Fusion categories and modular tensor categories seminar

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1 Introduction – Part I (Ehud "Udi" Meir)

In order to explain what a fusion category is, we will begin with an example.

Example 1. Suppose G is a finite group and $k = \overline{k}$ is a field with char $k = 0$. Consider the category $\mathcal{C} = \mathtt{Rep}(G)$, the finite-dimensional k-vector spaces with a G-action. What do we know about this category?

1. By *Maschke's theorem*, we know that $\mathcal{C}$ is *semisimple*: that is, every object is a sum of simple (indecomposable) objects.

2. It is k-linear: $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a k-vector space, and it has kernels and cokernels.

3. $\mathcal{C}$ is a monoidal category: if $V, W \in \mathcal{C}$, then we have $V \otimes W \in \mathcal{C}$ (by the diagonal action: $g(v \otimes w) = gv \otimes gw$)). This determines a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, which will satisfy certain axioms which we'll explore later. In particular, this monoidal structure admits a unit object $\mathbf{1}$.

4. If $V \in \mathcal{C}$, then we have a natural structure of a G-representation on $V^* = \operatorname{Hom}_k(V, k)$: given $f \in V^*$, the action is given by $(gf)(v) = f(g^{-1}v)$ (the inverse is just to make it a left action rather than a right action).

This is a primal example of what is known as a *fusion category*.

Definition 1. A *fusion category* is a category $\mathcal{C}$ such that:

1. $\mathcal{C}$ is k-linear (i.e. abelian with hom-sets actually k-vector spaces) and semisimple;

2. $\mathcal{C}$ has only a finite number of simple objects;

3. $\mathcal{C}$ is a monoidal category, i.e. we have some $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ with a unit object $\mathbf{1} \in \mathcal{C}$ and functorial isomorpsisms $V \otimes \mathbf{1} \cong V$, along with natural isomorphisms $\alpha_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$. However, this may not satisfy higher coherence conditions, e.g. the *pentagon axiom* (which I'm not going to draw here).

4. $\mathcal{C}$ is *rigid*, i.e. for any $V \in \mathcal{C}$ there exist V^* and ${}^*V$ along with maps $V^* \otimes V \to \mathbf{1}$, $\mathbf{1} \to V \otimes V^*$, $V \otimes {}^*V \to \mathbf{1}$, and $\mathbf{1} \to {}^*V \otimes V$ such that the composite
$$V \xrightarrow{\sim} \mathbf{1} \otimes V \xrightarrow{\mathrm{coev}_V} (V \otimes V^*) \otimes V \to V \otimes (V^* \otimes V) \xrightarrow{\mathrm{ev}_V} V \otimes 1 \xrightarrow{\sim} V$$
is equal to the identity. (This is the categorical way of expressing that objects are dual to each other.)

When we say that $\mathcal{C}$ is semisimple with a finite number of simple objects, we mean that there are only finitely many simple objects $S_1, \ldots, S_n$ (i.e. those with no nontrivial monomorphisms into them), and that any object $V \in \mathcal{C}$ can be written uniquely as
$$V \cong \bigoplus_{i=1}^{m} S_i^{\oplus m_i}.$$
It follows that for any simple object S, $\operatorname{Hom}(S, S) \cong k$. Another way of saying that $\mathcal{C}$ is semisimple is that any monomorphism $U \hookrightarrow X$ splits, so that $X \cong U \oplus V$ for some V.

Example 2. We now study *G-graded vector spaces*, which is in a sense dual to the previous example. Specifically, let G be a finite group. We define $\mathcal{C} = \mathtt{Vec}_G$ to have objects G-graded finite-dimensional vector spaces $V = \bigoplus_{g \in G} V_g$, with $\operatorname{Hom}(\bigoplus V_g, \bigoplus W_g)$ those linear homomorphisms f satisfying $f(V_g) \subseteq W_g$. The tensor product is given by $(V \otimes W)_g = \bigoplus_{ab=g} V_a \otimes W_b$. The simple objects are all of the form k_g for some $g \in G$. The duals of $V = \bigoplus_{g \in G} V_g$ are given by $(V^*)_g = (V_{g^{-1}})^*$ and ${}^*V = V^*$.

A key facet of this example is that when G is not abelian, the tensor product is not commutative. This brings us to the discussion of left- and right-duals. First, however, we must discuss what functors between fusion categories look like.

Definition 2. A *tensor functor* $F : \mathcal{C} \to \mathcal{D}$ of fusion categories is a k-linear functor (i.e. the maps on hom-spaces are k-linear) together with natural isomorphisms $f(X \otimes Y) \xrightarrow{\sim} f(X) \otimes f(Y)$, such that for any $X, Y, Z \in \mathcal{C}$, the diagram
$$\begin{array}{ccc}
F((X \otimes Y) \otimes Z) & \longrightarrow & F(X \otimes Y) \otimes F(Z) \\
\downarrow & & \downarrow \\
F(X \otimes (Y \otimes Z)) & & (F(X) \otimes F(Y)) \otimes F(Z) \\
\downarrow & & \downarrow \\
F(X) \otimes F(Y \otimes Z) & \longrightarrow & F(X) \otimes (F(Y) \otimes F(Z))
\end{array}$$
commutes.

Let us look at some specific examples.

Example 3. Let G, H be two finite groups and $\varphi : G \to H$ be a homomorphism. Then by restriction of scalars, we get a functor $F : \texttt{Rep}(H) \to \texttt{Rep}(G)$. For instance, if we take $G = \{1\}$ then this becomes $\texttt{Rep}(H) \to \texttt{Vec}$. This is a tensor functor, because the underlying vector space of the tensor product of H-reps is just the tensor product of the underlying vector spaces.

Example 4. Again let G, H be two finite groups and $\varphi : G \to H$ be a homomorphism. Then we get a functor $F : \texttt{Vec}_G \to \texttt{Vec}_H$, given by $F(k_g) = k_{\varphi(g)}$. Now, however, we must specify the isomorphism $F(k_a \otimes k_b) \to F(k_a) \otimes F(k_b)$. This runs as follows. First of all, $k_a \otimes k_b = k_{ab}$, so we must have $\alpha(a,b) : F(k_{ab}) = k_{\varphi(ab)} \to k_{\varphi(a)} \otimes k_{\varphi(b)} = k_{\varphi(a)\varphi(b)}$. This defines a function $\alpha : G \times G \to k^\times$. We would like it to satisfy

$$\begin{array}{ccc}
F((k_a \otimes k_b) \otimes k_c) & \xrightarrow{\alpha(ab,c)} & F(k_a \otimes k_b) \otimes F(k_c) \\
\downarrow & & \downarrow \alpha(a,b) \\
F(k_a \otimes (k_b \otimes k_c)) & & (F(k_a) \otimes F(k_b)) \otimes F(k_c) \\
\alpha(a,bc) \downarrow & & \downarrow \\
F(k_a) \otimes F(k_b \otimes k_c) & \xrightarrow{\alpha(b,c)} & F(k_a) \otimes (F(k_b) \otimes F(k_c)).
\end{array}$$

That is, we should have $\alpha(b,c)\alpha(a,bc) = \alpha(a,b)\alpha(ab,c)$; that is, α should be a *2-cocycle*. In fact, these functors are precisely parametrized by $H^2(G, k^\times)$.

This is actually a special case of *Ocneanu rigidity*: the cohomology group $H^2(G, k^\times)$ is a finite abelian group, and so in particular there are only a finite number of ways to equip such a functor with a tensor structure. This is not at all obvious from the original definition!

Definition 3. Let $\mathcal{C}$ be a fusion category with simple objects $S_1, \ldots, S_n$. The *Grothendieck ring* of $\mathcal{C}$ is $K_0(\mathcal{C}) = \bigoplus_i \mathbb{Z} \cdot S_i$; if $S_i \otimes S_j = \bigoplus_k N_{ij}^k S_k$, then the product in $K_0(\mathcal{C})$ is determined by $S_i \cdot S_j = \sum N_{ij}^k S_k$. (This agrees with the usual definition involving short exact sequences.)

Let us consider some examples.

Example 5. Let $\mathcal{C} = \texttt{Vec}_G$. Since the simple objects are k_g for $g \in G$, then $K_0(\mathcal{C}) \cong \mathbb{Z}G$, the usual group algebra. (It's important to note that in any case, $K_0(\mathcal{C})$ always comes with a distinguished basis with nonnegative structure constants).

Example 6. Let $\mathcal{C} = \texttt{Rep}(G)$. Then $K_0(G)$ is the *character ring* of G, i.e. $\bigoplus_\psi \mathbb{Z} \cdot \psi$, where ψ are the characters (i.e. one-dimensional representations) of G.

We can ask: For a given ring R with a chosen basis with positive structure constants, can we *categorify* to find a fusion category $\mathcal{C}$ with $K_0(\mathcal{C}) = R$ (with the same basis)? If so, in how many ways?

Example 7. Suppose we're looking at $\mathbb{Z}G$ with the usual basis. We know that $\mathcal{C}$ will have to have basis k_g, with $k_g \otimes k_h = k_{gh}$. But we must check further structure. For instance, we have

$$\begin{array}{ccc}
(k_g \otimes k_h) \otimes k_l & \xrightarrow{\sim} & k_g \otimes (k_h \otimes k_l) \\
\| & & \| \\
k_{gh} \otimes k_l & & k_g \otimes k_{hl} \\
\| & & \| \\
k_{ghl} & \xrightarrow{w(g,h,l)} & k_{ghl}
\end{array}$$

for some $w : G \times G \times G \to k^\times$. It turns out that the pentagon diagram commutes iff w is a *3-cocycle*. It turns out that such categories are actually parametrized by $H^3(G, k^\times)$, which is also a finite group.

