Goodwillie calculus Arbeitsgemeinschaft

Contents

1 Survey (Justin Noel)

We begin with Goodwillie's dictionary:

ordinary calculus	Goodwillie calculus
smooth manifold M	cofibrantly generated topological model category $\mathcal{C}$ (where finite holims commute with filtered hocolims) (or a differentiable ∞-category)
smooth map $f : M \to N$	topological functors $F : \mathcal{C} \to \mathcal{D}$ (preserving weak equivalences, and possibly also filtered colimits)
$\mathbb{R}$(as a vector space)	spectrally enriched category (or stable ∞-category)
$\mathbb{R}$(as a manifold)	spectra
$x \in M$	$c \in \mathcal{C}$
$T_x M$	stabilization of $\mathcal{C} \downarrow c$
Df	$D_1 F$(the linear approximation to a functor)
n^{th} Taylor polynomial of f	polynomial approximation $P_n F$
Taylor series of f	Taylor tower $\{P_n F\}_{n \geq 0}$ (which lives under F)
quadratic form $f(x) = ax^2$	$D_2 F = \mathrm{hofib}(P_2 F \to P_1 F)$ (a 2-homogeneous functor)
symmetric bilinear form	functor of 2 variables which is linear in each and equipped with a Σ_2-action
radius of convergence of f	ρ-analyticity of F

To start to explain all this, we give the following definition.

Definition 1. A functor is linear (or excisive) if it takes homotopy pushouts to homotopy pullbacks. It suffices to consider homotopy pushout squares and homotopy pullback squares.

The standard example is $\Sigma^\infty : \texttt{Top}_* \to \texttt{Spectra}$. Another example is the identity functor on $\texttt{Spectra}$. A non-example is $\texttt{Top}_* \to \texttt{Spectra}$ given by $X \mapsto \Sigma^\infty X^{\wedge 2}$.

The transformation $F \to P_1 F$ is the closest linear approximation to F.

Definition 2. A functor is n-excisive if it takes *strongly cocartesian* $(n+1)$*-cubes* to *cartesian* $(n+1)$*-cubes*; we'll see the precise definitions of these terms in a talk next time, but suffice it to say that this generalizes the previous definition.

To generalize the previous statement, the transformation $F \to P_n F$ is the closest n-excisive approximation to F.

Just as there is a correspondence between quadratic forms and symmetric bilinear forms, there is a correspondence between 2-homogeneous functors and appropriate functors of 2 variables (as above). Recall that a quadratic form Q corresponds to $f(x, y) = (Q(x + y) - Q(x) - Q(y))/2$. Algebraically, this is known as the first cross-effect of Q. In an analogous way, then, to a 2-homogeneous functor Q we associate the 2-variable functor $F(X, Y) = cr_1(X \coprod Y)_{h\Sigma_2}$. (Taking homotopy orbits corresponds to dividing by 2; more generally, on the classical side we'll be dividing by $n!$.) This is a nice way of getting at high derivatives of functors, since it relates them to a bunch of first partial derivatives of a multivariate functor.

Example 1. Consider the functor $\texttt{Top}_* \to \texttt{Spectra}$ given by $X \mapsto \Sigma^\infty X^{\wedge n}$. This is n-homogeneous. So the Taylor tower of the functor $X \mapsto \bigvee_{n \geq 0} C_n \wedge \Sigma^\infty X^{\wedge n}$ splits in the obvious way.

Example 2. The functor $\texttt{Top}_* \to \texttt{Spectra}$ given by $\Sigma^\infty X^{\times n}$ is polynomial of degree n, but not homogeneous. Indeed, stably, Cartesian products split into wedges and smashes.

Here is a surprising result.

Proposition 1. *Given a functor $F : \texttt{Top}_* \to \texttt{Top}_*$ which is reduced (i.e. $F(*) = *$), the n^{th} homogeneous approximation can be written as $D_n F(X) = \Omega^\infty((C_n \wedge X^{\wedge n})_{h\Sigma_n})$. We call C_n the n^{th} coefficient of the Taylor series. It is naturally a Σ_n-spectrum.*

This is all pretty amazing. For example, the identity functor $\mathrm{Id}_{\mathtt{Top}}$ has a Taylor tower whose fibers are actually indexed by spectra. It's a small theorem that this is 1-analytic, and hence its Taylor tower converges for all simply-connected X. That is, in this case the natural transformation $\mathrm{Id}_{\mathtt{Top}}(X) \to \lim(P_n \mathrm{Id}_{\mathtt{Top}})(X)$ is an equivalence. This implies that there exists a spectral sequence

$$E_1^{*,*} = \pi_* \Omega^\infty((C_n \wedge X^{\wedge n})_{h\Sigma_n}) \Rightarrow \pi_* X$$

which converges strongly. The input is a generalized equivariant homology theory. So somehow, stable equivariant data limits to unstable data!

The same tower filtration gives rise to strongly convergent spectral sequences computing E_*X (for any connective generalized homology theory E_*) and H^*X. These kind of things you don't get very often: for instance, homology doesn't generally interact so well with inverse limits.

It's extremely interesting to analyze the behavior of this particular functor. As Irakli will tell us later, we understand all its derivatives $\partial_* \mathrm{Id}$ (which can be described as Spanier-Whitehead duals of spaces in the decomposition of $\Omega \bigvee_{k\geq 1} S^j$) but we don't really understand its polynomial approximations.

Another amazing fact about this tower is that $(D_i \mathrm{Id}_{\mathtt{Top}})S^{2n+1}$ is p-locally contractible if $i \neq p^j$ for some j (but is interesting otherwise). Moreover, applying the n^{th} telescopic localization (i.e. taking v_n-periodic homotopy) kills everything above level p^n. At $n = 0$ we're just getting rational information, and at $n = 1$ this is hard but computable.

One of the other things we'll be working towards, which will appear on the last day, will be to make sense of the statement that after p-completion, "algebraic K-theory and TC differ by a constant". That is, there is a complicated map called the *cyclotomic trace map* running $K \to TC$; if we let F be the fiber, then we can equivalently write that $\partial_X F$ vanishes (for all X). This breaks down the computation of $K(R)$ (p-completed) to understanding the Cartesian square

$$\begin{tikzcd}
K(R) \arrow[r] \arrow[d] & TC(R) \arrow[d] \\
K(R_{\leq 0}) \arrow[r] & TC(R_{\leq 0}).
\end{tikzcd}$$

While this is still extraordinarily difficult (for example, even when $R_{\leq 0} = R_0 = \mathbb{Z}$, we don't know the algebraic K-theory) but it seems to be the best tool we've got so far.

2 First derivatives and basic examples (Tibor Macko)

Following Goodwillie's *Calculus I*, we will write $\mathcal{U}$ for the category of unbased spaces, $\mathcal{T}$ for the category of based spaces, and $\mathcal{S}p$ for the category of spectra. For $X \in \mathcal{U}$, we also have $\mathcal{U}_X$, the category of unbased spaces over X. Similarly, for $X \in \mathcal{U}$ we have $\mathcal{T}_X$, the category of spaces over X with a section (i.e. over and under X). So of course, $\mathcal{U}_* = \mathcal{U}$ and $\mathcal{T}_* = \mathcal{T}$.

Our functors will generally run $\mathcal{U} \to \mathcal{T}$ or $\mathcal{T} \to \mathcal{S}p$. If we restrict from $\mathcal{U}$ to $\mathcal{U}_X$, we will often just write $F|$. In this case, for $Y \xrightarrow{f} X \in \mathcal{U}_X$, we will write $F(Y)$ for $F(Y \xrightarrow{f} X)$.

All our functors will be homotopy functors, i.e. they will take weak equivalences to weak equivalences.

2.1 Linear functors

Definition 3. A functor $L : \mathcal{U}_X \to \mathcal{T}$ (or to $\mathcal{S}p$) is called *linear* if it is:

1. a homotopy functor;

2. excisive (i.e. it takes coCartesian squares to Cartesian squares);

3. reduced (i.e. $L(X \xrightarrow{\mathrm{id}} X) \simeq *$).

Definition 4. We say that a functor L satisfies the *limit axiom* if for all CW-complexes Y, $\pi_* L(Y) = \operatorname{colim} \pi_*(L(Y'))$, where Y' runs through the finite subcomplexes.

Example 3. Let $X = *$ and $\mathbb{C} \in \mathcal{S}p$. Then the functors $\mathcal{U} \to \mathcal{S}p$ and $\mathcal{U} \to \mathcal{T}$ given by $Y \mapsto \mathbb{C} \wedge Y_+$ and $Y \mapsto \Omega^\infty(\mathbb{C} \wedge Y_+)$ are both excisive. The functor $Y \mapsto \operatorname{hofib}(\mathbb{C} \wedge Y_+ \to \mathbb{C} \wedge_+ *)$ (possibly followed by Ω^∞) is linear.

Example 4. If $L : \mathcal{U} \to \mathcal{T}$ is linear, then $L(Y)$ is an infinite loopspace. Indeed, let $\mathcal{Y}$ denote the pushout square

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & CY \arrow[d] \\ CY \arrow[r] & SY. \end{tikzcd}$$

Then $L(\mathcal{Y})$ is the diagram

$$\begin{tikzcd} L(Y) \arrow[r] \arrow[d] & L(CY) \arrow[d] \\ L(CY) \arrow[r] & L(SY), \end{tikzcd}$$

which since L is reduced gives us $L(Y) \xrightarrow{\simeq} \Omega L(SY)$. Thus, given such an L, we can construct $\mathbb{L} : \mathcal{U} \to \mathcal{S}p$ via $\mathbb{L}(Y) = \{L(S^j Y)\}_j$, so that $L(Y) = \Omega^\infty \mathbb{L}(Y)$.

Note further that $h_*(Y) = \pi_{*+j} L(S^j Y)$ is a generalized homology theory; its coefficient spectrum is $\mathbb{L}_\mathbb{C} = \mathbb{L}(S^0) = \{L(S^j)\}_j$. Then there exists a natural transformation $\alpha : \operatorname{hofib}(\mathbb{L}_\mathbb{C} \wedge Y_+ \to \mathbb{C}_\mathbb{C} \wedge_+ *) \xrightarrow{\simeq} \mathbb{L}(Y)$.

Example 5. Now suppose $X \neq *$. If $F : \mathcal{U} \to \mathcal{T}$ or $F : \mathcal{U} \to \mathcal{S}p$ is excisive, we can construct $\widetilde{F} : \mathcal{U}_X \to \mathcal{T}$ by defining $\widetilde{F}(Y) = \operatorname{hofib}(F(Y) \to F(Y))$, a linear functor.

If $E \to X$ is a Serre fibration and we consider $Y \to X$ in $\mathcal{U}_X$, then we get

$$\begin{tikzcd} E \times_X Y \arrow[r] \arrow[d] & E \arrow[d] \\ Y \arrow[r] & X, \end{tikzcd}$$

and so the functors $Y \mapsto Q_+(E \times_X Y)$ and $Y \mapsto \operatorname{hofib}(Q_+(E \times_X Y) \to Q_+(E))$ is a linear functor $\mathcal{U}_X \to \mathcal{T}$.

