assume that $f = \gamma(j)$ for $j : A \rightarrowtail B$ and $f' = \gamma(j')$ for $j' : B \rightarrowtail D$

using the calculus of fractions. Now in $\mathcal{C}$ we have the diagram



and all the triangles here are elementary distinguished triangles. (The top right square and the square below it commute by naturality of δ. The map $j'j$ is a cofibration since it is the composition of cofibrations.)

3 Algebraic vs. topological triangulated categories

Definition 7. A triangulated category is *algebraic* if it embeds (fully faithfully) into the homotopy category $K(\mathcal{A})$ of an additive category $\mathcal{A}$.

Definition 8. A triangulated category is *topological* if it is equivalent to the homotopy category $Ho(\mathcal{C})$ for a stable cofibration category $\mathcal{C}$.

Note that algebraic implies topological.

Example. The category $\mathcal{D}(R\text{-}\mathbf{mod}) = K(R\text{-}\mathbf{mod})[q\text{-}iso^{-1}]$, the derived category of R-modules, is an algebraic triangulated category by *cofibrant resolution/approximation*, $\mathcal{D}(R\text{-}\mathbf{mod}) \hookrightarrow K(R\text{-}\mathbf{mod})$. A complex $C_\bullet$ is *cofibrant* if it admits a filtration by subcomplexes $0 = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n \subseteq \cup_n F_n = C$ such that $d(F_n) \subseteq F_{n-1}$ and F_n/F_{n-1} is dimensionwise projective. We take $D_\bullet \in \mathcal{D}(R\text{-}\mathbf{mod})$ to some $C_\bullet \in K(R\text{-}\mathbf{mod})$ cofibrant such that we have a quasi-isomorphism $C_\bullet \overset{\sim}{\to} D_\bullet$.

Example. We have $\underline{\mathbf{mod}}\text{-}R$ for Frobenius rings R. Define $K^{ac}(\mathbf{proj}\text{-}R)$ to be the homotopy category of acyclic complexes of projective modules, a subcategory of $K(\mathbf{mod}\text{-}R)$. The issue here is that an acyclic complex of projectives may not be contractible, and indeed this stable category exactly encodes that difference. The category $K^{ac}(\mathbf{proj}\text{-}R)$ admits a functor to $\underline{\mathbf{mod}}\text{-}R$ by sending $P_\bullet$ to $\ker(P_0 \overset{d}{\to} P_{-1}) = \mathrm{im}(P_1 \overset{d}{\to} P_0)$. This is an equivalence, with inverse functor *complete resolution*, which given a module M takes a projective resolution $P_\bullet \to M$ and an injective resolution $M \to I_\bullet$ and splices them into a complex $\cdots \to P_1 \to P_0 \to I_0 \to I_1 \to \cdots$.

There are very few examples of algebraic categories that are not topological: the category of finitely generated free $\mathbb{Z}/4$-modules is essentially the only example.

We now focus on two questions:

1. Do algebraic triangulated categories have special properties?

2. Is the stable homotopy category $\mathcal{SHC}$ algebraic?

The answer to the first question is yes. We need the following notation.

Definition 9. Given $X \in \mathcal{T}$ and $n \in \mathbb{Z}$, we write X/n for any cone of $X \overset{\cdot n}{\to} X$ (where the map is the sum of n copies of Id_X).

Lemma. *If $\mathcal{T}$ is algebraic, then $n \cdot X/n = 0$ in $\mathcal{T}(X/n, X/n)$.*

Proof. Without loss of generality, we assume $\mathcal{T} = K(\mathcal{A})$. (This may be a larger category, but it suffices to prove it in this setting.) Let X be a complex. Then we model $X/n = X \oplus X[1]$ with differential $d = \begin{pmatrix} d & n \\ 0 & -d \end{pmatrix}$. Observe that the degree-1 map $s : X/n \to X/n$ given by $s(x, y) = (0, x)$ (represented by $s = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$) satisfies $sd + ds = n \cdot \mathrm{Id}_{X/n}$. Thus $n \cdot X/n = 0$. □

Unfortunately, this does not work in general. Let us apply $\mathcal{T}(X/n, -)$ to the defining triangle:

$$0 \to \mathbb{Z}/n \otimes \mathcal{T}(X/n, X) \to \mathcal{T}(X/n, X/n) \to \mathcal{T}(X/n, \Sigma X) \to 0.$$

This shows that the middle group is annihilated by n^2, and in general this is the best we can do. (In the algebraic-but-not-topological example, $2 \cdot X/2 \neq 0$.)

Example. In $\mathcal{SHC}$, we have the *sphere spectrum* $\mathbb{S} = \{S^n, \sigma_n : S^n \wedge S^1 \overset{\cong}{\to} S^{n+1}\}$. Then $\mathbb{S} \overset{\cdot n}{\to} \mathbb{S} \to \mathbb{S}/n \to \Sigma\mathbb{S}$ gives us the *mod-n Moore spectrum* $\mathbb{S}/n$. (This has the formal properties of Moore spaces.) It turns out that $2 \cdot \mathbb{S}/2 \neq 0$ (which can be proved easily using Steenrod squares), and hence $\mathcal{SHC}$ is not algebraic. But this is an entirely 2-local phenomenon.