2 Introduction – Part II (Orit Davidovich)

2.1 Recap

We recall the definition of a fusion category. Let k be a field with char $k = 0$ and $k = \overline{k}$.

Definition 4. A *fusion category* is a k-linear, semi-simple, rigid, monoidal category such that:

- There are finitely many simple objects.
- The hom spaces are finite dimensional.
- $\mathrm{End}(\mathbf{1}) = k$.

Recall that *k-linearity* means that our category is enriched over $\mathtt{Vec}_k$, with the usual categorical operations $\oplus$, ker, coker, etc. Recall that *semi-simple* means that every object is a direct sum of finitely many *simple* objects (i.e. indecomposable objects, i.e. those admitting no nontrivial monomorphisms). Recall that *rigid* means that objects have left and right duals. Recall that *monoidal* means that we have a tensor product.

If we consider a fusion category $\mathcal{C}$ as a k-linear category, then it is automatically isomorphic to $\prod_{i\in I} \mathtt{Vec}_k$, where I is the set of isomorphism classes of simple objects of $\mathcal{C}$. However, the interesting structure comes on top. The monoidal structure gives us a multiplication on the abelian group $\mathbb{Z}^I$, and the rigidity gives us involutions on the ring.

Example 8. Let G be a finite group, and let $\mathcal{C} = \mathtt{Rep}\ G$ (finite-dimensional). We saw that this has the structure of a fusion category. The tensor product uses the diagonal action, and dualizing takes V to $V^\vee$ and acts via the inverse (to preserve the fact that we're looking at left actions).

Example 9. Let G be a finite group, and let $\mathcal{C} = \mathtt{Vec}_G$ be the category of G-graded finite-dimensional k-vector spaces. We also saw that this has the structure of a fusion category. The tensor product is obtained by *convolution*: $(V \otimes_W)_g = \bigoplus_{h\in G} V_{gh^{-1}} \otimes W_h$. Duals are defined by $(V^\vee)_g = (V_{g^{-1}})^\vee$.

At the end of the last lecture, we discussed a generalization of this latter example.

Example 10. We generalized by introducing a 3-cocycle $\omega \in H^3(G; k^\times)$ to gives us $\mathtt{Vec}_G^\omega$ (where $\omega \neq 1$). This gives us a choice of associators $\alpha_{g,h,i} : (V_g \otimes V_h') \otimes V_i'' \to V_g \otimes (V_h' \otimes V_i'')$ given by $\alpha_{g,h,i}(v \otimes v' \otimes v'') = \omega(g, h, i) \cdot v \otimes v' \otimes v''$. The fact that $d\omega = 0$ is equivalent to the statement that the pentagon diagram commutes.

Definition 5. The *Grothendieck ring* is an invariant of a fusion category. As a group, it is the free abelian group generated on isomorphism classes of simple objects; multiplication is given by $[x_i][x_j] = \sum_{a\in I} V_{ij}^a [x_a]$, where $V_{ij}^a = \dim(\mathrm{Hom}(X_a, X_i \otimes X_j)$.

Let's explore what the Grothendieck rings of the examples we've just seen are. First of all, $GR(\mathcal{C}) \otimes_{\mathbb{Z}} k$ is known as the *Verlinde algebra*. When $\mathcal{C} = \mathtt{Rep}_G$, then $GR(\mathcal{C}) \otimes_{\mathbb{Z}} k$ is the algebra of class functions. When $\mathcal{C} = \mathtt{Vec}_G$ or $\mathcal{C} = \mathtt{Vec}_G^\omega$, then $GR(\mathcal{C}) =\cong \mathbb{Z}G$. This is a very rough invariant; it doesn't detect associativity.

Proposition 1. *Given a group G and a field k, there is a bijection*

$$\{\text{fusion cat. } \mathcal{C} \text{ with } GR(\mathcal{C}) \cong \mathbb{Z}G\}/\text{iso.} \overset{\sim}{\leftrightarrow} H^3(G; k).$$

2.2 Commutativity

If a fusion category is supposed to be a categorification of the idea of a monoid, then the first thing we might ask for is a categorification of the notion of commutativity. We might first demand isomorphisms $V \otimes W \cong W \otimes V$, but we'll quickly realize that we actually need these to be functorial. We can rephrase this as asking for a natural isomorphism $\beta : \otimes \to \otimes^{op}$ (from the "tensor product" functor to the "tensor product in the opposite order" functor). Then, for any $\sigma \in \mathfrak{S}_n$ we have functorial isomorphisms

$$((V_1 \otimes V_2) \otimes \ldots) \otimes V_n \cong ((V_{\sigma(1)} \otimes V_{\sigma(2)}) \otimes \ldots) \otimes V_{\sigma(n)}.$$

Moreover, we want a canonical choice of this isomorphism. Thus, we are looking for *coherence*, or even perhaps *braided coherence*.

We have the following definition, which also may be viewed in some sense as a theorem.

Definition 6. Let $\mathcal{C}$ be a monoidal category with a natural isomorphism $\beta : \otimes \to \otimes^{op}$. We say that β is *coherent* (or *satisfies coherence*) if the following two diagrams commute. The first is

$$\begin{array}{ccccc}
(V \otimes V') \otimes V'' & \xrightarrow{\beta} & V'' \otimes (V \otimes V') & \xrightarrow{\alpha^{-1}} & (V'' \otimes V) \otimes V' \\
\downarrow \alpha & & & & \uparrow \beta \otimes 1 \\
V \otimes (V' \otimes V'') & \xrightarrow[\beta]{} & V \otimes (V'' \otimes V') & \xrightarrow[\alpha^{-1}]{} & (V \otimes V'') \otimes V',
\end{array}$$

and the second is obtained by replacing the upper and lower left horizontal arrows by β^{-1} and the right vertical arrow with $\beta^{-1} \otimes 1$. (These are sometimes called the *hexagon diagrams.*) Then, $\mathcal{C}$ equipped with such a β is called a *braided monoidal category*, and β is called its *braiding.*

It is convenient to rephrase this as follows. We define the *braid category* $\mathcal{B}$ to have objects $\mathbb{N} = \{0, 1, 2, \ldots\}$, with:

1. $\mathrm{Hom}_{\mathcal{B}}(n, n) = \{\text{braids on } n \text{ strands}\}/\text{isotopy}$, where composition is given by stacking braids (from bottom to top). (This is a group, called the n^{th} *braid group.*)
2. $\mathrm{Hom}_{\mathcal{B}}(n, m) = \emptyset$ if $n \neq m$.

This category has a monoidal structure: $m \otimes n = (m+n)$, and tensor product of morphisms is given by just putting the braids next to each other. It's easy to see that this is actually a *strict* monoidal structure. As suggested by the name, $\mathcal{B}$ also has a braiding: $\beta : m \otimes n \to n \otimes m$ is given by the morphism in $\mathrm{Hom}_{\mathcal{B}}((m+n), (m+n))$ which interchanges the obvious blocks of m and n dots, putting the strands from the first m dots to the last m dots *on top of* the strands from the last n dots to the first n dots.

Proposition 2. *$\mathcal{B}$ is a braided monoidal category.*

This is just a routine verification of the commutativity of the hexagon diagrams.

Theorem 1 (Joyal-Street)**.** *Let $\mathcal{C}$ be a braided monoidal category. Then there is an equivalence of categories $Fun^{\otimes,br}(\mathcal{B}, \mathcal{C}) \simeq \mathcal{C}$, given by $F \mapsto F(1)$.*

Thus, $\mathcal{B}$ is the braided monoidal category freely generated on a single object. This suggests the extremely fruitful idea of working diagramatically, with diagrams expressed as functors instead of actually being written out.

2.3 Graphical calculus

Suppose we are given a rigid braided monoidal category $\mathcal{C}$. The first thing we want to do is give pictorial representation to morphisms in $\mathcal{C}$. So, we will draw $f : x \to y$ as a vertical arrow through a box containing the letter f. Then, $f \otimes f'$ is drawn by putting these vertical arrows next to each other. More generally, if we have $f : x_1 \otimes \ldots \otimes x_n \to y_1 \otimes \ldots \otimes y_m$, we write f as a wide box with n inputs at the bottom and m outputs at the top.

We may consider an upward V^vee arrow as a downward V arrow. This manifests itself as follows. The evaluation is a functorial collection of maps $e : V^vee \otimes V \to 1$. Pictorially, we might represent this as a wide box with $V^\vee$ and V coming in and 1 coming out; however, we will often write this as a curved arrow running counterclockwise, starting up and ending down. Dually, we might write the coevaluation $c : 1 \to V \otimes V^\vee$, instead of as a box with 1 coming in and V and $V^\vee$ going out, as a curved arrow running clockwise, starting down and ending up.

Next, the braiding gives $\beta : x \otimes y \to y \otimes x$; instead of writing this as a box with x and y going in and y and x coming out, we simply drawn an X with the x-arrow going over the y-arrow. (Then, β^{-1} is given by putting the y-arrow over the x-arrow.)

Now, the rigidity axioms simplify dramatically. The axiom that

$$V \cong 1 \otimes V \overset{c \otimes 1}{\to} V \otimes V^\vee \otimes V \overset{1 \otimes e}{\to} V \otimes 1 \cong V$$

is equal to id_V is now simply interpreted as straightening out a squiggling arrow (of the shape $y = -(x^3 - x)$, going upwards) into an upwards vertical arrow.