Example 6. Suppose $L : \mathcal{U}_X \to \mathcal{T}$ is linear. Recall that this means that $L(Y)$ is an infinite loopspace, but this functor is no longer reduced. So define the diagram $\mathcal{Y}$ by

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & C_X Y \arrow[d] \\ C_X Y \arrow[r] & S_X Y \end{tikzcd}$$

(where $C_X Y = \mathrm{cyl}(Y \to X)$ and $S_X Y = C_X Y \cup_Y C_X Y$). Then $L(\mathcal{Y})$ will be given as

$$\begin{tikzcd} L(Y) \arrow[r] \arrow[d] & L(C_X Y) \arrow[d] \\ L(C_X Y) \arrow[r] & L(S_X Y), \end{tikzcd}$$

a Cartesian diagram, and hence we get $L(Y) \xrightarrow{\simeq} \Omega L(S_X Y)$.

Now the question is: Is such a functor determined by a single spectrum? The answer is no; instead, we must move to *fiberwise* (or *parametrized*) spectra. Write $\mathbb{L} : \mathcal{U}_X \to \mathcal{S}p$ for the functor $Y \mapsto \mathbb{L}(Y) = \{L(S_X^j Y)\}_j$.

Here is a statement which we will make more precise later: $\mathcal{L}$ is determined by its values $\mathbb{L}(Y)$ for $Y = X \vee_x S^0$, for all $x \in X$. (The projection down to X is the obvious one.) So we get $\mathbb{L}_{\mathbb{C},x} = \mathbb{L}(X \vee_x S^0) = \{L(X \vee_x S^j)\}_j$.

Proposition 2. *Let $\eta : L \to M$ be a natural transformation of linear functors $\mathcal{U}_X \to \mathcal{T}$ (or to $\mathcal{S}p$) such that η induces an equivalence $\eta_* : \mathbb{L}_{\mathbb{C},x} \xrightarrow{\simeq} \mathbb{M}_{\mathbb{C},x}$ for all $x \in X$. Then $\eta : L(Y) \xrightarrow{\simeq} M(Y)$ for all $Y \in \mathcal{U}_X$.*

Proof. First of all, it is enough to prove this for $\mathbb{L} \to \mathbb{M}$. Moreover, it is enough to prove this on $\mathcal{T}_X \to \mathcal{S}p$ (for reasons outside the scope of this talk). Now, note that by definition this statement is true for $Y = X \coprod * = X \vee_x S^0$. Since our functors are homotopy invariant, then it's true for $Y = X \coprod D^n$. Using coCartesian squares and a Mayer-Vietoris argument, it's then true for $X \coprod S^n$. Now if we have $X \to Y \to X \in \mathcal{T}_X$, if we assume Y is a relative CW-complex, we can do induction on the relative cels. So if it is true for Y', then the pushout diagram

$$\begin{tikzcd} X \coprod S^{n-1} \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \coprod D^n \arrow[r] & Y \end{tikzcd}$$

allows us to see that it must still hold when we add a new cell. Applying the limit axiom completes the argument. ☐

The moral is that even though we didn't say how to reconstruct a linear functor from its coefficient spectrum, we can still see that the coefficient spectrum determines the linear functor.

Remark 1. The forgetful functor $\mathcal{T}_X \to \mathcal{U}_X$ induces a functor between functor categories, and it's an incredibly useful fact that this is an equivalence on linear functors.

Definition 5. A homotopy functor $L : \mathcal{U}_X^n \to \mathcal{T}$ (or to $\mathcal{S}p$ is $(1, \ldots, 1)$-*linear* (or -*excisive*, or -*reduced*) if it is such in each variable.

Example 7. The functor $(X_1, \ldots, X_n) \mapsto \mathbb{C} \wedge X_{1+} \wedge \ldots \wedge X_{n+}$ is $(1, \ldots, 1)$-excisive. Hence we get $\mathbb{L} : \mathcal{U}_X^n \to \mathcal{S}p$, and for all $x_1, \ldots, x_n \in X$, we get $\mathbb{L}(X \vee_{x_1} S^0, \ldots, X \vee_{x_n} S^0)$ as before.

2.2 Approximation by linear functors

Suppose $F : \mathcal{U}_X \to \mathcal{T}$ (or to $\mathcal{S}p$) is any homotopy functor. We'd like to associate $P_X F : \mathcal{U}_X \to \mathcal{T}$ (or to $\mathcal{S}p$) called the *first differential*, and $D_X F : \mathcal{U}_X \to \mathcal{T}$ (or to $\mathcal{S}p$) called the *1-jet*. Of course, we want to subject these to some good conditions.

Definition 6. We say that F is *stably excisive* (with constants $c, \kappa \in \mathbb{Z}$) if it satisfies the $E(c,\kappa)$ property: for all coCartesian squares

$$\begin{tikzcd} Y \arrow[r, "f_1"] \arrow[d, "f_2"'] & Y_1 \arrow[d] \\ Y_2 \arrow[r] & Y_{12} \end{tikzcd}$$

with f_i being k_i-connected and $k_i \geq \kappa$, then the diagram

$$\begin{tikzcd} F(Y) \arrow[r] \arrow[d] & F(Y_1) \arrow[d] \\ F(Y_2) \arrow[r] & F(Y_{12}) \end{tikzcd}$$

is $(k_1 + k_2 - c)$-cartesian. So it's stronger to have lower numbers. (If F satisfies the $E(c,\kappa)$ property for all $c, \kappa \in \mathbb{Z}$, then F is excisive.)

Proposition 3. *$\mathrm{Id}_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ satisfies $E(1,\kappa)$ for all κ; this is the Blakers-Massey theorem.*

Proposition 4. *If K is a finite CW-complex, then $Y \mapsto Q_+(Map(K,Y))$ satisfies $E(2\dim(K),\kappa)$ for any κ.*

Proposition 5. *The Waldhasen functor $Y \mapsto A(Y)$ satisfies $E(1,2)$.*

We make the following notational definitions in order to say somthing about stably excisive functors.

Definition 7. If $F : \mathcal{U} \to \mathcal{T}$ (or to $\mathcal{S}p$) is a homotopy functor, define $T_X F : \mathcal{U}_X \to \mathcal{T}$ (or to $\mathcal{S}p$) by $Y \mapsto \mathrm{holim}(F(C_X Y) \to F(S_X Y) \leftarrow F(C_X Y))$, and let $t : F_X(Y) \to T_X F(Y)$ be induced by

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & C_X Y \arrow[d] \\ C_X Y \arrow[r] & S_X Y. \end{tikzcd}$$

This is good, but if we iterate it we get something fantastic:

$$P_X F(Y) = \mathrm{hocolim}(F_X(Y) \to T_X F(Y) \to T_X^2 F(Y) \to \cdots).$$

We write $p : F_X(Y) \to P_X F(Y)$, and we write $D_X F(Y) = \mathrm{hofib}(P_X F(Y) \to P_X F(X))$.

Remark 2. Note that in these definitions, we first stabilized and then reduced, but in fact we could've first reduced and then stabilized.

Proposition 6. *If F is stably excisive, then $P_X F$ is excisive and $D_X F$ is linear.*

Proof. First, observe that if F satisfies stable excision with $E(c,\kappa)$, then $T_X F$ satisfies stable excision with $E(c-1,\kappa-1)$. So if $\mathcal{Y}$ is any coCartesian square, then we can estimate the connectivity of the square $T_X F(Y)$ by studying its *total fiber.* Since taking fibers commutes with taking loops, we get

$$\mathrm{tfib}(T_X F(\mathcal{Y})) \simeq \mathrm{tfib}(\Omega F(S_X \mathcal{Y})) \simeq \Omega \mathrm{tfib}(F(S_X \mathcal{Y})).$$

The middle square is $(k_1 + k_2 - (c-1))$-connected, and the last square is then $((k_1 + 1 + k_2 + 1 - c) - 1)$-connected. $\square$

The construction (and proof) is basically what makes all of Goodwillie calculus tick.

Definition 8. For all $x \in X$, let $\partial_x F(X)$ denote the coefficient spectrum of $D_X F$ at $x \in X$. This is called the *first derivative* of F at $x \in X$.

2.3 Examples

We now turn to some examples.

Example 8. If F is excisive, then $F(X) = \Omega^\infty(\mathbb{C} \wedge X_+)$, and $\partial_x F(X) = \mathbb{C}$.

Example 9. Consider the identity functor $1 : \mathcal{T} \to \mathcal{T}$. Given a point $\xi \in X$, we can compute the differential to be $D_{(X,\xi)}1(Y \xrightarrow{f} X) = Q(\mathrm{hofib}_\xi(Y \xrightarrow{f} X))$. Now that we're pointed, this should depend on *two* points in X, and indeed we have $(\partial_x 1)(X, \xi) \simeq \Sigma^\infty_+ P_{x,\xi}(X)$, where P denotes paths from x to ξ.

More interesting is to study the mapping space functor.

Example 10. Let K be a finite CW complex. Consider $F : \mathcal{U}_X \to \mathcal{T}$ given by $Y \mapsto Q_+ Map(K, Y)$. One can check that this is excisive. (When $K = S^1$, then of course $Map(K, Y) = \Lambda Y$.) Given a space X, we define a fibration $E(K, X) = K \times Map(K, K) \to K \times X$ given by $(k, f) \mapsto (k, f(k))$. This is functorial in the rather obvious way, and we define $E_Y(K, X)$ via the pullback diagram

$$\begin{tikzcd}
E_Y(K,X) \arrow[r] \arrow[d] & E(K,X) \arrow[d] \\
K \times Y \arrow[r] \arrow[d] & K \times X \arrow[d] \\
K \arrow[r, equal] & K.
\end{tikzcd}$$

Now, we have the space of sections of $E_Y(K, X) \to K$, which gives rise to the *spectrum* of sections $\Gamma_K(E_Y(K, X))$. We claim that this is excisive as a functor of Y. Roughly, this comes from a cellular induction.

In fact, we have the nontrivial theorem.

Theorem 1. *$P_X F(Y) = \Gamma_K(E_Y(K, X))$ and $\partial_x F(X) = \Gamma_K(E_x(K, X)) = \{(k, f) : f(k) = x\} \subset K \times Map(K, X)$.*

Corollary 1. *In the case $K = S^1$, we get*

$$\partial_x Q_+(\Lambda X) = Map(S^1, \Sigma^\infty_+ Loops_x X) \simeq \Sigma^\infty_+ \Omega_x X \times Loops \Sigma^\infty_+ \Sigma_x X.$$

This is an interesting result, for the following reason. It's related to talk 5.2 on Waldhausen A-theory, which as $\partial_x A(X) = \Sigma^\infty_+ \Omega_x X$. And it's reated to talk 6.1, where the trace map $A(X) \to Q_+(\Lambda X)$ induces an equivalence $\partial_x A(X) \to \partial_x Q_+(\Lambda X) \to Susp^\infty_+ \Omega_x X$.

3 Homotopy (co)limits and n-excisive functors (Aaron Mazel-Gee)

3.1 Homotopy (co)limits

In this section, we will work in a bicomplete simplicial model category $\mathcal{C}$. That is:

- *(bicomplete)* $\mathcal{C}$ has (small and sequential) limits and colimits.
- *(simplicial)* $\mathcal{C}$ is simplicially enriched, i.e. hom objects are simplicial sets instead of just sets.
- *(model)* $\mathcal{C}$ is a model category.
- *[convention]* By convention, implicit in this terminology is that $\mathcal{C}$ is simplicially bitensored, meaning simplicially tensored and cotensored, meaning that for $X \in \mathcal{C}$ and $S \in \mathtt{sSet} = \mathtt{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$ we have $X \otimes S \in \mathcal{C}$ and $\mathrm{Hom}(S, X) = X^S \in \mathcal{C}$ satisfying the usual exponentiation rules.

Of course, `Top` (or really `sSet`) is the primordial example.