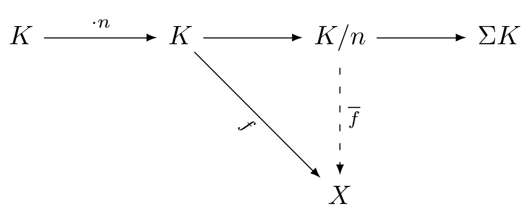
Proposition. *Let $\mathcal{T}$ be topological and let n be an odd number. Then $n \cdot X/n = 0$ for all $X \in \mathcal{T}$.*

Proof. With some work, one can construct an exact functor $F : \mathcal{SHC} \to \mathcal{T}$ such that $F(\mathbb{S}) = X$. (This makes serious use of the assumption that $\mathcal{T}$ is topological.) From this, we can apply F to the defining triangle for the mod-n Moore spectrum, obtaining $X \overset{\cdot n}{\to} X \to F(\mathbb{S}/n) = X/n \to \Sigma X$. This means that $n \cdot \mathrm{Id}_{X/n} = F(n \cdot \mathrm{Id}_{\mathbb{S}/n}) = F(0)$. So we have actually reduced to a universal example. □

This leads to the following question:

3. Let p be an odd prime. Is $\mathcal{SHC}_{(p)}$ algebraic?

Definition 10. Let $\mathcal{T}$ be an arbitrary triangulated category, and let $n \geq 2$. Suppose $X \in \mathcal{T}$. We define the *$n - order$* of X, $n - ord(X) \in \{0, 1, 2, 3, \ldots, \} \cup \{\infty\}$, by the following inductive process. First, $n - ord(X) \geq 0$ for all X. For $k \geq 1$, X has $n - ord(X) \geq k$ iff for all $K \in \mathcal{T}$ and all $f : K \to X$ there is an extension $\overline{f} : K/n \to X$ making the diagram



commute, such that some (and hence any) cone of $\overline{f}$ satisfies $n - ord(Cone(\overline{f})) \geq k - 1$. We define $n - ord((\mathcal{T}) = n - ord(0)$.

As it turns out, the n-order is invariant under isomorphism and shift.

Example. If $n - ord(X) \geq 1$, then for all $f : K \to X$ there is an extension $\overline{f} : K/n \to X$ satisfying a vacuous condition. This holds iff $n \cdot f = 0$ for all $f : K \to X$, iff $n \cdot X = 0$. Intuitively, the n-order measures how "strongly" $n \cdot X = 0$ holds.

For an object $X \in \mathcal{S} \subseteq \mathcal{T}$ (for $\mathcal{S}$ a full triangulated subcategory), $n - ord^{\mathcal{S}}(X) \geq n - ord^{\mathcal{T}}(X)$. (In $\mathcal{T}$, there are more ways to fail the condition.) In particular, we can apply this to $X = 0$ to obtain $n - ord(\mathcal{S}) \geq n - ord(\mathcal{T})$.

If $\mathcal{T}$ is $\mathbb{Z}[\frac{1}{n}]$-linear, then $n - ord(\mathcal{T}) = \infty$. The same holds if $\mathcal{T}$ is $\mathbb{Z}/n$-linear. Thus n-order is useless if $\mathcal{T}$ is k-linear for some field k. So this is really an arithmetic invariant.

Theorem (Main results)**.**

1. *Let $\mathcal{T}$ be algebraic. Then $n - ord(X/n) = \infty$ for all X. (This generalizes (quite strongly) the property that we proved earlier that $n \cdot X/n = 0$, since this is equivalent to saying that $n - ord(X/n) \geq 1$.)*

2. *Let $\mathcal{T}$ be topological and p be a prime. Then $p - ord(X/p) \geq p - 2$. (This is contentless at $p = 2$. For $p \neq 2$, this gives that $p \cdot X/p = 0$. Thus, topology need not be quite as strong as algebra.)*

3. *In $\mathcal{T} = \mathcal{SHC}$ (which is a topological example), $p - ord(\mathbb{S}/p) = p - 2$, so this is the best possible global bound.*

The issue here is that the mod-2 Moore spectrum does not admit a multiplication, not even in the stable homotopy category. The mod-3 Moore spectrum has a homotopy commutative multiplication, but it is not homotopy associative. For $p \geq 5$, the mod-p Moore spectrum has a homotopy associative and homotopy commutative multiplication, and the coherence gets better as p increases. However, the multiplications on mod-p Moore spectra are never A_∞. But this result has nothing to do with smash products. So this picks out the question of how much this phenomenon persists when we forget smash products.

Proof sketch of part 1. If $\mathcal{T}$ is algebraic, then it has what we call a *mod-n reduction*, i.e. there is some other triangulated homotopy category $\mathcal{T}/n$ such that we have an adjunction of exact functors $\rho_* : \mathcal{T} \leftrightarrows \mathcal{T}/n : \rho^*$ such that for every X we have a distinguished triangle $X \xrightarrow{\cdot n} X \xrightarrow{\text{unit}} \rho^*(\rho_* X) \to \Sigma X$.

The key part of the proof is the following claim: If $\mathcal{T}$ has a mod-n resolution, then $n - ord(X/n) = \infty$ for all $X \in \mathcal{T}$. We show that $n - ord(\rho^* Z) \geq k$ for all k and all $Z \in \mathcal{T}/n$ by induction on k. The base case is tautological. For $k \geq 1$, given a map $f : K \to \rho^*(Z)$, we adjoin to get $\hat{f} : \rho_*(K) \to Z$ and adjoin again to get $\rho^* \hat{f} : \rho^*(\rho_* K) \to \rho^*(Z)$. We claim that $\overline{f} = \rho^* \hat{f}$ is an extension of f. To see this, choose a distinguished triangle $\rho_* K \to Z \to C \to \Sigma \rho_* K$. Then apply ρ^* to get the distinguished triangle $\rho^*(\rho_* K) \to \rho^* Z \to \rho^* C \to \Sigma \rho^* C$. This first map is an extension of f, so $n - ord(\rho^* C) \geq k - 1$ by induction, so $n - ord(\rho^* Z) \geq k$.

□