This is great, but we have the problem of consistency. (See picture.) Suppose we have V coming out at the top. then the bottom hook is a coevaluation, so $V^\vee$ comes in, but then the evaluation is on $V^\vee$, so it has input $V^{\vee\vee}$, so ultimately we see that this has $V^{\vee\vee}$ as an input. One way of fixing this is to choose isomorphisms $V \cong V^{\vee\vee}$, i.e. a monoidal natural isomorphism $\delta : 1 \to (-)^{\vee\vee}$. This is called a *pivotal* structure. So, we can precompose our picture with the box δ going from V to $V^{\vee\vee}$. Now, we're working up to isotopy, and this looks like it's isotopic to a vertical line. However, there's no reason this should be equal to the identity. So instead we fatten our lines to *ribbons*, and we call this a *twist*. This twist of V is denoted θ_V. (See picture.) In general, $\theta_V \neq \mathrm{id}_V$. A rigid braided monoidal pivotal category is called a *ribbon category*.

Thus, we have strengthened our graphical calculus into the calculus of *directed ribbon graphs*.

Theorem 2 (Reshetikhin-Turaev)**.** *Let $\mathcal{C}$ be a ribbon category, and write $R_{\mathcal{C}}$ for the category of directed ribbon graphs labelled by objects of $\mathcal{C}$. Then there is a unique monoidal functor $R_{\mathcal{C}} \to \mathcal{C}$.*

2.4 Modularity

By definition, we call a category *premodular* if it is a ribbon fusion category. Now, for any object X, we can braid twice against any other object V, and we can ask: To what extent is this equal to the identity? (See picture.) This has important quantum physical interpretations. We say X is *transparent* if for all V, $\beta_{X,V} \circ \beta_{V,X} = 1_{V\otimes X}$. To what extent is X *transparent*? (Can we tell if we've sent our favorite particle around another?) On the one extreme we have *symmetry*, which is by definition the case that every simple X is transparent (so $\beta_X^2 = 1$ for all X). On the other extreme we have *modularity*, which is by definition the situation where only the unit object is transparent.

3 Dimensions in fusion categories (Orit)

3.1 Dimensions

Let $\mathcal{C} = \mathtt{Vec}_k$. Then we have a notion of *trace*: given $f : V \to V$, we can define $\mathrm{tr} f$ via

$$k \to V \otimes V^* \stackrel{f\otimes 1}{\to} V \otimes V^* \to k$$

given by

$$1 \mapsto \sum e_i \otimes e^i \mapsto \sum f(e_i) \otimes e^i \mapsto \sum e^i(f(e_i)).$$

We would like to generalize this to fusion categories. So, we take the trace of $f : V \to V^{**}$ by looking at the map

$$1 \stackrel{\mathrm{coev}}{\to} V \otimes V^* \stackrel{f\otimes 1}{\to} V \otimes V^{**} \otimes V^* \stackrel{\mathrm{ev}}{\to} 1,$$

which runs in exactly the same way.

Remark 1. The functor $(\ )^{**}$ is monoidal, and hence $\mathrm{tr}(f \otimes g) = \mathrm{tr}(f) \cdot \mathrm{tr}(g)$ for $f : V \to V^{**}$ and $g : W \to W^{**}$.

Now, $V \cong V^{**}$, but not necessarily canonically or monoidally in $\mathcal{C}$. Thus we make the following definition.

Definition 7. For $V \in \mathcal{C}$ simple, fix any isomorphism $f_V : V \to V^{**}$. We then defined the *squared norm* of V to be $|V|^2 = \mathrm{tr}(f_V) \cdot \mathrm{tr}((f_V^*)^{-1})$ (where $f_V^* : V^{***} \to V^*$, so $(f_V^*)^{-1} : V^* \to (V^*)^{**}$). This is only defined for simple objects!

Remark 2. Of course, $|V|^2$ does not depend on our choice of f_V. In general, given f, using our graphical calculus we can write f^* as [PICTURE].

Moreover, $|1|^2 = 1$, and $|V \otimes W|^2 = |V|^2 \cdot |W|^2$.

Definition 8. We define the *global dimension* of a fusion category $\mathcal{C}$ to be the sum of the squared norm $|V|^2$ for all simple $V \in \mathcal{C}$.

Example 11. Let G be a finite group, and let $\mathcal{C} = \mathtt{Rep}(G)$. Then $|V|^2 = (\dim V)^2$. Hence, $\dim \mathcal{C} = |G|$.

Example 12. When $\mathcal{C} = \mathtt{Vec}_G$, then $|V|^2 = 1$ so $\dim \mathcal{C} = |G|$.

This equality of dimensions is not a coincidence! It comes from the fact that these two examples are *Morita equivalent*, which we will discuss later.

Example 13. Let $\mathcal{C}$ be the *Fibonacci category*. Its simple objects are 1 and X (with $X \equiv X^* \equiv {}^*X$), with $X \otimes X = 1 \oplus X$. Then $|X|^2 = ((1+\sqrt{5})/2)^2$, and hence $\dim \mathcal{C} = 1^2 + ((1+\sqrt{5})/2)^2$.

Example 14. Let H be a finite-dimensional semi-simple *Hopf algebra* over k. (The precise definition isn't important; what we need to know is that this is the object whose representation theory gives you a fusion category.) (The first example is the special case where $H = kG$.) This comes with an *antipode* $\delta : H \to H$, which is invertible; we know that $\delta^2(x) = gxg^{-1}$ for some group-like element $g \in H$. This gives us a canonical identification $V \cong V^{**}$. (In the first example, $\delta^2 = \mathrm{id}$.) Then $\dim \mathtt{Rep}(H) = \mathrm{tr}(\delta^2) = \dim H$.

Recall that for a fusion category $\mathcal{C}$, we said that $\mathcal{C}$ has a *pivotal strucure* if we have some monoidal equivalence $\varepsilon : 1_{\mathcal{C}} \to (\ )^{**}$. (Recall that this was our fix for the issue of wanting strands to correspond to a single object, but then the twist θ_V would go from V^{**} to V, so we precompose it with ε_V.)

Definition 9. Let $\mathcal{C}$ be a pivotal fusion category and let $X \in \mathcal{C}$. The *quantum dimension* of X, $\mathrm{qdim}\ X \in \mathrm{End}(1)$, is given pictorially by [PICTURE] (which is often written without the ε), i.e. $e_V \circ \varepsilon_V \otimes 1 \circ c_V : 1 \to 1$. Equivalently, $\mathrm{qdim}\ V = \mathrm{tr}(\varepsilon_V)$.

Proposition 3. *Let $\mathcal{C}$ be a pivotal fusion category. Then:*

1. $|V|^2 = (\mathrm{qdim}\ V) \cdot (\mathrm{qdim}\ V^*)$.
2. *If $k = \mathbb{C}$, then* $\mathrm{qdim}\ V^* = \overline{\mathrm{qdim}\ V}$.

Proof. First statement first. By definition, $|V|^2 = \mathrm{tr}(\varepsilon_V) \cdot \mathrm{tr}((\varepsilon_V^*)^{-1})$. Because ε is monoidal and we're working in the semi-simple setting, then $(\varepsilon_V^*)^{-1} = \varepsilon_{V^*}$. The statement immediately follows.

Now the second statement. We again use the monoidality of the pivotality. Let $X_i, X_j \in \mathcal{C}$ be simple. Then $\varepsilon_{X_i} \otimes \varepsilon_{X_j} = \varepsilon_{X_i \otimes X_j} : X_i \otimes X_j \to X_i^{**} \otimes X_j^{**}$. Let us decompose this to some

$$\bigoplus_k N_{ij}^k X_k \to \bigoplus_k N_{ij}^k X_k^*.$$

Hence, $\varepsilon_{X_i} \otimes \varepsilon_{X_j} = \sum_k N_{ij}^k \varepsilon_k$. Applying trace on both sides, we obtain $(\mathrm{qdim}\ X_i) \cdot (\mathrm{qdim}\ X_j) = \sum_k N_{ij}^k \cdot \mathrm{qdim}\ X_k$.

Given our ordering on the simple objects, we can write the vector $\vec{d} = (\ldots, \mathrm{qdim}\ X_i, \ldots)$. Then we have the i^{th} *fusion matrix* N_i given by $(N_i)_j^k = \dim \mathrm{Hom}(X_k, X_i \otimes X_j)$. This yields that $\mathrm{qdim}\ X_i \cdot \vec{d} = N_i \vec{d}$. (This, by the way, implies that quantum dimension must be an algebraic number.)

Now, if we left-multiply by the Hermitian conjugate $(\vec{d})^\dagger$, we get $\mathrm{qdim}\ X_i \cdot \|\vec{d}\|^2 = (\vec{d}^\dagger) N_i \vec{d}$. Thus $N_i^t = N_{i^*}$ (the fusion matrix for X_i^*), then $\mathrm{qdim}\ X_i \cdot \|\vec{d}\|^2 = (N_{i^*}\vec{d})^\dagger \vec{d} = \overline{\mathrm{qdim}\ X_i^*} \cdot \|\vec{d}\|^2$. □

We introduce an auxiliary definition.

Definition 10. Let $\mathcal{C}$ be a pivotal fusion category. We say that $\mathcal{C}$ is *spherical* if $\mathrm{qdim}\ V = \mathrm{qdim}\ V^*$ for all $V \in \mathcal{C}$.