People usually begin with homotopy colimits and then just say "the story of homotopy limits is dual" for the sake of brevity. So that they don't get the short end of the stick yet again, we'll instead begin with homotopy limits and then briefly outline what one needs to change to dualize to homotopy colimits.

3.1.1 Homotopy limits

The motivation for homotopy limits is that ordinary limits of diagrams aren't homotopy invariant. The standard example is that the morphism of corners $(* \to X \leftarrow PX) \longrightarrow (* \to X \leftarrow *)$ consists of equivalences (where PX denotes a based pathspace), but taking limits gives $\Omega X \to *$. Of course, since we're taking a pullback, it's probably a good idea to demand that our maps be fibrations. Indeed, a homotopy limit will represent "homotopy-coherent systems of maps to a diagram", so it'd make sense to want to be able to lift homotopies back through the morphisms in our diagram. So for an arbitrary corner $(Y \xrightarrow{f} X \xleftarrow{g} Z)$, we make the ad hoc definition

$$\mathrm{holim}(Y \xrightarrow{f} X \xleftarrow{g} Z) = \lim(P_f \to X \leftarrow P_g) \cong \{(y, \alpha, z) \in Y \times X^I \times Z : f(y) = \alpha(0), \alpha(1) = z\},$$

where $P_f = \{(y, \omega) \in Y \times X^I : f(y) = \omega(0)\}$ is the usual pathspace construction turning f into a fibration.

To test this, let's consider the diagram $(X \xrightarrow{f} Y \xrightarrow{g} Z)$. This time, to get the holim we might think that we should take

$$\lim(P_f \to P_g \to Z) \cong \{(x, \alpha, y, \beta, z) \in X \times Y^I \times Y \times Z^I \times Z : f(x) = \alpha(0), \alpha(1) = y, g(y) = \beta(0), \beta(1) = z\}.$$

But in fact, there's the hidden morphism $X \xrightarrow{gf} Z$ that we didn't write down, and so instead we might think that we should require a path $\gamma \in Z^I$ from $gf(x)$ to z too; indeed, to give a canonical and robust definition we really have no choice. But this doesn't give somethink equivalent! On the other hand, the fact that $gf = g \circ f$ suggests that in order for our holim to be encoding "homotopy-coherence", we should furthermore require a "higher homotopy" $\delta \in Z^{(\Delta^2)}$ witnessing a homotopy $g(\alpha) \cdot \beta \simeq \gamma$ rel endpoints.

And this is what leads us to the correct general definition. First, we have a few preliminary definitions.

Definition 9. The *cosimplicial indexing category*, denoted by $\mathbf{\Delta}$, is the category whose objects are the nonempty finite ordered sets $[n] = \{0, \ldots, n\}$ and whose morphisms are (weakly) increasing maps. (We can also consider $[n]$ as a poset and hence as a category, and then $\mathbf{\Delta}$ is a full subcategory of `Cat` in the obvious way.) The morphisms $d^i : [n] \to [n+1]$ (for $0 \leq i \leq n+1$; these are the *codegeneracy maps*) and $s^i : [n] \to [n-1]$ (for $0 \leq i \leq n-1$; these are the *coface maps*) generate all the morphisms in $\mathbf{\Delta}$.

Definition 10. A *cosimplical object* in $\mathcal{C}$ is a functor $X : \mathbf{\Delta} \to \mathcal{C}$. These form the functor category $\mathtt{c}\mathcal{C} = \mathcal{C}^{\mathbf{\Delta}}$.

Definition 11. Given a cosimplicial object $X \in \mathtt{c}\mathcal{C}$, its *corealization* (or *totalization*) is the object of $\mathcal{C}$ given by

$$\mathrm{Tot}(X) = \mathrm{eq}\left(\prod_{[n]} (X_n)^{(\Delta^n)} \rightrightarrows \prod_{[s] \xrightarrow{\varphi} [t]} (X_t)^{(\Delta^s)} \right),$$

where the arrows are induced by the maps $\Delta^s \to X_s \xrightarrow{X(\varphi)} X_t$ and $\Delta^s \xrightarrow{\varphi_*} \Delta^t \to X_t$.

So a point $x \in \mathrm{Tot}(X)$ should be thought of as the following data:

- a point $x_0 : \Delta^0 \to X_0$;
- a path $x_1 : \Delta^1 \to X_1$ connecting the two images of x_0, whose composition to X_0 is $\Delta^1 \xrightarrow{s^0} \Delta^0 \xrightarrow{x_0} X_0$;
- a triangle $x_2 : \Delta^2 \to X_2$ interpolating between the three images of x_1 (and hence between the three images of x_0), whose two compositions to X_1 are $\Delta^2 \xrightarrow{s^i} \Delta^1 \xrightarrow{x_1} X_1$;
- a tetrahedron $x_3 : \Delta^3 \to X_3$ interpolating between the four images of x_2 (and hence between the six images of x_1, and hence between the four images of x_0), whose three compositions to X_2 are $\Delta^3 \xrightarrow{s^i} \Delta^2 \xrightarrow{x_2} X_2$;
- etc.

Thus, x picks out a point in X_0, along with all possible higher homotopies between its $(n+1)$ images in X_n (for all n), such that an n-dimensional homotopy (i.e. a homotopy parametrized by Δ^n) can only be nondegenerate in X_n and above.

Remark 3. In good (but not all) cases, a morphism in $\mathsf{c}\mathcal{C}$ consisting of weak equivalences induces an equivalence of corealizations. (A sufficient condition is for both the source and target to be *Reedy fibrant*.)

Remark 4. Via the unique maps $\Delta^n \to \Delta^0$, we get a morphism of diagrams

$$\left(\prod_{[n]} X_n \rightrightarrows \prod_{[s]\to[t]} X_t\right) = \left(\prod_{[n]} (X_n)^{(\Delta^0)} \rightrightarrows \prod_{[s]\to[t]} (X_t)^{(\Delta^0)}\right) \longrightarrow \left(\prod_{[n]} (X_n)^{(\Delta^n)} \rightrightarrows \prod_{[s]\to[t]} (X_t)^{(\Delta^s)}\right),$$

which induces a map of equalizers. But the equalizer of the former is just $\lim X = \mathrm{eq}(X_0 \rightrightarrows X_1)$, so we get a canonical map $\lim X \to \mathrm{Tot}(X)$. In the case that $\mathcal{C} = \mathtt{Top}$, this takes a point $x \in \mathrm{eq}(X_0 \rightrightarrows X_1) \subset X_0$ to the point $x \in X_0$, along with the constant path at $d^0(x) \in X_1$, along with the constant triangle at $d^0(d^0(x)) = d^1(d^0(x)) \in X_2$, etc.

We are now ready to return to the original problem of constructing holims.

Definition 12. Let $\mathcal{I}$ be any small indexing category, and let $D : \mathcal{I} \to \mathcal{C}$ be a diagram. We define the *cosimplicial replacement* of D, denoted $\mathrm{crep}(D) \in \mathsf{c}\mathcal{C}$, by setting

$$\mathrm{crep}(D)_n = \prod_{i_0 \to \cdots \to i_n} D(i_n).$$

The coface and codegeneracy maps are "the only thing they could be". (It's clear if you write it out. Maps to a product are determined by what they do on each factor, and everything but d^n ends up being described by identity maps anyways (while changing the indexing tuples of morphisms, of course).)

Definition 13. Let $\mathcal{I}$ be any small indexing category, and let $D : \mathcal{I} \to \mathcal{C}$ be a diagram. We define the *homotopy limit* of D to be

$$\mathrm{holim}_{\mathcal{I}} D = \mathrm{Tot}(\mathrm{crep}(D)).$$

So a point of $\mathrm{holim}_{\mathcal{I}} D$ is a collection of maps $\Delta^n \to D(i_n)$, one for each n-chain of morphisms $i_0 \to \cdots \to i_n$ in $\mathcal{I}$, which provide a compatible system of (higher) homotopies between the various ways of arriving at the object $D(i_n)$.

Remark 5. Unwinding the definitions, we see that we can write this as

$$\operatorname{holim} D = \operatorname{eq}\left(\prod_i (D(i))^{B(\mathcal{I}_{\downarrow i})^{op}} \rightrightarrows \prod_{s \to t} (D(t))^{B(\mathcal{I}_{\downarrow s})^{op}} \right).$$

(Recall that for any category $\mathcal{D}$, we define the *classifying space* by $B\mathcal{D} = |N\mathcal{D}|$, where $N\mathcal{D} \in \mathtt{sSet}$ is given by $(N\mathcal{D})_n = \{i_n \leftarrow \cdots \leftarrow i_0\}$. We'll define these symbols in the next section.) In this context, the map $\operatorname{holim} D \to \operatorname{hocolim} D$ is induced by the unique maps from the classifying spaces to Δ^0. This is a nice point of view, since it illustrates how we need more fat mapping into $D(i)$ depending on the size and shape of the overcategory $\mathcal{I}_{\downarrow i}$.

Exercise 1. Show that this gives our ad hoc constructions when applied to the diagrams $(Y \to X \leftarrow Z)$ and $(X \to Y \to Z)$.

Exercise 2. Make precise the claim that "the hocolim corepresents homotopy-coherent systems of maps to a diagram" (from a single object). Note that the homotopies themselves will be encoded in such a datum.

Remark 6. From Remark 4, we have a map $\lim D = \operatorname{eq}(\operatorname{crep}(D)_0 \rightrightarrows \operatorname{crep}(D)_1) \to \operatorname{Tot}(\operatorname{crep}(D)) = \operatorname{holim} D$. This is precisely the map guaranteed by the on-the-nose coherent (and hence homotopy-coherent) system of maps to the diagram from its limit. Of course, the homotopies are all constant here.

Lastly, here are a few comparison results for holims.

Proposition 7. *If $D, D' : \mathcal{I} \to \mathcal{C}$ both consist of fibrant objects and $D \to D'$ is an objectwise equivalence, then* $\operatorname{holim} D \xrightarrow{\sim} \operatorname{holim} D'$. *(Note that all objects of* `Top` *are fibrant!)*

Proposition 8. *If $D : \mathcal{I} \to \mathcal{C}$, J is another small indexing diagram, and $u : \mathcal{J} \to \mathcal{I}$ is a functor, then the induced map* $\operatorname{holim}_{\mathcal{I}} D \to \operatorname{holim}_{\mathcal{J}} u^* D$ *is an equivalence if u is* homotopy initial *(a/k/a* homotopy cofinal*, a/k/a* homotopy left cofinal*), i.e. for every object $i \in \mathcal{I}$, the category $\mathcal{J} \times_{\mathcal{I}} (\mathcal{I}_{\downarrow i})$ is nonempty and contractible.*

Corollary 2. *If $\mathcal{I}$ has an initial object a, then for every diagram $D : \mathcal{I} \to \mathcal{C}$, the map $D(a) \cong \lim_{\mathcal{I}} D \to \operatorname{holim}_{\mathcal{I}} D$ is an equivalence.*

3.1.2 Homotopy colimits

The story of hocolims is dual to the story of holims, so we'll simply outline the dual exposition.