By the above proposition, if $k = \mathbb{C}$ then this implies that $\mathrm{qdim}\ V \in \mathbb{R} \subset \mathbb{C}$ for all V. Since we have a Galois action, then in fact $\mathrm{qdim}\ V$ will have to be *totally real* (i.e., its orbit under the Galois action is contained in $\mathbb{R}$). (Given a fusion category over $\mathbb{C}$, we can obtain an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, i.e. for any σ we obtain a new fusion category $\mathcal{C}^\sigma$ by keeping our objects but twisting the hom-spaces by $- \otimes_{\mathbb{C},\sigma} \mathbb{C}$. If for $X \in \mathcal{C}$ we write the associated object $X^\sigma \in \mathcal{C}^\sigma$, then we have the formula $\mathrm{qdim}\ X^\sigma = \sigma(\mathrm{qdim}\ X)$.)

Remark 3. Given any fusion category $\mathcal{C}$, we can construct a *pivotal cover* $F : \widetilde{\mathcal{C}} \to \mathcal{C}$ (which is automatically spherical). This functor preserves squared norms. So when $k = \mathbb{C}$, then square-norms of simple objects are positive in $\widetilde{\mathcal{C}}$, and hence this is true for $\mathcal{C}$ too, which implies that $\dim \mathcal{C} \geq 1$ (with equality iff $\mathcal{C} \simeq \mathtt{Vec}_{\mathbb{C}}$).

3.2 Modularity

Recall that we defined a *ribbon category* to be a braided rigid monoidal category with pivotal structure, and a *premodular category* to be a ribbon fusion category. Then we defined a *modular category* to be a premodular category where only the object 1 is "transparent".

We will rewrite this as follows. Assume that $\mathcal{C}$ is premodular. Let I be the set of isomorphism classes of simple objects in $\mathcal{C}$. We define a square matrix $\widetilde{S}$ of dimension $|I|$: writing $\{x_i\}_{i\in I}$ for the representations, we define $\widetilde{S}_{ij} = \mathrm{qtr}(\beta_{x_j,x_i} \circ \beta_{x_i,x_j})$ i.e. [PICTURE].

Now we can say that a ribbon fusion category is *modular* if its $\widetilde{S}$-matrix is invertible. This gave us an ordering of ribbon fusion categories by $\mathrm{rk}(\widetilde{S})$, when this is 1 we call it "symmetric".

Let us write $T = \mathrm{diag}(\theta_i)$ be the diagonal matrix of $\theta_i \mathrm{id}_{X_i} = \theta_{X_i}$. Then we have the following result.

Proposition 4. *Under a certain rescaling of $\widetilde{S}$ and T giving matrices s and t, we have the relations $(st)^3 = s^2$ and $s^4 = 1$. Note that these are the defining relations for*

$$\hat{s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \hat{t} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

in $SL_2(\mathbb{Z})$.

Thus, a modular category gives rise to a *modular representation.* (The S in $\widetilde{S}$ just comes from the S of $SL_2(\mathbb{Z})$.)

3.3 Drinfeld centers

(These are a/k/a "Drinfeld doubles" or "quantum doubles".)

Suppose $\mathcal{C}$ be a rigid monoidal category. The *Drinfeld center* of $\mathcal{C}$ is a category $Z(\mathcal{C})$ whose objects are pairs (X,ϕ) where $X \in \mathcal{C}$ is an object and $\phi : X \otimes - \xrightarrow{\cong} - \otimes X$ is a natural isomorphism which satisfies the *braid relations* that $(1 \otimes \phi_Z) \circ (\phi_Y \otimes 1) : X \otimes Y \otimes Z \to Y \otimes Z \otimes X$ is equal to $\varphi_{Y\otimes Z} : X \otimes (Y \otimes Z) \to (Y \otimes Z) \otimes X$.) In $Z(\mathcal{C})$, a morphism from (X,ϕ) to (X',ϕ') is given by a morphism $f : X \to X'$ such that $\phi'_Y \circ (f \otimes 1_Y) : X \otimes Y \to X' \otimes Y$ equals $(1_Y \otimes f) \circ \varphi_Y : X \otimes Y \to Y \otimes X'$. Pictorially, [PICTURES].

Proposition 5. *$Z(\mathcal{C})$ is rigid, monoidal, and braided (namely $\beta_{(X,\phi),(X',\phi')} = \phi_{X'}$).*

Example 15. Let $\mathcal{C} = \mathtt{Vec}_G$. We claim that $Z(\mathcal{C})$ is the category of G-equivariant sheaves on G, i.e. $G//G$-representations (where $G//G$ is defined by the conjugation action of G on G; this is a groupoid, or really a *quiver*). Now, we only need to know what happens with $\phi : V \otimes - \to - \otimes V$ on simple objects, i.e. $\phi_x : V \otimes k_x \to k_x \otimes V$. Hence $(\phi_x)_{gx} : V_g \to V_{x^{-1}gx}$. The braiding relation is precisely what guarantees that [PICTURE] is a representation.

It turns out that as braided monoidal categories, $Z(\mathtt{Rep}(G)) \simeq \mathtt{Vec}_G$. This can (very nearly) be taken as a definition of Morita equivalence; the equality of global dimensions falls out as a consequence.

4 Frobenius-Perron dimension and module categories (Udi)

4.1 Frobenius-Perron dimension

As motivation, let V be a finite-dimensional representation of the finite group G. Then, $V \otimes kG \simeq V_{tr} \otimes kG$, where V_{tr} has the same underlying vector space as V but with trivial G-action. Explicitly, we can take $v \otimes 1 \mapsto v \otimes 1$ and $v \otimes g \mapsto g^{-1}v \otimes g$. This actually gives us an isomorphism of representations. We want to see how this looks inside $K_0(\mathtt{Rep}(G))$. There, $[V][kG] = \dim(V) \cdot [kG]$. Thus, $\dim V$ is an eigenvalue of multiplication by $[V]$, and $[kG]$ is a common eigenvector for all $[V]$. In general, we won't always explicitly have this setup, but we can build off the following structure theorem.

Theorem 3 (Frobenius-Perron)**.** *Suppose A is an $n \times n$ matrix with nonnegative entries. Then:*

1. *There is some $\lambda > 0$ such that λ is an eigenvalue of A.*

2. *Of all the largest-in-absolute-value eigenvalues of A, at least one is positive.*

Thus, we make the following definition.

Definition 11. Let $V \in \mathcal{C}$, and denote by A_V the matrix of left multiplication by $[V]$ on $K_0(\mathcal{C})$. We call the largest positive eigenvalue of A_V the *Frobenius-Perron dimension* of V. We denote this by FPdim V. Note that this need not be an integer (though it is always algebraic).

Proposition 6. *Consider the element*

$$R = \sum_{V \text{ simple}} \mathrm{FPdim}\ V \cdot [V] \in K_0(\mathcal{C}) \otimes \mathbb{C}$$

Then R is a common eigenvector: for all $V \in \mathcal{C}$, we have $[V] \cdot R = \mathrm{FPdim}\ V \cdot R$.

We can think of this element R as our generalization of $[kG] \in K_0(\mathtt{Rep}(G))$

Corollary 1. FPdim *is multiplicative, i.e.* $\mathrm{FPdim}(V \otimes W) = \mathrm{FPdim}\ V \cdot \mathrm{FPdim}\ W$.

Of course, $\mathrm{FPdim}(V \oplus W) = \mathrm{FPdim}V + \mathrm{FPdim}W$, so in fact we have a ring homomorphism $\mathrm{FPdim} : K_0(\mathcal{C}) \to \mathbb{R}$.

Proposition 7. FPdim *is the unique homomorphism which sends all simple objects to positive numbers.*

Definition 12. We define $\mathrm{FPdim}\ \mathcal{C} = \sum_{V \text{ simple}} (\mathrm{FPdim}\ V)^2$, and we say that $\mathcal{C}$ is *integral* if $\mathrm{FPdim}\ \mathcal{C} \in \mathbb{Z}$.

Example 16. Let p be a prime number. Consider the category $\mathtt{Vec}_{C_p}$, where C_p is a cyclic group of order p. Any simple object is of the form k_g for $g \in C_p$, so $\mathrm{FPdim}(\mathtt{Vec}_{C_p}) = p$. More generally, $\mathrm{FPdim}(\mathtt{Vec}_G^\omega) = |G|$ (where $\omega \in H^3(G, k^\times)$ is any twist of the associator).

Theorem 4 (E-N-O)**.** *Any fusion category $\mathcal{C}$ with* $\mathrm{FPdim}\ \mathcal{C} = p$ *(for p prime) is of the form* $\mathtt{Vec}_{C_p}^\omega$.

In fact, there is a more general classification for $\mathrm{FPdim}\ \mathcal{C} = p^n$ – they are all "group-theoretic" – but that is beyond the scope of this talk.

Note that all of this only depends on the Grothendieck ring of the category.

4.2 Module categories and Morita equivalence

As fusion categories are a categorification of the notion of rings, we have the notion of *module categories*, on which fusion categories act.

Definition 13. Let $\mathcal{C}$ be a fusion category. We say that a category M is a *module category* over $\mathcal{C}$ if the following conditions hold:

1. M is k-linear.
2. M is semisimple with a finite number of isomorphism classes of simple objects.
3. We have a functor $\mathcal{C} \times M \to M$, denoted $(X, M) \mapsto X \otimes M$, along with natural isomorphisms $\beta : (X \otimes Y) \otimes M \xrightarrow{\cong} (X \otimes (Y \otimes M)$ such that the pentagon diagram commutes.

Example 17. The fusion category $\mathcal{C}$ can be taken as a $\mathcal{C}$-module category.

Example 18. Suppose $\mathcal{C} = \mathtt{Rep}(H)$ for H a Hopf algebra. Then $M = \mathtt{Vec}$ is a $\mathcal{C}$-module category, with action $\mathcal{C} \times M \to M$ given by $(V, W) \mapsto V \otimes W$ (forgetting that V is a representation).

Example 19. Suppose G is a finite group, and let $\mathcal{C} = \mathtt{Rep}(G)$. (Since $\mathtt{Rep}(G) = \mathtt{Rep}(kG)$, the previous example is an example of a $\mathcal{C}$-module.) Let $A \leq G$ be a subgroup, and let $M = \mathtt{Rep}(A)$. Then we have the module structure $\mathcal{C} \times M \to M$ given by $(V, W) \mapsto V \otimes W$.

Example 20. Let A be a finite group, and write $kA = \mathrm{span}\{U_a\}_{a \in A}$, where $U_a U_b = U_{ab}$. More generally, for $\alpha \in H^2(A, k^\times)$ we have $\alpha : A \times A \to k^\times$, and then we get $k^\alpha A = \mathrm{span}\{U_a\}_{a \in A}$ given by $U_a U_b = \alpha(a, b) U_{ab}$. Now if $A \leq G$ and $\mathcal{C} = \mathtt{Rep}(G)$ and $M = \mathtt{Rep}(k^\alpha A)$, then we have an action $\mathcal{C} \times M \to M$ by $(V, W) \mapsto V \otimes W$, where $a(v \otimes w) = a \cdot v \otimes U_a \cdot w$.

The most important thing about the previous example is that this actually gives all the indecomposable module categories of $\mathtt{Rep}(G)$.

In a sense we will make precise, we have described all module categories. We explore this presently.

Definition 14. An object $A \in \mathcal{C}$ is called an *algebra* if it has an associative multiplication $M : A \otimes A \to A$ with unit $u : 1 \to A$. Given an algebra $A \in \mathcal{C}$, a *right A-module* is an object $X \in \mathcal{C}$ with a map $X \otimes A \to X$ satisfying the usual axioms.

Example 21. Let $\mathcal{C} = \mathtt{Vec}_G$, the category of G-graded vector spaces. Suppose $A \leq G$ and $\alpha \in H^2(A, k^\times)$. We've already discussed $k^\alpha A$, which is actually a G-graded algebra (in the usual sense):

$$(k^\alpha A)_g = \begin{cases} 0, & g \notin A \\ \text{span } U_g, & g \in A. \end{cases}$$

Hence $k^\alpha A \in \mathtt{Vec}_G$ is an algebra.

From this, we can construct module categories. Let $\mathcal{C}$ be any fusion category, let $A \in \mathcal{C}$ be an algebra, and let X be a right A-module. Then $Y \otimes X$ is a right A-module by $(Y \otimes X) \otimes A \xrightarrow{\sim} Y \otimes (X \otimes A) \to Y \otimes X$. However, recall that the definition of module category includes a semisimplicity condition. So, we make the following definition.

Definition 15. If $A \in \mathcal{C}$ is an algebra, we say A is *semisimple* if the category $\mathtt{Mod}_{\mathcal{C}} A$ is semisimple.

Proposition 8. *If $A \in \mathcal{C}$ is a semisimple algebra, then $\mathtt{Mod}_{\mathcal{C}} A$ is a $\mathcal{C}$-module category, given by the above formula.*

Theorem 5 (Ostrik)**.** *Given any fusion category $\mathcal{C}$ any any $\mathcal{C}$-module category M, there exists a semisimple algebra $A \in \mathcal{C}$ and an equivalence $M \simeq \mathtt{Mod}_{\mathcal{C}} A$.*

Outline of proof. Let $\mathcal{M}$ be a $\mathcal{C}$-module category, and let $M, N \in \mathcal{M}$. We need to construct $\underline{\mathrm{Hom}}(M, N)$, so that for $X \in \mathcal{C}$ we have an exponential adjunction $\mathrm{Hom}_{\mathcal{M}}(X \otimes M), N) \cong \mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}(M, N))$. Equivalently, we need an evaluation $ev_{M,N} : \underline{\mathrm{Hom}}(M, N) \otimes M \to N$. To specify this as an object of $\mathcal{C}$, we just need to say how many times each simple in $\mathcal{C}$ shows up; in general this might not be defined, but fusion categories have so much structure that by "abstract nonsense" these actually always do exist.

Now, given $M \in \mathcal{M}$, consider $\underline{\mathrm{Hom}}(M, M) \in \mathcal{C}$. We have a canonical evaluation map $\underline{\mathrm{Hom}}(M, M) \otimes M \to M$. To put an algebra structure on $\underline{\mathrm{Hom}}(M, M)$, we need a map $\underline{\mathrm{Hom}}(M, M) \otimes \underline{\mathrm{Hom}}(M, M) \to \underline{\mathrm{Hom}}(M, M)$. But this is the same as a map $\underline{\mathrm{Hom}}(M, M) \otimes \underline{\mathrm{Hom}}(M, M) \otimes M \to M$, which is given by iterating the evaluation map. The unit map $1 \to \underline{\mathrm{Hom}}(M, M)$ is equivalent to the identity map $1 \otimes M \to M$. For any $N \in \mathcal{M}$, we can put a $\underline{\mathrm{Hom}}(M, M)$-module structure on $\underline{\mathrm{Hom}}(M, N)$ by essentially the same trick of applying the evaluation map twice. So, we get a functor $\mathcal{M} \to \mathtt{Mod}_{\mathcal{C}} \underline{\mathrm{Hom}}(M, M)$. Of course, this need not be an equivalence. However, Ostrik proves that this is an equivalence iff M is a *generator* of $\mathcal{M}$, i.e. any $N \in \mathcal{M}$ is a direct summand of $X \otimes M$ for some $X \in \mathcal{C}$. (These always exist, for instance we can take a direct sum of all the simple objects of $\mathcal{M}$.) □

5 On Morita equivalence for fusion categories (Udi)

Recall that if $\mathcal{C}$ is a fusion category, a *module category* M over $\mathcal{C}$ is a category M such that:

- M is k-linear;
- M is semisimple and has a finite number of simple objects;
- we have an action functor $\mathcal{C} \times M \to M$ satisfying the usual axioms.

Example 22. If G is a finite group and $\mathcal{C} = \mathtt{Vec}_G$, then any subgroup H and 2-cocyclce $\psi \in H^2(H, k^\times)$ gives a module category: we consider the algebra $k^\psi H$ inside of $\mathcal{C}$, and set $M = \mathtt{Mod}_{\mathcal{C}} k^\psi H$: given $X \in M$ nad $Y \in k^\psi H$, we have a (right) action $X \otimes k^\psi H \to X$ and moreover this agrees with $(X \otimes Y) \otimes k^\psi H \to Y \otimes (X \otimes k^\psi H) \to Y \otimes X$.

Recall the theorem that we saw last time.

Theorem 6 (Ostrik)**.** *If $\mathcal{C}$ is any fusion category and M is any module category over $\mathcal{C}$, then there is an algebra $A \in \mathcal{C}$ such that $M \cong \mathtt{Mod}_{\mathcal{C}} A$.*

Today, we will define the notion of *Morita equivalence*. Recall that back in the decategorified world, we had the following.

Definition 16. Let A and B be two algebras over a field k. We say that A and B are *Morita equivalent* if the abelian categories $\mathtt{Rep}_A$ and $\mathtt{Rep}_B$ are equivalent.

Example 23. The simplest example is when $A = k$ and $B = M_n(k)$.

Proposition 9. *A and B are Morita equivalent iff there is some $P \in \mathtt{Mod}_A$ (with some nice properties: it's projective, and it's a generator if $\mathtt{Mod}_A$, i.e. any object is a quotient of copies of P) such that $B \cong \mathrm{End}_A(P)$ as rings. (This is precisely the connecting $A-B$-bimodule for the usual definition.) In general, if A and B are Morita equivalent then $Z(A) \cong Z(B)$.*

So, how will we categorify this?

Definition 17. Let $\mathcal{C}$ be a fusion category and M be a module category over $\mathcal{C}$. We define the *dual* of $\mathcal{C}$ with respect to M to be

$$\mathcal{C}^*_M = \mathrm{Fun}_{\mathcal{C}}(M, M) = \{F \in \mathrm{Fun}(M, M) : \text{for all } X \in \mathcal{C}, m \in M \text{ we have } \gamma_{X,m} : F(X \otimes m) \xrightarrow{\sim} X \otimes F(m)\}.$$

(Of course, we automatically assume all our functors to be additive; we're working in the category of abelian categories.) This has a natural tensor structure, given by composition.

Example 24. Let G be a finite group, and let $\mathcal{C} = \mathtt{Vec}_G$ be the category of G-graded vector spaces, and let $M = \mathtt{Vec}$. Note that M only has only one simple object. So given a $\mathcal{C}$-linear functor $F : M \to M$, we just look at $F(k) = V$. Now, the simple objects of $\mathcal{C}$ are k_g for $g \in G$. So, we just need to specify $F(k_g \otimes k) \to k_g \otimes F(k)$ – but these are both just $F(k)$. So ultimately, for all $g \in G$ we obtain $T_g : V \to V$ such that $T_g \circ T_h = T_{gh}$. That is, the $\mathcal{C}^*_M = \mathtt{Rep}(G)$. It's not hard to check that the tensor structures agree, too. In fact, we see that $\mathcal{C}^*_M$ is also a fusion category *if we assume that* char $k = 0$ (because we need semisimplicity).

In general, if H is a Hopf algebra, $M = \mathtt{Vec}$ is a module category over $\mathcal{C} = \mathtt{Rep}(H)$ (as a special case of the above), then $\mathcal{C}^*_M = \mathtt{Rep}(H^*)$.