- The standard counterexample to the homotopy invariance of colimits is that the morphism of corners $(CX \leftarrow X \to CX) \longrightarrow (* \leftarrow X \to *)$ consists of equivalences, but taking colimits gives $\Sigma X \to *$. This time, we'll want to replace our maps by cofibrations; indeed, a hocolim is meant to corepresent "homotopy-coherent systems of maps off a diagram", so it'd make sense to want to be able extend homotopies along the morphisms in our diagram. Once again, we'll run into subtleties with the diagram $(X \to Y \to Z)$, which will make us realize that we need to take compositions into account.
- A *simplical object* in $\mathcal{C}$ is a functor $X : \boldsymbol{\Delta}^{\mathbf{op}} \to \mathcal{C}$. These form the functor category $\mathtt{s}\mathcal{C} = \mathcal{C}^{\boldsymbol{\Delta}^{\mathbf{op}}}$.
- A simplicial object $X \in \mathtt{s}\mathcal{C}$ has a (*geometric*) *realization*, which is the object of $\mathcal{C}$ defined as

  $$|X| = \operatorname{coeq}\left(\coprod_{[s] \to [t]} X_t \otimes \Delta^s \rightrightarrows \coprod_{[n]} X_n \otimes \Delta^n \right) = \left(\coprod_{[n]} X_n \otimes \Delta^n \right) \Big/ \begin{array}{l} (d_i x, t) \sim (x, d^i t) \\ (s_i x, t) \sim (x, s^i t) \end{array}$$

  (whenever the last expression makes sense in $\mathcal{C}$).
- A morphism in $\mathtt{s}\mathcal{C}$ consisting of weak equivalences induces an equivalence of realizations when the source and target are *Reedy cofibrant*.

- We get a morphism $|X| \to \operatorname{colim} X$ by projecting away the simplices in the coequalizer diagram. In the case that $X \in \mathtt{sSet} \subset \mathtt{sTop}$, this is just the π_0 map.
- The *simplicial replacement* of a diagram $D : \mathcal{I} \to \mathcal{C}$ is the simplicial object $\operatorname{srep}(D) \in \mathtt{s}\mathcal{C}$ defined by

$$\operatorname{srep}(D)_n = \coprod_{i_0 \leftarrow \cdots \leftarrow i_n} D(i_n).$$

(This is reasonable notation since $\operatorname{srep}(D)$ is a simplicial object so it's a functor off $\mathbf{\Delta^{op}}$, so the subscripts above correspond with the elements of $[n] \in \mathbf{\Delta^{op}}$ in a way compatible with the morphisms.) Then, we define the *homotopy colimit* of D to be

$$\operatorname{hocolim}_{\mathcal{I}} D = |\operatorname{srep}(D)|.$$

So to obtain $\operatorname{hocolim}_{\mathcal{I}} D$, we gluing together a bunch of objects of the form $D(i_n) \otimes \Delta^n$, one for each n-chain of morphisms $i_0 \leftarrow \cdots \leftarrow i_n$. This can be rewritten as

$$\operatorname{hocolim} D = \operatorname{coeq}\left(\coprod_{s \to t} D(s) \otimes B(\mathcal{I}_{t\downarrow})^{op} \rightrightarrows \coprod_{i} D(i) \otimes B(\mathcal{I}_{i\downarrow})^{op} \right);$$

now, we're fattening $D(i)$ according to the size and shape of the undercategory $\mathcal{I}_{i\downarrow}$.
- If $D, D' : \mathcal{I} \to \mathcal{C}$ both consist of cofibrant objects and $D \to D'$ is an objectwise equivalence, then $\operatorname{hocolim} D \xrightarrow{\sim} \operatorname{hocolim} D'$. If $\mathcal{C} = \mathtt{Top}$, then this is true without the cofibrancy condition (which is not to say that all objects of $\mathtt{Top}$ are cofibrant!).
- If $D : \mathcal{I} \to \mathcal{C}$, $\mathcal{J}$ is another small indexing diagram, and $u : \mathcal{J} \to \mathcal{I}$ is a functor, then the induced map $\operatorname{hocolim}_{\mathcal{J}} u^* D \to \operatorname{hocolim}_{\mathcal{I}} D$ is an equivalence if u is *homotopy terminal* (a/k/a *homotopy final*, a/k/a *homotopy right cofinal*), i.e. for every object $i \in \mathcal{I}$, the category $\mathcal{J} \times_{\mathcal{I}} (\mathcal{I}_{i\downarrow})$ is nonempty and contractible. As a special case, if $\mathcal{I}$ has a terminal object z, then for every diagram $D : \mathcal{I} \to \mathcal{C}$, the map $\operatorname{hocolim}_{\mathcal{I}} D \to \operatorname{colim}_{\mathcal{I}} D \cong D(z)$ is an equivalence.

3.1.3 An aside on the derived functor perspective

Suppose we have a diagram $D : \mathcal{I} \to \mathcal{C}$. Write QD for the diagram obtained by replacing each $D(i)$ with $\operatorname{hocolim}_{\mathcal{I}_{\downarrow i}} u_i^* D$, where $u_i : \mathcal{I}_{\downarrow i} \to \mathcal{I}$ is the evident functor; note that there's a natural equivalence $QD \xrightarrow{\sim} D$ since $(i \xrightarrow{\mathrm{id}} i) \in \mathcal{I}_{\downarrow i}$ is a terminal object. Then $\operatorname{colim} QD \cong \operatorname{hocolim} D$, and moreover $\mathcal{C}^{\mathcal{I}}(QD, D')$ is naturally homeomorphic to the space of homotopy-coherent maps $D \to D'$ for any diagram $D' : \mathcal{I} \to \mathcal{C}$. Thus, we've found a replacement diagram which corepresents homotopy-coherent maps to other diagrams (instead of to a single object, which is the special case obtained by viewing a single object as a constant diagram).

In fact, it turns out that there's a fibrant model structure on $\mathcal{C}^{\mathcal{I}}$ (so weak equivalences and fibrations are checked objectwise) in which Q is a functorial cofibrant replacement functor. There's a Quillen pair $\operatorname{colim} : \mathcal{C}^{\mathcal{I}} \rightleftarrows \mathcal{C} : \mathit{const.}$, and since colim is the left adjoint, we obtain its (left-)derived functor by applying it to cofibrant replacements. In this sense, one can say that “homotopy colimit is the derived functor of colimit”. Of course, the dual story is true for homotopy limits. Also, this should alert us to the fact that what we've given is only one model for the homotopy (co)limit; indeed, we could've used any (co)fibrant replacements.

Example 11. Let $\mathbb{N}$ be the natural numbers, considered as poset and hence as a category. Then $\mathcal{C}^{\mathbb{N}}$ is the diagram category of directed sequences. Given $D \in \mathcal{C}^{\mathbb{N}}$, we can form the *mapping telescope replacement diagram* by

$$D_{tel}(n) = \operatorname{coeq}\left(\coprod_{i=1}^{n-1} D(i) \rightrightarrows \coprod_{i=1}^{n} D(i) \otimes I \right).$$

Clearly there's an equivalence $D_{tel} \xrightarrow{\sim} D$, and one can check that D_{tel} is cofibrant (i.e. it satisfies the left lifting property against trivial fibrations). So this is indeed a cofibrant replacement, and hence $\operatorname{colim}_{\mathbb{N}} D_{tel} \simeq \operatorname{hocolim}_{\mathbb{N}} D$. This is a much smaller model than the one above, and indeed it's the one that we should be keeping in mind.

The bar construction (which we probably won't actually discuss, but which is outlined below) provides a particularly nice cofibrant replacement functor. Roughly, the idea is to recognize the colimit as a tensor product and then mimic the algebraic bar construction. (Recall that the realization of the 2-sided bar construction is (usually) the (left-)derived tensor product.)

3.1.4 An aside on the $\mathcal{I}$-module perspective

There is an incredibly slick way to write down (and generalize) everything we've said so far about homotopy (co)limits, due to Hollender and Vogt. Of course, being slick, this has the advantage that certain facts become much cleaner and clearer and it's generally easier to prove things this way, but on the flipside it has the disadvantage that there's so much wrapped up in the notation that it can be somewhat daunting to unwind.

Definition 14. Given diagrams $X : \mathcal{I} \to \mathcal{C}$ and $W : \mathcal{I}^{op} \to \mathcal{C}$, we define their *tensor product* to be

$$W \otimes_{\mathcal{I}} X = \operatorname{coeq}\left(\coprod_{s \to t} W_t \otimes X_s \rightrightarrows \coprod_i W_i \otimes X_i \right).$$

(This is an example of a *coend*.) We think of X as a left $\mathcal{I}$-module and W as a right $\mathcal{I}$-module.

Our primary reason for doing this is the following. If we denote by $* : \mathcal{I}^{op} \to \mathcal{C}$ the constant diagram at the terminal object of $\mathcal{C}$, then $* \otimes_{\mathcal{I}} X \cong \operatorname{colim} X$. However, this also subsumes a few other constructions we've seen.

Example 12. If $X \in \mathtt{sTop}$ and $\Delta^* : \mathbf{\Delta} \to \mathtt{Top}$ is the standard inclusion, then $X \otimes_{\mathbf{\Delta}} \Delta^* \cong |X|$.

Example 13. If $D : \mathcal{I} \to \mathcal{C}$ and $B(\mathcal{I}_{-\downarrow})^{op} : \mathcal{I}^{op} \to \mathcal{C}$ is given by $i \mapsto B(\mathcal{I}_{i\downarrow})^{op}$, then $B(\mathcal{I}_{-\downarrow})^{op} \otimes_{\mathcal{I}} D \cong \operatorname{hocolim}_{\mathcal{I}} D$.

Recall that if R is a ring with $M \in \mathtt{Mod}\text{-}R$ and $N \in R\text{-}\mathtt{Mod}$, then $M \otimes_R N = \operatorname{coeq}(M \otimes R \otimes N \rightrightarrows M \otimes N)$. Via the unit map $\mathbb{Z} \to R$, this becomes the 1-truncation of the *two-sided bar construction*, a simplicial abelian group $B_\bullet(M, R, N)$ defined by $B_n(M, R, N) = M \otimes R^{\otimes n} \otimes N$. Under mild cofibrancy hypotheses, $|B_\bullet(M, R, N)| = M \otimes^{\mathbb{L}}_R N$. Now that we have identified colimits as tensor products, we can repeat this same story.

Definition 15. Given diagrams $X : \mathcal{I} \to \mathcal{C}$ and $W : \mathcal{I}^{op} \to \mathcal{C}$, we define their *two-sided bar construction* to be $B_\bullet(W, \mathcal{I}, X) \in \mathtt{s}\mathcal{C}$, given by

$$B_n(W, \mathcal{I}, X) = \coprod_{i_0 \leftarrow \cdots \leftarrow i_n} W(i_0) \otimes X(i_n)$$

(with completely obvious face and degeneracy maps, except that d_0 and d_n are only mostly obvious). This is covariantly functorial in both X and W.

Now, $B_\bullet(*, \mathcal{I}, X) \cong \operatorname{srep}(X)$. If we write $B(W, \mathcal{I}, X) = |B_\bullet(W, \mathcal{I}, X)|$, then there's a natural map $B(W, \mathcal{I}, X) \to \operatorname{coeq}(B_1(W, \mathcal{I}, X) \rightrightarrows B_0(W, \mathcal{I}, X)) = W \otimes_{\mathcal{I}} X$, which becomes the natural map $\operatorname{hocolim} X \to \operatorname{colim} X$ when we set $W = *$. As in the algebraic setting, the two-sided bar construction is a "derived tensor product".

Example 14. $B_\bullet(*, \mathcal{I}, *) = N(\mathcal{I}^{op})$. Since $B\mathcal{I} = |N\mathcal{I}|$, this should be thought of as analogous to the "freeification" of a group action on a single point.

Remark 7. If we consider $X : \mathcal{I} \to \mathcal{C}$ instead as $X' : \mathcal{I}^{op} \to \mathcal{C}^{op}$, then $B(*, \mathcal{I}, X) \cong B(X', \mathcal{I}, *)$. For this reason, the definition of simplicial replacement given above depended on a few choices, but the resulting hocolims are well-defined up to natural isomorphism.

Remark 8. This theory of modules extends quite easily to bimodules. With this language, it takes a single line to prove that our rewriting of the hocolim in terms of classifying space is equivalent to the original definition. This also affords us a simple definition of the cofibrant replacement functor $Q : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}^{\mathcal{I}}$.

There is a dual story of course, where we have a "function space" construction which is adjoint to the tensor product construction; this admits a homotopical replacement as well, which is called the *cobar construction*.

3.1.5 An aside on the ∞-category perspective

Everything one can say about simplicial model categories can be ported over to quasicategories (a/k/a ∞-categories, a/k/a $(\infty, 1)$-categories, a/k/a inner Kan complexes), and some of the future talks will use this latter framework. The most important thing for us to know is that ho(co)lims in a simplicial model category correspond to ordinary (co)lims in quasicategories. Indeed, one might say that "∞-categories don't even know what 'on-the-nose' means".

We give only slightly more details. In the case of unenriched categories, we have the left Kan extension diagram

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, hook, "i"] \arrow[d, hook, "y"'] & \mathtt{Cat} \\ \mathtt{sSet}, \arrow[ur, dashed, "\tau"'] & \end{tikzcd}$$

in which τ is uniquely determined by the facts that it makes the diagram commute (i.e. $\tau(\Delta^n) = [n]$) and that it commutes with colimits. (This follows from the general fact that every object of the presheaf category $\mathtt{sSet} = \mathtt{Set}^{\mathbf{\Delta}^{op}} = PreSh_{\mathtt{Set}}(\mathbf{\Delta})$ is a colimit of representable presheaves on $\mathbf{\Delta}$.) This determines an adjunction $\tau : \mathtt{sSet} \rightleftarrows \mathtt{Cat} : N$, where N is the *nerve* functor. In the case of simplicially enriched categories, there is an analogous inclusion $i_{\mathbf{\Delta}} : \mathbf{\Delta} \to \mathtt{Cat}_{\mathbf{\Delta}}$ which lifts i in the sense that $\mathrm{Hom}_{i_{\mathbf{\Delta}}([n])}(j, k)$ is contractible if $j \leq k$ and empty otherwise. Again, we have a left Kan extension diagram

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, hook, "i_{\mathbf{\Delta}}"] \arrow[d, hook, "y"'] & \mathtt{Cat}_{\mathbf{\Delta}} \\ \mathtt{sSet}, \arrow[ur, dashed, "\mathfrak{C}"'] & \end{tikzcd}$$

which determines an adjunction $\mathfrak{C} : \mathtt{sSet} \rightleftarrows \mathtt{Cat}_{\mathbf{\Delta}} : N_{hc}$, where N_{hc} is the *homotopy-coherent nerve* functor. In fact, this entire construction lifts the previous one in the sense that the diagram

$$\begin{tikzcd} \mathtt{sSet} \arrow[r, shift left, "\mathfrak{C}"] \arrow[d, equal] & \mathtt{Cat}_{\mathbf{\Delta}} \arrow[l, shift left, "N_{hc}"] \arrow[d, shift left, "\pi_0"] \\ \mathtt{sSet} \arrow[r, shift left, "\tau"] & \mathtt{Cat} \arrow[l, shift left, "N"] \arrow[u, hook, shift left] \end{tikzcd}$$

commutes.

Now, the adjoint pair $\mathfrak{C} \dashv N_{hc}$ is in fact a Quillen equivalence (when one endows $\mathtt{Cat}_{\mathbf{\Delta}}$ with the *Bergner model structure*, in which weak equivalences are those functors that are essentially surjective and homotopically fully faithful (with respect to the Quillen model structure), and $\mathtt{sSet}$ with the *Joyal model structure*, in which the weak equivalences are those that become such in $\mathtt{Cat}_{\mathbf{\Delta}}$). Moreover, quasicategories are precisely the fibrant objects of $\mathtt{sSet}$ (namely, their inner horns can be filled).

Thus, given a simplicial model category $\mathcal{C}$, one takes the full simplicial subcategory $\mathcal{C}^0$ of bifibrant objects and then applies N_{hc} to get a fibrant simplicial set, i.e. a quasicategory. Note that $\mathcal{C}^0$ is fibrant by the "corner axioms" for a simplicial model category (which ensures that $N_{hc}(\mathcal{C}^0)$ is fibrant) and indeed the inclusion $\mathcal{C}^0 \to \mathcal{C}$ is a fibrant replacement. (There is a technical condition which ensures that a quasicategory actually corresponds to a simplicial model category.)

For a simplicial model category $\mathcal{C}$ and a diagram $D : \mathcal{I} \to \mathcal{C}$, we therefore obtain the diagram

$$\begin{tikzcd}
N(\mathcal{I}) = N_{hc}(\mathcal{I}) \arrow[r, "N_{hc}(D)"] \arrow[dr, dashed] & N_{hc}(\mathcal{C}) \\
& N_{hc}(\mathcal{C}^0). \arrow[u, "\wr"']
\end{tikzcd}$$

Under this correspondence, every homotopy (co)limit diagram in $\mathcal{C}$ yields an ∞-categorical (co)limit diagram in $N_{hc}(\mathcal{C}^0)$, and conversely every ∞-categorical (co)limit diagram in $N_{hc}(\mathcal{C}^0)$ rigidifies to a homotopy (co)limit diagram in $\mathcal{C}$. (Recall that a limit of a functor of ∞-categories is a terminal object in the ∞-category of cones on the induced diagram; a terminal object is an object with a contractible space of maps from any other object, and the ∞-category of these is always empty or contractible. Analogously for colimits.)

3.1.6 Important examples for Goodwillie calculus: cubes and orbits

In Goodwillie calculus, the main ho(co)lims we'll care about are those of cubical and "punctured" cubical diagrams, as well as hocolims of directed diagrams. We'll also care about homotopy orbits of a group action (a homotopy colimit construction). Let's address these each in turn.

Definition 16. Let S be a finite set with $|S| = n$. Then a functor $X : \mathscr{P}(S) \to \mathcal{C}$ is called an *n-cube* in $\mathcal{C}$. Here $\mathscr{P}(S)$ is the power set of S, considered as a poset and hence as a category. We write $\mathscr{P}_0(S) = \mathscr{P}(S) - \{\emptyset\}$ and $\mathscr{P}_1(S) = \mathscr{P}(S) - \{S\}$; diagrams of these shapes are called *punctured* cubes. Often we will take $S = \underline{n} = \{1, \ldots, n\}$.

So 0-cubes are just objects, 1-cubes are just morphisms, and 2-cubes are just commutative squares. It's sometimes helpful to think of an $(n+1)$-cube as a morphism of n-cubes, or more generally of an $(m+n)$-cube as an m-cube of n-cubes.

Note that cubes have initial and terminal objects, so their (homotopy) (co)limits are uninteresting. This is why we introduced punctured cubes, though.

Definition 17. An n-cube $X : \mathscr{P}(S) \to \mathcal{C}$ is called *cartesian* if the natural map $X(\emptyset) \to \operatorname{holim}_{\mathscr{P}_0(S)} X$ is an equivalence, and *cocartesian* if the natural map $\operatorname{holim}_{\mathscr{P}_1(S)} X \to X(S)$ is an equivalence. More generally, X is called *k-cartesian* if the former map is k-connected, and *k-cocartesian* if the latter map is k-connected.

Definition 18. An n-cube $X : \mathscr{P}(S) \to \mathcal{C}$ is called *strongly cocartesian* if the restriction $X|_{\mathscr{P}(T)} : \mathscr{P}(T) \to \mathcal{C}$ is cocartesian for all $T \subseteq S$ with $|T| \geq 2$.

So vacuously, all 0- and 1-cubes are strongly cocartesian. For $n \geq 2$, any strongly cocartesian n-cube $X : \mathscr{P}(S) \to \mathcal{C}$ is determined up to equivalence by $X(T)$ for $T \subset S$ with $|T| \leq 1$: the rest of the cube can be obtained by homotopy pushouts. If we assume our maps $X(\emptyset) \to X(\{i\})$ for $i \in S$ are cofibrations, then we can just take ordinary pushouts.

Next, we've already discussed directed hocolims but there is a fact about them that will be needed in future talks, so we state it now for the record.

Proposition 9 (HTT, 7.3.4.7)**.** *If $\mathcal{C} = \texttt{Top}$ or $\mathcal{C} = \texttt{Spectra}$ (or more generally $\mathcal{C}$ is any ∞-topos), then finite holims commute with directed hocolims. More precisely, if $\mathcal{I}$ is a finite indexing category and $D \in \mathcal{C}^{\mathcal{I}\times\mathbb{N}} \cong (\mathcal{C}^{\mathcal{I}})^{\mathbb{N}} \cong (\mathcal{C}^{\mathbb{N}})^{\mathcal{I}}$, then*

$$\operatorname{hocolim}_{\mathbb{N}} \left(\operatorname{holim}_{\mathcal{I}\times\{n\}} D\right) \simeq \operatorname{holim}_{\mathcal{I}} \left(\operatorname{hocolim}_{\{i\}\times\mathbb{N}} D\right).$$

In the case that $\mathcal{C} = \texttt{Spectra}$ and $\mathcal{I} = \mathscr{P}(\underline{2})$, this follows simply from the fact that a commutative square of spectra is cartesian iff it is cocartesian; this specializes to the fact that fiber sequences and cofiber sequences are the same thing.

Lastly, let us say a word about homotopy orbits. Suppose $X \in \mathcal{C}$ and G is a finite group (or at least a discrete group; for us, G will actually always be a symmetric group). A G-action on X is the same thing as a functor $a : G \to \mathcal{C}$ landing at X, where G is considered as a one-object category. Then the *homotopy orbits* object is defined by $X_{hG} = \operatorname{hocolim}_G a$.

Remark 9. We always have an object $EG \in \mathcal{C}$, which comes equipped with a G-action. Chasing through the definitions, one can verify that $X_{hG} = (X \otimes EG)_G$. This gives the usual definition of homotopy orbits in $\texttt{Top}$, and specializes to the fact that $*_{hG} = (EG)_G = BG$.

Example 15. In $\texttt{Top}$, we get the explicit model $B\mathbb{Z}/2 = *_{h\mathbb{Z}/2} = (S^\infty)_{\mathbb{Z}/2} = \mathbb{R}P^\infty$. (Incidentally, this is a simple counterexample to the reasonable-sounding claim that the homotopy colimit is equivalent to the ordinary colimit if all the maps are cofibrations, since $*_{\mathbb{Z}/2} = *$. The correct statement is wrapped up in our exposition of the derived functor perspective above.)

This concludes our foray into the world of ho(co)lims.

3.2 n-excisive functors

In what follows, $\mathcal{C}$ and $\mathcal{D}$ will be categories of the sort that we studied in the last section; the ones we'll actually care about in the end are $\texttt{Top}$, $\texttt{Top}_*$, and $\texttt{Spectra}$. We will only consider *homotopy* functors (i.e. functors that preserve equivalences).