If char $H = 0$ and $\dim H < \infty$, there is a deep theorem (by Larson and Radford) saying that H is semisimple iff H^* is semisimple. We have the following generalization to fusion categories.

Theorem 7 (Etingof, Nikshych, Ostrik)**.** *If $\mathcal{C}$ is a fusion category and M is an indecomposable module category (i.e. it cannot be written as a direct sum of sub-module categories), then $\mathcal{C}^*_M$ is also a fusion category.*

Definition 18. In this situation, we call $\mathcal{C}$ and $\mathcal{C}^*_M$ *Morita equivalent* fusion categories.

Let us unravel part of the definition for $\mathcal{C}^*_M$ to be a fusion category.

- The unit object in $\mathcal{C}^*_M = \mathrm{Fun}_{\mathcal{C}}(M, M)$ is the identity functor. (If M were decomposable, this would be decomposable, which violates the definitions.)
- Duals come from (left and right) adjoint functors. (Since M is sufficiently nice, these automatically exist.)
- It's not so easy to see that $\mathcal{C}^*_M$ only has a finite number of simple objects; cf. the following theorem.

Supppose that $M \cong \mathtt{Mod}_{\mathcal{C}} A$ for some algebra $A \in \mathcal{C}$. We have the functor $F : (\mathtt{Bimod}_{\mathcal{C}} A)^{op} \to \mathcal{C}^*_M$ given by $X \rightsquigarrow F_X$, where $F_X(T) = T \otimes_A X$. (The *op* is because $F_{X \otimes_A Y}(T) = T \otimes_A (X \otimes_A Y) = F_Y \circ F_X(T)$.) In the classical case this is an equivalence.

Theorem 8 (Ostrik)**.** *This functor F is an equivalence of tensor categories.*

So, for the last bullet point above, it suffices (and is much easier) to prove that $\mathtt{Bimod}_{\mathcal{C}} A$ has a finite number of simple objects. Indeed, if $S \in \mathtt{Bimod}_{\mathcal{C}} A$ is simple, then there exists a simple object $X \in \mathcal{C}$ and a map $A \otimes X \otimes A \twoheadrightarrow S$ of A-bimodules. That is, S needn't be a simple object of $\mathcal{C}$, but we can take X to be a simple constituent of S, and $A \otimes X \otimes A$ has the obvious (free) A-bimodule structure; the map will certainly be nonzero, but since S is simple then it must be epic. Now, there are only a finite number of simple objects of $\mathcal{C}$, and it's not hard to see that moreover for each such X, $A \otimes X \otimes A$ only admits a finite number of simple quotients.

(Of course, the really hard part of the above E-N-O theorem is to show that $\mathcal{C}^*_M$ is semisimple. If $\mathcal{C}$ is a representation category of a Hopf algebra and $M = \mathtt{Vec}$, then we can use what we have already seen; otherwise, there is a generalization of the theorem of Larson and Radford, whichs says that any fusion category is the representation category of a *weak* Hopf algebra, and this is what E-N-O use to finally prove the result.)

Proposition 10. *The relation $\mathcal{C} \sim \mathcal{C}_M^*$ is in fact an equivalence relation.*

Proof. For identity, if we consider $\mathcal{C}$ as a $\mathcal{C}$-module category, we'll get that $\mathcal{C}_\mathcal{C}^* \cong \mathcal{C}$. For symmetry, if $\mathcal{C}$ is a fusion category with indecomposable module category M, note that M is also a $\mathcal{C}_M^*$-module category (with the evaluation action $(F, m) \mapsto F(m)$). A "double-centralizer theorem" (due to E-N-O) tells us that $(\mathcal{C}_M^*)_M^* \cong \mathcal{C}$. Transitivity is similar. $\square$

Just as in the classical case, we have a 1-to-1 correspondence between module categories over $\mathcal{C}$ and module categories over $\mathcal{C}_M^*$. Namely, if N is a module category over $\mathcal{C}$, then we obtain the $\mathcal{C}_M^*$-module category $\mathrm{Fun}_\mathcal{C}(M, N)$. (This associates left modules with right modules and vice versa, but we won't dwell on this minor issue.)

In the classical case if A and B are Morita equivalent rings then $Z(A) \cong Z(B)$; parallelly, if $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent fusion categories, then $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{Z}(\mathcal{D})$ as *braided* fusion categories. (Recall that we saw the notion of center previously.) This is actually kind of surprising, because recall that now we have $\mathcal{Z}(\mathcal{C}) \twoheadrightarrow \mathcal{C}$ (as opposed to $Z(A) \subset A$). The converse holds, too: if $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{Z}(\mathcal{D})$ as braided fusion categories, then $\mathcal{C} \sim \mathcal{D}$.

We'd like to try and explain how and why the center enters the picture here. (It's analogous to the same argument for rings.) We begin with a definition.

Definition 19. If $\mathcal{C}$ and $\mathcal{D}$ are fusion categories, we define a new fusion category $\mathcal{C} \boxtimes \mathcal{D}$, their *(external) tensor product*, which has simple objects $X \boxtimes Y$ for $X \in \mathcal{C}$ and $H \in \mathcal{D}$ both simple.

Now, in particular, we can take $\mathcal{D} = \mathcal{C}^{op}$. Then, $\mathcal{C} \boxtimes \mathcal{C}^{op}$ has the module category $M = \mathcal{C}$ given by $(X \boxtimes Y) \cdot Z = X \otimes Z \otimes Y$. Now, a functor $\mathrm{Fun}_{\mathcal{C}\boxtimes\mathcal{C}^{op}}(M, M)$ needs to commute with the action of $\mathcal{C} \boxtimes \mathcal{C}^{op}$, i.e. it needs to commute with both the left and right actions of $\mathcal{C}$. For example, if $F(1) = V \in \mathcal{C}$, then it must be that $F(X) = F(X \otimes 1) = X \otimes F(1) = X \otimes V$. We also need $F(1 \otimes X) \to F(1) \otimes X$, which is just $X \otimes V \to V \otimes X$. For this reason, objects of $\mathrm{Fun}_{\mathcal{C}\boxtimes\mathcal{C}^{op}}(M, M)$ will be precisely objects $V \in \mathcal{C}$ equipped with isomorphisms $V \otimes X \to X \otimes V$ for all $X \in \mathcal{C}$; that is, they will be *central objects*. Thus, $\mathrm{Fun}_{\mathcal{C}\boxtimes\mathcal{C}^{op}}(M, M) \cong \mathcal{Z}(\mathcal{C})$.

Lastly, let us just point out that given $\mathcal{C}$ and M, we can consider M as a module category over $\mathcal{C} \boxtimes (\mathcal{C}_M^*)^{op}$, and then we obtain

$$\mathcal{Z}(\mathcal{C}_M^*) \cong (\mathcal{C} \boxtimes (\mathcal{C}_M^*)^{op})_M^* \cong \mathcal{Z}(C).$$

6 Ocneanu rigidity (Orit)

6.1 Unitary fusion categories

Our proof of Ocneanu rigidity will be applicable to *unitary* fusion categories, so this is our starting point.

Definition 20. A *unitary fusion category* is a fusion category $\mathcal{C}$ defined over $\mathbb{C}$, with all structure isomorphisms for simple objects being unitary (in a sense we'll see in a second), equipped with a conjugation $\mathrm{Hom}(x, y) \to \mathrm{Hom}(y, x)$, denoted $f \mapsto \overline{f}$, for all $x, y \in \mathcal{C}$, which is:

1. *anti-linear*: $\overline{(\lambda \cdot f)} = \overline{\lambda} \cdot \overline{f}$;
2. *contravariant*: $\overline{g \circ f} = \overline{f} \circ \overline{g}$;
3. *monoidal* $\overline{f \otimes g} = \overline{f} \otimes \overline{g}$;
4. *positive*: $\overline{f} \circ f = 0$ iff $f = 0$;
5. *involutive*: $\overline{\overline{f}} = f$.

In particular, for any simple $x \in \mathcal{C}$, we have $\mathrm{Hom}(x, x) \to \mathrm{Hom}(x, x)$ which is canonically the conjugation on $\mathbb{C}$.

To talk about adjoints, as a consequence we have an inner product on hom-sets: if $x, y \in \mathcal{C}$ are simple, then for any $f, g \in \mathrm{Hom}(x, y \otimes z)$ (called the *splitting state space* in physics; analogously, $\mathrm{Hom}(y \otimes z, x)$ is called the *fusion state space*), we set $\langle f, g \rangle = \overline{g} \circ f \in \mathrm{End}(x) = \mathbb{C}$. More generally, for any $f, g \in \mathrm{Hom}(x \otimes y \otimes z, a \otimes b)$, we (omitting tensor product symbols among objects of $\mathcal{C}$) define the canonical isomorphism

$$\bigoplus_{c,d} \mathrm{Hom}(xy, c) \otimes \mathrm{Hom}(cz, d) \otimes \mathrm{Hom}(d, ab) \overset{\cong}{\to} \mathrm{Hom}(xyz, ab).$$

This can be illustrated diagramatically as PICTURE.

It is with respect to this inner product that we demand our structure isomorphisms be unitary.

Example 25. $\mathtt{Vec}_G$ and $\mathtt{Rep}(G)$ are both unitary, as are quantum fusion/modular categories. However, not all fusion categories are unitary!