Definition 19. A functor $F : \mathcal{C} \to \mathcal{D}$ is called *n-excisive* (or *polynomial of degree at most n*) if whenever X is a strongly cocartesian $(n+1)$-cube in $\mathcal{C}$, $F(X)$ is a cartesian cube in $\mathcal{D}$. (A useful mnemonic is that a polynomial of degree at most n is determined by its values on $n+1$ distinct points.)

Proposition 10. *If F is n-excisive then it is also $(n+k)$-excisive for all $k \geq 0$.*

Proof. Clearly it suffices to prove the statement for $k = 1$. Consider an $(n+2)$-cube X as a morphism $Y \to Z$ of $(n+1)$-cubes. If X is strongly cocartesian, then so are Y and Z. By assumption, this means that $F(Y)$ and $F(Z)$ are cartesian. But this implies that $F(X)$ is cartesian too, by an easy lemma [Calc II, 1.6]. □

At the lowest level, F is 0-excisive iff $F(X) \to F(*)$ is an equivalence for all $X \in \mathcal{C}$. In this case we say that F is *homotopy constant*. F is 1-excisive iff it takes homotopy pushout squares to homotopy pullback squares. Following existing terminology, then, we often simply say *excisive* for 1-excisive. The next proposition illustrates why the terminology makes sense; excision and Mayer-Vietoris are both "locality" axioms.

Proposition 11. *If E is any spectrum and $\mathcal{C} = \texttt{Top}$ or $\mathcal{C} = \texttt{Spectra}$, then the functors $F : \mathcal{C} \to \mathcal{D}$ given by $X \mapsto E \wedge X_+$ and $X \mapsto \Omega^\infty(E \wedge X_+)$ are excisive.*

Proof. Suppose that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

is a cocartesian square in $\mathcal{C}$. Then $W \simeq \mathrm{hocolim}(Y \xleftarrow{f} X \xrightarrow{g} Z)$. We can decompose this hocolim in the obvious way, so that $U = M_f \simeq Y$, $V = M_g \simeq Z$, and $U \cap V = X \times I \simeq X$. This decomposition yields a Mayer-Vietoris sequence for the homology theory E_* (where $E_*(U \cup V) \cong E_* W$ via the assumed equivalence). Note that for either choice of F, we can write $E_* A = \pi_* F(A)$ for any $A \in \mathcal{C}$ (where we mean stable homotopy of spectra or unstable homotopy of spaces). Meanwhile, by the following lemma, there is a long exact sequence in homotopy for a homotopy pullback square which has the exact same shape as a Mayer-Vietoris sequence. Then, applying F to the map $X \to \mathrm{holim} = \mathrm{holim}(F(Z) \to F(W) \leftarrow F(Z))$ induces the morphism of long exact sequences

$$\begin{tikzcd} \cdots \arrow[r] & \pi_{n+1}F(Y) \oplus \pi_{n+1}F(Z) \arrow[r] \arrow[d, equal] & \pi_{n+1}F(W) \arrow[r] \arrow[d, equal] & \pi_n F(X) \arrow[r] \arrow[d] & \pi_n F(Y) \oplus \pi_n F(Z) \arrow[r] \arrow[d, equal] & \pi_n F(W) \arrow[r] \arrow[d, equal] & \cdots \\ \cdots \arrow[r] & \pi_{n+1}F(Y) \oplus \pi_{n+1}F(Z) \arrow[r] & \pi_{n+1}F(W) \arrow[r] & \pi_n \,\mathrm{holim} \arrow[r] & \pi_n F(Y) \oplus \pi_n F(Z) \arrow[r] & \pi_n F(W) \arrow[r] & \cdots , \end{tikzcd}$$

and so the five lemma implies that the middle map is an isomorphism. Hence $F(X) \to \mathrm{holim}$ is an equivalence. □

Lemma 1. *If $A = \mathrm{holim}(C \to D \leftarrow B)$, then there is a long exact sequence*

$$\cdots \to \pi_n(A) \to \pi_n(B) \oplus \pi_n(C) \to \pi_n(D) \to \pi_{n-1}(A) \to \cdots .$$

Proof. Recall that we consider $A \subset B \times D^I \times C$. Then we have an (honest) pullback diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & D^I \arrow[d] \\ B \times C \arrow[r] & D \times D \end{tikzcd}$$

The right vertical map is a fibration with fiber ΩD, so the left vertical map is as well. The long exact sequence in homotopy for this fibration is the desired long exact sequence. □

Corollary 3. *We can take $E = S$, and then the functors $\Sigma^\infty_+ : \texttt{Top} \to \texttt{Spectra}$ and $\Omega^\infty \Sigma^\infty_+ : \texttt{Top} \to \texttt{Top}_*$ are excisive.*

In fact, we will see in another talk that if $F : \mathcal{C} \to \mathcal{D}$ is 1-excisive and *finitary* (i.e. commutes with filtered holims, a/k/a *continuous*), then there necessarily exist *coefficient spectra* C_0 and C_1 so that either $F(X) \simeq C_0 \vee (C_1 \wedge X)$ or $F(X) \simeq \Omega^\infty(C_0 \vee (C_1 \wedge X))$.

Exercise 3. If E is any spectrum, show that the Bousfield localization functor $L_E : \mathtt{Spectra} \to \mathtt{Spectra}$ is excisive. (Use that L_E preserves (co)fiber sequences, and that a square in spectra is cartesian iff it is cocartesian.) Incidentally, finitary Bousfield localizations are usually called *smashing*; in this case, $C_0 \simeq *$ and $C_1 \simeq L_E S$.

We now work towards a slightly less trivial result.

Lemma 2. *Suppose $\mathcal{A} = \bigcup_{s\in S} \mathcal{A}_s$ is a covering of a poset by subposets which are all either concave or convex. Then for any functor $F : \mathcal{A} \to \mathcal{D}$, the cube defined by*

$$\begin{aligned} T &\mapsto \operatorname{holim}\left(F|_{\bigcap_{s\in T}\mathcal{A}_s}\right) \\ \emptyset &\mapsto \operatorname{holim}(F) \end{aligned}$$

is cartesian.

This is proved by an easy induction.

Lemma 3. *If $F : \mathcal{C} \to \mathcal{D}$ is n-excisive, then for any strongly cocartesian m-cube $X : \mathscr{P}(S) \to \mathcal{C}$, the natural map*

$$F(X(\emptyset)) \to \operatorname{holim}_{\{T\in\mathscr{P}(S):|S-T|\leq n\}} F(X(T))$$

is an equivalence.

Proof. For $m \leq n$, this is true since the holim of the target is indexed over all of $\mathscr{P}(S)$, which has the initial object $\emptyset$. For $m = n+1$, this is true by definition of n-excisiveness. For $m > n+1$, we prove the statement by induction on m.

Define a cube $Y : \mathscr{P}(S) \to \mathcal{D}$ by setting

$$Y(T) = \operatorname{holim}_{\{U\in\mathscr{P}(S):U\supset T,|S-U|\leq n\}} F(X(U))$$

for each $T \subset S$. Then there is a morphism of cubes $F(X) \to Y$ given at $T \in \mathscr{P}(S)$ by

$$F(X(T)) \xrightarrow{\sim} \operatorname{holim}_{\{U\in\mathscr{P}(S):U\supset T\}} F(X(U)) \to Y(T),$$

where the second map is the restriction.

Our goal is to show that this is an equivalence at $T = \emptyset$. By the inductive hypothesis, it is an equivalence at all $T \in \mathscr{P}_0(S)$, so this will follow if both $F(X)$ and Y are cartesian. But $F(X)$ is cartesian since n-excisiveness implies m-excisiveness for all $m \geq n$, and it is routine to verify that Y is cartesian from the previous lemma by taking $\mathcal{A} = \mathscr{P}(S)$ and $\mathcal{A}_s = \{U \in \mathscr{P}(S) : s \in U, |S-U| \leq n\}$. ☐

Proposition 12. *Suppose $F : \mathcal{C}^r \to \mathcal{D}$ is n_i-excisive in the i^{th} variable. Then $F \circ \Delta : \mathcal{C} \to \mathcal{D}$ is n-excisive, where $n = \sum n_i$.*

Proof. Let $X : \mathscr{P}(S) \to \mathcal{C}$ be a strongly cocartesian cube in $\mathcal{C}$ with $|S| = r > n$. Consider the morphism of cubes given at $U \in \mathscr{P}(S)$ by

$$F\circ\Delta(X(U)) = F(X(U),\ldots,X(U)) \to \operatorname{holim}_{\{(T_1,\ldots,T_r)\in\mathscr{P}(S)^r:T_i\supset U,|S-T_i|\leq n_i\}} F(X(T_1),\ldots,X(T_r)).$$

We will first show that this is an equivalence, and then we will show that the target cube is cartesian.

At any fixed $U \in \mathscr{P}(S)$, we prove that the map is an equivalence by applying the latter lemma r times. To explain, we first define the cube $X' : \mathscr{P}(S-U) \to \mathcal{C}$ by $X'(T') = X(T'\cup U)$, so in particular $X'(\emptyset) = X(U)$. Then the target can be rewritten as

$$\operatorname{holim}_{\{T_1'\in\mathscr{P}(S-U):|(S-U)-T_1'|\leq n_1\}}\left(\cdots\left(\operatorname{holim}_{\{T_r'\in\mathscr{P}(S-U):|(S-U)-T_r'|\leq n_r\}} F(X'(T_1'),\ldots,X'(T_r'))\right)\cdots\right),$$

and the canonical map from $F(X(U),\ldots,X(U)) = F(X'(\emptyset),\ldots,X'(\emptyset))$ is an equivalence.

To complete the proof, we apply the former lemma to show that the target cube is cartesian, taking $\mathcal{A} = \{(T_i) \in \mathscr{P}(S)^r : |S - T_i| \leq n_i\}$ and $\mathcal{A}_s = \{(T_1,\ldots,T_r) \in \mathcal{A} : s \in T_i\}$. $\square$

Corollary 4. *For any spectrum E, the functors $X \mapsto E \wedge (X^n_+) = E \wedge (X_+)^{\wedge n}$ and $X \mapsto \Omega^\infty(E \wedge X^n_+)$ off* `Top` *and the functors $X \mapsto E \wedge X^{\wedge n}$ and $X \mapsto \Omega^\infty(E \wedge X^{\wedge n})$ off* $\texttt{Top}_*$ *are all n-excisive. (In particular, we can take $E = S$.)*

3.3 The generalized Blakers-Massey theorem

The idea of the Blakers-Massey theorem is as follows. Although homotopy doesn't satisfy excision, it does in the so-called *stable range*. (Indeed, stable homotopy is a homology theory!) Precisely, we have the following statement.

Theorem 2 (Blakers-Massey)**.** *Let $(X; A, B, x)$ be a triad such that $(A, A \cap B)$ is an n-connected relative CW complex (for $n \geq 1$) and $(B, A \cap B)$ is an m-connected relative CW complex. Then $\pi_r(A, A \cap B, x) \to \pi_r(X, B, x)$ is an isomorphism for $1 \leq r < m + n$ and an epimorphism for $r = m + n$.*

(Recall that the n^{th} *relative homotopy* of a pointed pair is defined as homotopy classes of maps from the pointed pair $(D^n, S^{n-1}, *)$.) In the context of Goodwillie calculus, we can view this as giving us a partial answer to the question of comparing k-cartesianness with k-cocartesianness: the Blakers-Massey theorem tells us that these notions coincide in the stable range.

Theorem 3 (Blakers-Massey, take 2)**.** *Let $X : \underline{2} \to$* `Top` *be a square of spaces. If X is cocartesian and $X(\emptyset) \to X(\{i\})$ is k_i-connected, then X is $(k_1 + k_2 - 1)$-cartesian. Dually, if X is cartesian and $X(\{i\}) \to X(\underline{2})$ is k_i-connected, then X is $(k_1 + k_2 + 1)$-cocartesian.*

The first assertion is the one which is equivalent to the previous statement; this is also called the *homotopy excision theorem*, since it's telling us how highly connected the map $X(\emptyset) \to \mathrm{holim}(X(\{2\}) \to X(\underline{2}) \leftarrow X(\{1\}))$ is. Recall that the homotopy of this holim sits in a sort of Mayer-Vietoris sequence, so the homotopy of the source does too up through a certain dimension.

The Blakers-Massey theorem implies the following result, which is really the entire reason for the existence of stable homotopy theory.

Corollary 5 (Freudenthal suspension theorem)**.** *For every n-connected CW complex X, the suspension homomorphism $\pi_r(X) \to \pi_{r+1}(\Sigma X)$ is an isomorphism for $1 \leq r \leq 2n$ and an epimorphism for $r = 2n + 1$.*

The Blakers-Massey theorem is also called the *triad connectivity theorem*, which suggests the appropriate generalization. An $(n+1)$-*ad* is a tuple $(X, \{X_s\}_{s \in S})$ of a space X and n subspaces X_s, where $|S| = n$. This determines an n-cube $\mathcal{X} : \mathscr{P}(S) \to$ `Top` by setting $\mathcal{X}(S) = X$ and $\mathcal{X}(S - T) = \bigcap_{s \in T} X_s$ for $T \subsetneq S$. In fact, to give an $(n+1)$-ad is precisely the same thing as to give an n-cube such that:

- all the maps $\mathcal{X}(T) \to \mathcal{X}(S)$ are inclusions of subspaces, and
- $\mathcal{X}(T \cap U) = \mathcal{X}(T) \cap \mathcal{X}(U)$.

Goodwillie provides the following generalization of the Blakers-Massey theorem.

Theorem 4 (generalized Blakers-Massey)**.** *Let $X : \mathscr{P}(S) \to$* `Top` *be an n-cube of spaces, with $n \geq 1$. If X is strongly cocartesian and $X(\emptyset) \to X(\{s\})$ is k_s-connected for each $s \in S$, then X is k-cartesian with $k = 1 - n + \sum k_s$. Dually, if X is strongly cartesian and $X(S - \{s\}) \to X(S)$ is k_s-connected for each $s \in S$, then X is k-cocartesian with $k = n - 1 + \sum k_s$.*

Remark 10. Of course, the formula for the holim of a punctured n-cube reduced to a particularly nice form when $n = 2$, but it becomes already fairly intractable when $n = 3$. Nevertheless, this gives us a stable range in which our n-cube behaves as if it were both cartesian and cocartesian. Let's see what this buys us.

Recall that, dual to the skeletal filtration of a realization, the totalization of a cosimplicial space $K : \mathbf{\Delta} \to \mathtt{Top}$ admits a map to a tower of fibrations, called the *coskeletal cofiltration*, the q^{th} space of which is

$$\operatorname{cosk}^q(K) = \operatorname{eq}\left(\prod_{\{[n]:n\le q\}} (K_n)^{(\Delta^n)} \rightrightarrows \prod_{\{[s]\to[t]:s,t\le q\}} (K_t)^{(\Delta^s)}\right).$$

If we write $F_q = \operatorname{fib}(\operatorname{cosk}^q(K) \to \operatorname{cosk}^{q-1}(K))$, then taking homotopy yields a first-quadrant spectral sequence of the form $E_1^{p,q} = \pi_p(F_q) \Rightarrow \pi_p(\operatorname{Tot}(K))$ with differentials $d_r^{p,q} : E_r^{p,q} \to E_r^{p-1,q+r}$.

Now, if $K = \operatorname{crep}(D)$ for some diagram $D : \mathcal{I} \to \mathtt{Top}$, then of course $\operatorname{Tot}(K) = \operatorname{holim}_{\mathcal{I}} D$. So in our stable range, $\pi_* \operatorname{holim}_{\mathscr{P}_0(S)} X \cong \pi_*(X(\emptyset))$, so the vertical stripe roughly given by $0 \le p \le k$ (where our cube is k-cartesian) should be thought of as analogous to the partial Mayer-Vietoris sequence for homotopy when $n = 2$. (Note that the homotopy of the spaces in the diagram show up in the F_q. In fact,

$$F_q = \Omega^q\left(\prod_{i_0\to\cdots\to i_q} D(i_q)\right),$$

but unfortunately the signature of our spectral sequence precludes this fact from implying strong convergence.)

4 The Taylor tower (Eric Peterson)

Just as one can define derivatives and approximating polynomials for smooth functions on spaces with smooth structure, there is a wholly analogous construction for certain functors between model categories with certain extra properties. We define these objects, investigate some simple examples, and consider an associated spectral sequence.

4.1 Secant and tangent curves

Before we get started on talking about finding polynomial approximations to functors, let's spend a few minutes revisiting the story for smooth functions on the real line. Differential calculus begins with the following construction: select a function f, a special point $x_0 \in \mathbb{R}$, and some other point $x_1 \in \mathbb{R}$. The *secant line* corresponding to this data is the unique line interpolating the pairs $(x_0, f(x_0))$ and $(x_1, f(x_1))$, for which we can write down the equation

$$T_1 f = f(x_1) \cdot \frac{x - x_0}{x_1 - x_0} + f(x_0) \cdot \frac{x - x_1}{x_0 - x_1}.$$

Then, when we bring *limits* into the picture. Letting x_0 and x_1 tend toward 0, we find $(P_1 f)(x)$, the linear (i.e., first order) approximation to f at 0.

Of course, it is possible to build interpolating polynomials through as many points as we'd like: for any set of $(n + 1)$ points in the plane that share no x-coordinates among them, there is a unique interpolating polynomial of degree n that passes through each of them. The formula given above for the interpolating line generalizes to Lagrange's formula:

$$y = \sum_{i=0}^{n} f(x_i) \prod_{j\ne i} \frac{x - x_j}{x_i - x_j}.$$

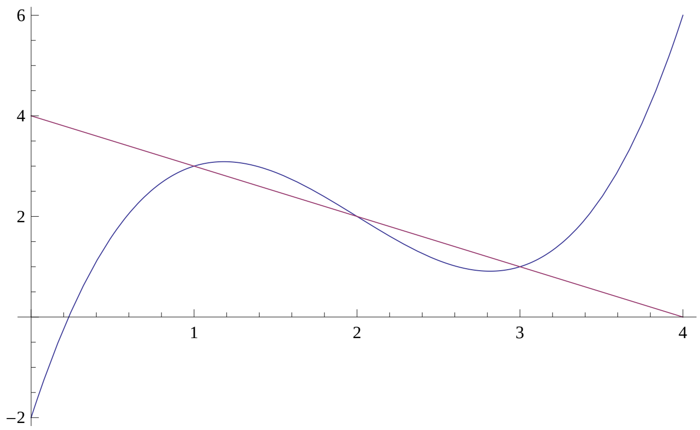

Figure 1: The secant line through 2 and 3 on a cubic.

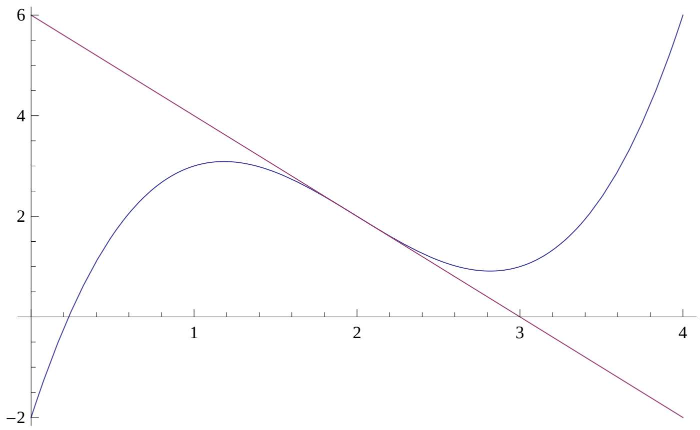

Figure 2: The tangent line through 2 on a cubic.

As an example, let's set $n = 2$, so that we build "secant parabolas," and then pick $f(x) = e^x$ along with the points with x-values h, 0, and $-h$ to test. Just as before, we can let these three points cluster toward 0 to attempt to build a tangent parabola — in the example, this means taking the limit $h \to 0$. If we expand out the Lagrange formula above, we get

$$\begin{aligned}(P_2 \exp)(x) &= \lim_{h\to 0} (T_2 \exp)(x)) \\ &= \lim_{h\to 0}\left(\frac{e^h - 2 + e^{-h}}{2h^2}\cdot x^2 + \frac{e^h - e^{-h}}{2h} + 1\right) \\ &= \left(\lim_{h\to 0}\frac{e^h - 2 + e^{-h}}{2h^2}\right)\cdot x^2 + \left(\lim_{h\to 0}\frac{e^h - e^{-h}}{2h}\right)\cdot x + \left(\lim_{h\to 0} 1\right)\cdot 1.\end{aligned}$$

Each of these limits can individually be calculated to be $\frac{1}{2}$, 1, and 1, giving $(P_2^0 \exp)(x) = \frac{1}{2}x^2 + x + 1$.

Now it's time to get excited, since you recognize this polynomial from elsewhere. Calculus students studying Taylor series learn the formula

$$(P_n f)(x) = \sum_{i=0}^{n} \frac{f^{(i)}(0)x^i}{i!},$$

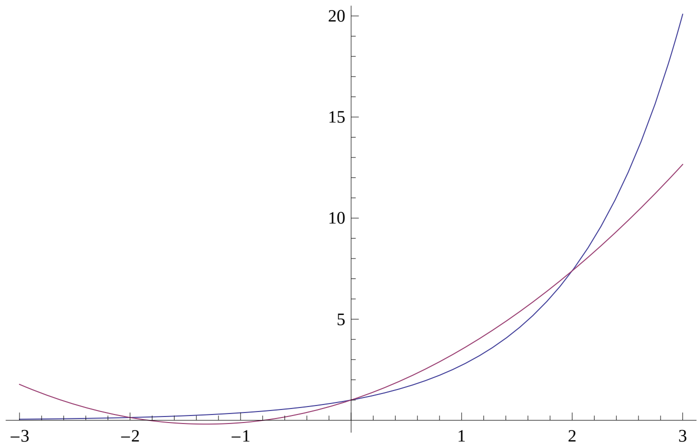

Figure 3: The secant parabola through $h = 2$, 0, and -2 on the exponential.