Let us describe Yang-Lee theory, which is not unitary. The simple objects are 1 and x, and the fusion rules are determined by $x \otimes x \cong 1 \oplus x$. (So far, this is identical to the Fibonacci theory.) The S and T matrices, however (which are closely tied to the braiding), are given by

$$S = -\frac{1}{\sqrt{3-\phi}} \begin{pmatrix} 1 & 1-\phi \\ 1-\phi & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\pi i/5} \end{pmatrix}.$$

(Here, ϕ is the golden ratio.)

However, we have the following fact (Kitaev): Unitary fusion categories are pivotal and spherical. We also have the fact (ENO): In unitary fusion categories, $\dim(\mathcal{C}) = \operatorname{FPdim}(\mathcal{C})$. An immediate consequence is that for any simple $x \in \mathcal{C}$, $\operatorname{qdim}(x) = \operatorname{FPdim}(x) > 0$. But in the theory described above, the object x has that $\operatorname{qdim}(x) = 1 - \phi < 0$, so that theory cannot be unitary. (Note that for unitary theories, quantum dimension is determined by the fusion rules alone; if we were to play the game we've done in the past with the highest eigenvalue of the associated operator, we'd get the quantum dimension of x for the Fibonacci theory.)

Remark 4. Sometimes (in fact, "often times") the Galois twist of a unitary theory is not unitary. This is a rich source of non-unitary theories.

In light of the above remark, we now examine the associator $(xy)z \to x(yz)$. (**From now on today, the letters $x, y, z, \dots, a, b, c, \dots$ will denote simple objects, and we'll omit tensor product signs.**) For each $u \in \mathcal{C}$ (simple), we have the pushforward $\operatorname{Hom}(u, (xy)z) \to \operatorname{Hom}(u, x(yz))$, which decomposes as

$$\begin{array}{ccc} \operatorname{Hom}(u,(xy)z) & \longrightarrow & \operatorname{Hom}(u,x(yz)) \\ \uparrow \wr & & \uparrow \wr \\ \bigoplus_c \operatorname{Hom}(u,cz)\otimes\operatorname{Hom}(c,xy) & \overset{F_u^{xyz}}{\dashrightarrow} & \bigoplus_d \operatorname{Hom}(u,xd)\otimes\operatorname{Hom}(d,yz). \end{array}$$

[SEE PICTURE] We call this a *unitary F-matrix*.

We claim that these matrices capture the data of the associator. Note that we're witnessing a decategorification, from natural transformations to unitary morphisms of Hilbert spaces.

In the language of unitary F-matrices, we can reformulate the *pentagon axiom*: diagramatically it looks precisely like the one for the associahedron written in terms of rooted trees. [SEE PICTURE]

$$F_u^{xyt} \circ F_u^{pzw} = (\mathrm{id}_x \otimes F_s^{yzw}) \circ F_u^{xrw} \circ (F_q^{xyz} \otimes \mathrm{id}_w).$$

6.2 Algebro-linear data of fusion categories

Now, we'd like to introduce some degrees of freedom into this situation. First, we make choices of orthonormal bases for splitting spaces. (Physicists call this *gauge freedom*.) Then, the pentagon axiom simply becomes an equality of matrices:

$$\sum FF = \sum (1 \otimes F) F (F \otimes F) \ + \ \text{complicated indexology.}$$

Thus, (once we've fixed some fusion rules) a fusion category can be thought of as a solution to a system of algebraic equations: item the pentagon axiom, the triangle axioms, and the unitarity axioms. All of these give rise to an affine algebraic variety over $\mathbb{R}$ (because of the unitarity axioms, which involve complex conjugates). We will denote this by X. We'd like to think of the points of X as unitary fusion categories, but since we made an arbitrary choice of orthonormal bases, these also come with a choice of bases for splitting spaces. Dave sez: This is really just an atlas for an algebraic stack.

Now, the gauge group is of course given by $G = \prod_{(a,b,c)\in\mathcal{C}^3} U(N_{ab}^c)$, a product of unitary groups (recall that $N_{ab}^c = \dim \operatorname{Hom}(c, ab)$). Thus we have an algebraic action of G on X. We can now say what we mean by *rigidity*: X/G is a discrete set.

However, note that not all systems of fusion rules give rise to *any* unitary fusion categories. For instance, it can be showed that a fusion category with two simple objects 1 and x, and with fusion rule $x^2 = 1 \oplus 3x$ cannot be made unitary.

Remark 5. This whole story can be carried over to the general case, replacing $U(n)$ with $GL(n)$.

6.3 Davydov-Yetter cohomology

To prove Ocneanu rigidity, we will define a "tangent complex" to X and show that its cohomology vanishes in positive degrees; as a result, we'll have that X/G is discrete. (We will follow the paper of A. Kitaev, a condensed matter physicist, called *Anyons....* This works for the unitary case; the general case is in E-N-O in *On fusion categories.*)

We will denote by $\Gamma_c^{ab} : \mathrm{Hom}(c, ab) \to \mathrm{Hom}(c, ab)$ the basis change matrices; these are unitary isomorphisms. Now, we haven't mentioned it yet, but in addition to the unitary F-matrices are parameters α and β (which we'll avoid thinking about but nevertheless include or completeness); thus we think of points on X as triples (F, α, β). Thus, suppose we have two points $(F, \alpha, \beta), (F', \alpha', \beta') \in X$. Then, a basis change from one point to the other takes the form

$$\begin{aligned} F_u^{abc}(\sum_e \Gamma_e^{ab} \otimes \Gamma_u^{ec}) &= (\sum_f \Gamma_u^{af} \otimes \Gamma_f^{bc})(F')_u^{abc} \\ \alpha_x &= \Gamma_x^{x1} \gamma \alpha'_x \\ \beta_x &= \Gamma_x^{1x} \gamma \beta'_x. \end{aligned}$$

Remark 6. This γ is part of the pair (Γ, γ) which in categorical terms amounts to a *monoidal functor.* To see this, let $\Gamma^{ab} = [PICTURE]$, where $\{\psi_i\}$ is an orthonormal basis for $\mathrm{Hom}(d, ab)$. This is a morphism in $\mathrm{End}(ab)$. Now, part of the data of a monoidal functor is a morphism $F(a) \otimes F(b) \to F(a \otimes b)$; thus, the underlying functor is the identity (although of course the monoidal functor itself need not be trivial). Now, we also must have $F(1) \to 1$, and this is where γ comes from.

Now, what if we have two monoidal functors (Γ, γ) and (Φ, φ)? We know there should be a monoidal natural isomorphism between them. We need to write this in linear-algebraic terms. Now, a monoidal natural transformation is just a collection of numbers $\{h_x\}$, where $h_x : x \to x$ (where as always, x is simple). This ends up giving rise to two additional equations:

$$\begin{aligned} \Phi_c^{ab} &= \frac{h_c}{h_a h_b} \Gamma_c^{ab} \\ \varphi &= h_1 \gamma. \end{aligned}$$

We now introduce infinitesimal deformations:

1. Since we're taking small deformations of the identity, we have $h_a \approx 1 - iX_a$ for $X_a \in \mathbb{R}$.
2. Deforming the identity functor we have $\Gamma_c^{ab} \approx 1 - iY_c^{ab}$ for $Y_c^{ab} : \mathrm{Hom}(c, ab) \to \mathrm{Hom}(c, ab)$ Hermitian.
3. Deforming the fusion structure we have $F_u^{abc} = F_u^{abc}(1 - iZ_u^{abc})$ for $Z_u^{abc} : \bigoplus_e \mathrm{Hom}(u, ec) \otimes \mathrm{Hom}(e, ab) \to \bigoplus_e \mathrm{Hom}(u, ec) \otimes \mathrm{Hom}(e, ab)$ Hermitian.

We should think of X, Y, and Z as infinitesimal. (We're ignoring α, β, and γ, but they don't figure in in any essential way.)

Now, let's see what we get when we feed 1 and 2 into 3. Up to first order,

$$Y_c^{ab} = (X_b - X_c + X_a) \cdot 1_{\mathrm{Hom}(c,ab)}.$$

Similarly to before, we can introduce an operator Y^{ab} (and its analog on the RHS), i.e. [PICTURE]; this smells a whole lot like group cohomology, and indeed this gives rise to the *tangent complex* and *Davydov-Yetter cohomology.* This has

$$C^n = \bigoplus_{(a_1, \ldots, a_n)} \mathrm{Hom}(a_1 \cdots a_n, a_1 \cdots a_n);$$

in particular, we have $\{Y^{ab}\} \in C^2 = \bigoplus_{(a,b)} \mathrm{Hom}(ab, ab)$. This also suggests what our boundary operator should be. We define $d_0^n : C^n \to C^{n+1}$ by adding in id_{a_1} on the left, and for $0 < k \le n$ we fuse and split two interior objects, and for d_{n+1}^n we add on $\mathrm{id}_{a_{n+1}}$ on the right; of course we set $d^n = \sum (-1)^k d_k^n$.

Now, from equation 3 above we get $Y = dX$; from equation 1 we get $Z = dY$; from the pentagon axiom we get $dZ = 0$. So for example, if $Y = 0$, then we're not deforming Γ, and h becomes a small automorphism of the identity automorphism id_e of the identity functor e. Now, since $dX = 0$ then this must be monoidal. By equation 1, then if we set $Z = 0$ then the fusion structure is fixed; in this case, $dY = 0$ iff Y is monoidal. Now, if we are allowing $Z \neq 0$, we still have that $Z = dY$, so what we're getting is *trivially* equivalent to our original fusion theory, obtained simply by a basis change. So for instance:

- H^3 classifies deformations of the tensor structure (up to basis change);
- H^2 classifies monoidal deformations of the identity functor (up to natural isomorphism);
- H^1 classifies deformations of the identity natural transformation of id_e.

From here, we can obtain rigidity as a consequence of the following theorem.

Theorem 9. *$H^*(C, d)$ vanishes in positive degrees.*

This is *Ocneanu rigidity* (which actually just follows from $H^3 = 0$). It says that one can take a fusion category and apply small deformations of all the structure constants, but one will always obtain something equivalent. The proof is not difficult (especially for those familiar with group cohomology, in which context this is expected since for G a finite group and k a field of characteristic 0, $H^*(G, k) = 0$ in positive degrees), but we will not cover it presently.

7 Group theoretical fusion categories (Udi)

Recall that if $\mathcal{C}$ is a fusion category and M is an indecomposable module category, then we define $\mathcal{C}_M^* = Fun_{\mathcal{C}}(M, M)$. This is a monoidal category (with respect to composition of functors), and in fact it's also a fusion category. Moreover, there exists an algebra object $A \in \mathcal{C}$ such that $M \cong \mathtt{Mod}_{\mathcal{C}} A$, and there is an equivalence $\mathcal{C}_M^* \cong \mathtt{Bimod}_{\mathcal{C}} A$. (If $A \in \mathcal{C}$ is an algebra, then $\mathtt{Mod}_{\mathcal{C}} A$, the category of right A-modules in $\mathcal{C}$, is a $\mathcal{C}$-module category.)

Definition 21. We say that a fusion category $\mathcal{C}$ is *group theoretical* if it is Morita equivalent to a pointed category, i.e. there exists a $\mathcal{C}$-module category M, a finite group G, and a 3-cocycle $\omega \in H^3(G, k^\times)$ such that $\mathcal{C}_M^* \cong \mathtt{Vec}_G^\omega$.

Example 26. Let G be a finite group which acts on another finite group N (by automorphisms). Recall that we can construct the semi-direct product $G \ltimes N$. We now describe a new category. Its objects are N-graded vector spaces which are simultaneously G-representations, compatible in the following sense: if $n \in N$ and $v \in V_n$, then $g(v) \in V_{g(n)}$ for all $g \in G$. Morphisms are required to respect all the structure. We have notions of tensor product for N-graded vector spaces and G-representations, and these fit together in the appropriate way.

Now, consider the category $\mathcal{D} = \mathtt{Vec}_{G \ltimes N}$. We will show that this is dual to $\mathcal{C}$. To do so, we consider the algebra $A = kG \in \mathcal{D}$. Write $M = \mathtt{Mod}_{\mathcal{D}} A$, and consider $\mathcal{D}_M^* \cong \mathtt{Bimod}_{\mathcal{D}} A$. We will describe a functor $F : \mathcal{C} \to \mathcal{D}_M^*$, assigning a compatible N-graded G-rep to an A-bimodule in $\mathcal{D}$. Suppose $V = \bigoplus V_n \in \mathcal{C}$. We set $(F(V))_{(g,n)} = V_n$, with kG-bimodule structure given as follows. Write $kG = \mathrm{span}\{U_g\}$. Then, we easily define the left action by defining $U_g \otimes F(V)_{(h,n)} \to F(V)_{(gh,n)}$ to be $\mathrm{id} : U_g \otimes V_n \to V_n$. For the right action, we need to define $F(V)_{(h,n)} \otimes U_g \to F(V)_{(hg, g^{-1}ng)}$, which is a map $V_n \otimes U_g \to V_{g^{-1}ng}$. Luckily, we already have a map $g^{-1} : V_n \to V_{g^{-1}ng}$, which all works out by the compatibility conditions for $V \in \mathcal{C}$.

Other examples of group-theoretical fusion categories will be generalizations of this one.

Here are some facts.

Theorem 10. *All fusion categories with Frobenius-Perron dimension p^n (for p prime) are group theoretical.*

Theorem 11. *All fusion categories with Frobenius-Perron dimension pqr (for p, q, r distinct primes) are group theoretical.*

Given this, it is natural to ask: What does a "general" group theoretical fusion category look like?

To answer this, we first need to understand what a "general" module category over $\mathtt{Vec}_G^\omega$ looks like. Above, we took module categories for a group algebra. In fact, this is always the situation, except that we may actually need to work with a *twisted* group algebra. Suppose $H < G$ is a subgroup. Can we put an algebra structure on kH (i.e. can we make it a monoid object in $\mathtt{Vec}_G^\omega$)? Of course, the only obstruction is the fact that we've got ω. In fact, this will work iff $\omega|_H$ is trivial. Let's work this out. Write $A = \operatorname{span}\{U_h\}_{h\in H}$. We must have $U_{h_1} \cdot U_{h_2} = \alpha(h_1, h_2)U_{h_1h_2}$. To test against the twisted associativity of $\mathtt{Vec}_G^\omega$, we need that

$$(U_{h_1} \cdot U_{h_2}) \cdot U_{h_3} = \alpha(h_1, h_2)U_{h_1h_2} \cdot U_{h_3} = \alpha(h_1, h_2)\alpha(h_1h_2, h_3)U_{h_1h_2h_3}$$

is the same thing as

$$\omega(h_1, h_2h_3)U_{h_1} \cdot (U_{h_2} \cdot U_{h_3}) = \omega(h_1, h_2, h_3)\alpha(h_2, h_3)U_{h_1} \cdot U_{h_2h_3} = \omega(h_1, h_2, h_3)\alpha(h_2, h_3)\alpha(h_1, h_2h_3)U_{h_1h_2h_3}.$$

That is, we need

$$(\partial\alpha)(h_1, h_2, h_3) = \frac{\alpha(h_1, h_2)\alpha(h_1h_2, h_3)}{\alpha(h_2, h_3)\alpha(h_1, h_2h_3)} = \omega(h_1, h_2, h_3).$$

This is precisely the statement that $\omega|_H \equiv 1 \in H^3(H, k^\times)$. To summarize: For every $H < G$ such that $\omega|_H$ is trivial, and for every $\alpha \in C^2(H, k^\times)$ such that $\partial\alpha = \omega|_H$, we get an algebra object in $\mathtt{Vec}_G^\omega$, and hence a module category which we denote $M(H, \alpha)$. To ease notation, we write $\mathcal{C}(G, \omega, H, \alpha) = (\mathtt{Vec}_G^\omega)^*_{M(H,\alpha)}$.

Theorem 12. *All module categories over $\mathtt{Vec}_G^\omega$ are of the form $M(H, \alpha)$.*

Now, suppose we have the category $\mathcal{C}(G, \omega, H, \alpha)$. These are just the bimodules for $A = k^\alpha H$ inside $\mathcal{C}$. To understand this category, we'd like to understand its simple objects, i.e. the simple A-bimodules in $\mathcal{C}$. Let's begin with the even more basic question: What are the simple A-modules in $\mathcal{C}$? In general, this can be quite tricky. But in the graded world, the extra structure drastically reduces the possibilities.

Proposition 11. *Suppose $m \in \mathcal{C}$ is a simple right A-module. Then the support of m is contained in some coset gH.*

Proof. Given a homogeneous vector $v \in m$ with $|v| = g$, then $v \cdot k^\alpha H$ must be all of m by the simplicity of m, and it is supported on gH. In fact, for any coset gH, the translation of A by g gives us an irreducible module; in this way, simple A-modules are in bijection with the set G/H. (Note that $k_g \otimes A$ is irreducible; if $V \subset k_g \otimes A$, then V contains some homogeneous nonzero element $v \in k_g \otimes A$ with $|v| = gh$, then it contains everything.) ☐

Returning to the original question, suppose that X is a simple A-bimodule. By the argument above, X has support on some *double* coset HgH. However, we must still determine how the two actions interact with each other (this is how some actual representation theory will come into play). Let us write $X = \bigoplus_{a\in HgH} X_a$. For each $h \in H$, we must have $L_h : U_h \otimes X_a = X_a \xrightarrow{\sim} X_{ha}$ and $R_h : X_a \otimes U_h = X_a \xrightarrow{\sim} X_{ah}$.

The compatibility conditions internal to the category $\mathtt{Vec}_G^\omega$ twists the condition that these actions actually commute with each other. First of all, we must have $L_{h_1}L_{h_2} = \text{scalar} \cdot L_{h_1h_2}$ and $R_{h_1}R_{h_2} = \text{scalar} \cdot R_{h_2h_1}$. The first scalar will be something like $\alpha(h_1, h_2)\omega(h_1, h_2, a)$, and the second will be similar. But using the left action, we can say that we know exactly how to identify any two vector spaces whose gradings are in the same right coset, and the same is true with left and right reversed. So in fact, this reduces the problem to looking at group actions on a *single* vector space. To be concrete, let's consider $L_h : X_a \to H_{ha}$ as identifications (for all h). Then we have maps $R_{h_2} : X_a \to X_{ah_2}$. We need to worry if $ah_1 = h_2a$, or equivalently $ah_1a^{-1} = h_2$, or equivalently $ah_1a^{-1} \in aHa^{-1} \cap H$. Let's consider the subgroup $gHg^{-1} \cap H$. If $h \in gHg^{-1} \cap H$, then we get $R_{g^{-1}h^{-1}g}L_h : X_g \to X_{hg} \to X_g$; that is, we get a *projective* representation of $gHg^{-1} \cap H$, i.e. an action twisted by a cocycle determined by α and ω. On the other hand, once we have a projective representation we can turn it into a bimodule, and we get the following result.

Proposition 12. *Simple A-bimodules in $\mathcal{C}$ with support in HgH correspond bijectively to simple representations of $k^\beta(gHg^{-1} \cap H)$, where β depends on α and ω. (The explicit formula for β is a bit ugly.)*