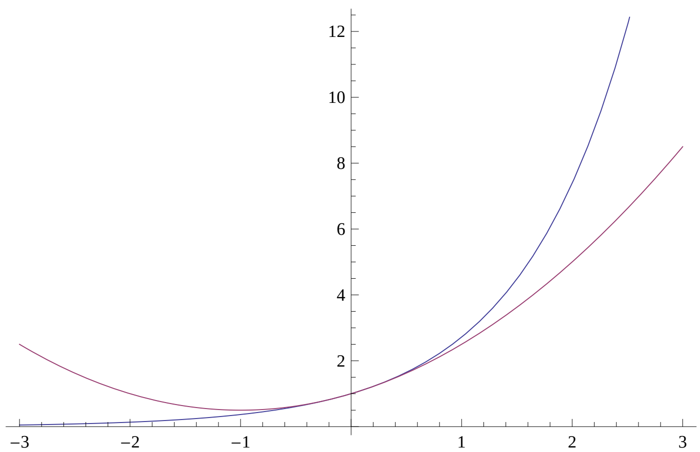

Figure 4: The tangent parabola at 0 on the exponential.

and using this one finds that the beginning of the expansion for the exponential function looks like $e^x = 1 + x + \frac{1}{2}x^2 + \cdots$, exactly matching what we found above. It turns out that this is not an accident — for a smooth function f, these two definitions of $P_n f$ coincide.[1]

These two approaches have their merits and dismerits. What's nice about the summation definition is that it turns out to be very computable; we have an extremely successful theory for computing the global derivatives of common smooth functions. What's nice about the geometric definintion is that it requires very little added machinery — specifically, we made decisions about what "polynomial interpolation" and "limit" should mean, then approximations of all orders immediately followed. This means that it is *portable* in a sense very important to us. We previously heard about linear functors, which means in this talk we should be all set to talk about polynomial approximations of higher order.[2]

[1] I learned this analogy-crucial fact from Randy McCarthy. As an unimportant side remark, he memorably described the limit $x_0, \ldots, x_n \to 0$ as "crashing toward the basepoint."

[2] To explain the moral value in the table, Taylor polynomials are often justified to students by drawing their graphs and noticing that they look quite similar to the graphs of the original functions near their centers. This is actually built in to the geometric construction, rather than a "Hmm, that's curious." side-remark.

Summation formula	Geometric definition
Some clear properties: $P_n P_{n+k} = P_n$	Moral value
Requires (iterated) derivatives	Awkward to compute
Computable	*Portable*

4.2 Interpolation for functors

Now we're going to rephrase this set-up to give a differential calculus of functors, so our goal are:

1. Decide what the words "interpolating polynomial" and "limit" mean.
2. Figure out how to evaluate $P_n F$ away from the basepoint, at an arbitrary X.

From here on, we'll fix a source ∞-category C and a target ∞-category D, and consider functors $F : \mathsf{C} \to \mathsf{D}$. We'll want...

- ...finite colimits to exist in C.
- ...for C to have a final object. This will be our notion of "basepoint."
- ...finite limits and directed colimits to exist in D.
- ...for these finite limits and directed colimits in D to commute.

That I'm picking an ∞-categorical setting is mostly a matter of abbreviation; the only real homotopy theory we'll use are facts about homotopy co/limits, which are entirely equivalent to ∞-categorical co/limits, and this way I am free to forget to say "homotopy" before "colimit" without causing catastrophe.

Recall that a linear functor (or a functor that is polynomial of degree 1, or a 1-excisive functor) is one that carries homotopy pushout squares to homotopy pullback squares. There is an obvious mode of generalization here: a homotopy pushout square models the decomposition of the pushout into 2 spaces, with their possibly nontrivial intersection marked at the opposite corner. Let's replace the pushout square in this set-up with a sort of pushout hypercube, trading a 2-parameter condition defining a degree 1-polynomial for an $(n+1)$-parameter condition defining a degree n polynomial.

To this end, let's briefly recount the definitions of Cartesian, co-Cartesian, and strongly co-Cartesian:

- A cubical diagram $\mathcal{X}$ of dimension n is a diagram indexed by the lattice of subsets $T \subseteq S$ of a finite set S of cardinality n, i.e., it is indexed by the partially ordered powerset $\mathcal{P}S$.
- Let $\mathcal{P}_0 S$ denote the full subcategory of $\mathcal{P}S$ of subsets of positive cardinality. An n-cube $\mathcal{X}$ is said to be Cartesian if the limit of $\mathcal{X}$ restricted to $\mathcal{P}_0 S$ agrees with $\mathcal{X}(\emptyset)$.
- An n-cube $\mathcal{X}$ is said to be co-Cartesian if it satisfies the dual condition. Let $\mathcal{P}_1 S$ denote the full subcategory of $\mathcal{P}S$ of proper subsets of S. Then, $\mathcal{X}$ is co-Cartesian if the colimit of the restriction of $\mathcal{X}$ to $\mathcal{P}_1 S$ agrees with $\mathcal{X}(S)$.
- Finally, an n-cube $\mathcal{X}$ indexed by subsets of a set S is said to be strongly co-Cartesian if for every choice of $T \subseteq S$ with $|T| > 1$, the restriction of $\mathcal{X}$ to subsets of T gives a co-Cartesian cube. Equivalently, $\mathcal{X}$ is strongly co-Cartesian when it is the Kan extension of its restriction to the verticesof cardinality 1.

We now have the language for our major definition: F is polynomial of degree n (or n-excisive) if it takes strongly co-Cartesian $(n+1)$-cubes to Cartesian $(n+1)$-cubes. This is meant to be in direct analogy with the classical situation, where Lagrange's formula tells you that if you know the value of a polynomial of degree n at $(n+1)$ sample points, then you can reconstruct the whole thing and sample it at any other point

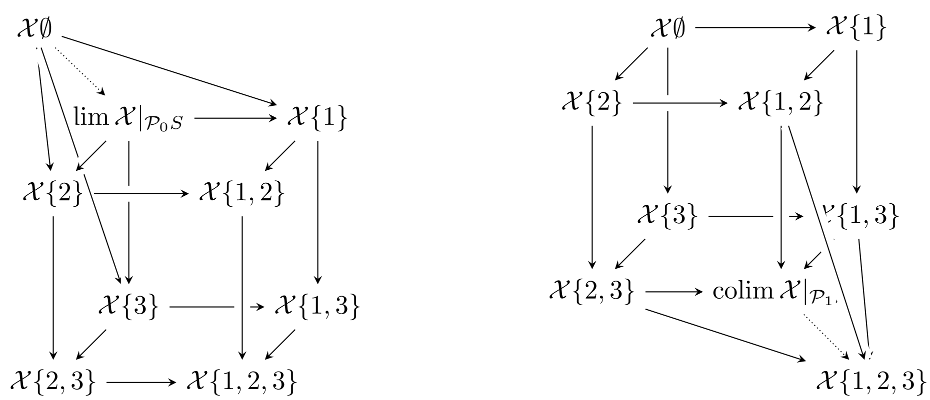

Figure 5: The Cartesian and co-Cartesian conditions for cubes indexed by $S = \{1, 2, 3\}$.

you choose. If we build a strongly co-Cartesian $(n+1)$-cube around a certain space X, then F is degree n if the value of F on the rest of the cube is enough to recover FX through this fixed method of interpolation.

Two remarks are in order. First, choosing strongly co-Cartesian over merely co-Cartesian is important, because we want an analogue of the statement that polynomials of order n are also of order m for $m \geq n$. The proof of this came up in the previous talk. Second, let's check an edge case. Intuitively, if F is polynomial of degree 0, then it ought to be (locally) constant, based on our experience with real functions. By our definitions, such a functor F takes strongly co-Cartesian 1-cubes to Cartesian 1-cubes. A 1-cube is merely an arrow, and the condition for strong co-Cartesianness is vacuous, so all arrows count as strongly co-Cartesian 1-cubes. Then, the image of any arrow under F must be Cartesian, meaning that the source of the arrow must be weakly equivalent to the limit of the diagram picking out the target of the arrow — but the limit of a one object, one arrow diagram is the object itself, and hence F must take all arrows to weak equivalences.

Getting back to it, there is one such strongly co-Cartesian cube with particularly good properties: fixing a space X, let $\mathcal{X}$ be the cube indexed by subsets $T \subseteq S$, $|S| = n+1$, whose vertex at T is the join of X and T — i.e., the $|T|$-pointed cone on X. Again, if F were polynomial of degree n, then $F(X)$ would be equivalent to the limit of the punctured cube $F \circ \mathcal{X}|_{\mathcal{P}_0 S}$, but at the very least we can record what F "ought to be" by setting

$$(T_n F)(X) = \lim F \circ \mathcal{X}|_{\mathcal{P}_0 S}.$$

This comes with a natural map $(t_n F) : F \to T_n F$ by universality of the limit.

Let's pause for a moment to further our analogy, though we'll have to restrict our setup a bit so that C has a sensible notion of homotopy groups, as for $\mathsf{C} = \mathsf{Spaces}$ or $\mathsf{C} = \mathsf{Spectra}$. Recall that $1 \in \mathsf{C}$ is the "center" of our construction, and that each object X comes with a map $X \to 1$. The connectivity k of this map measures the similarity of X to 1, and we think of the reciprocal $\frac{1}{k}$ as measuring their "nearness". If $X \to 1$ is k-connected, then notice that the objects $\mathcal{X}|_{\mathcal{P}_0 S}(R) \to 1$ are all at least $(k+1)$-connected. If our construction is supposed to be working toward "Taylor expanding around the basepoint" and we take "the basepoint" here to mean the final object, then studying $F\mathcal{X}|_{\mathcal{P}_0 S}$ means approximating the value of F at X by interpolating by values closer to the basepoint than X itself.

Of course, in the classical setup with secants, it wasn't sufficient to merely pick interpolation points nearer than the point at which you wanted to sample, there was an extra limiting step where we let the interpolation points cluster at the base. The same is true here: $T_n F$ does not have to be n-excisive, but it is "better," as its action on k-connected objects is determined by F's action on $(k+1)$-connected objects. Our analogue of clustering at the base is to iterate this construction: by applying T_n successively, we build a sequence

$$F \xrightarrow{t_n F} T_n F \xrightarrow{t_n(T_n F)} T_n T_n F \to \cdots \to P_n F,$$

yielding a functor in the colimit whose action is determined by the value of F on "very connected objects." This functor P_nF also comes with a natural map $p_nF : F \to P_nF$, and it will be our analogue of the Taylor polynomial of degree n.

4.3 Properties of T_n and P_n

Since T_n and P_n are defined in terms of each other and of co/limits, some basic facts about co/limits produce a variety of interaction properties of these functors.

- Because T_n is exactly defined to use our interpolation scheme to guess what $F(X)$ would be if F were n-excisive, when F actually *is* n-excisive it guesses correctly. So, for n-excisive F, $t_nF : F \to T_nF$ is a weak equivalence. In turn, when F is n-excisive, $p_nF : F \to P_nF$ is also a weak equivalence.

- We've assumed that finite limits and sequential colimits in our target category commute. Our functors T_n and P_m are exactly defined in terms of finite limits and sequential colimits, so we have the commutation law $T_nP_m = P_mT_n$. In particular, this means that $T_nP_nF = P_nT_nF = P_nF$, and so at the very least P_nF behaves as though it were n-excisive when we check the particular interpolation scheme used to build T_n. It also means that P_n and T_n preserve fiber sequences, which are themselves defined by a limit condition.

- In fact, P_nF is *actually* n-excisive! There is a technical lemma used to show this, which for now we will state rather than prove: for any strongly co-Cartesian $(n+1)$-cube $\mathcal{X}$, the map of cubes $(t_nF\mathcal{X}) : F(\mathcal{X}) \to T_nF\mathcal{X}$ factors as $F(\mathcal{X}) \to \mathcal{Y} \to T_nF\mathcal{X}$, where $\mathcal{Y}$ is a Cartesian cube. The construction of $\mathcal{Y}$ is not obvious, and its existence is why we picked the cube of cones rather than some other cube. I'll go through the proof at the end of the talk if there's time; if not, Rezk provides a slick proof of this fact. In any event, once we have $\mathcal{Y}$, then for any strongly co-Cartesian cube $\mathcal{X}$, $P_n\mathcal{X}$ is defined as the directed colimit of

$$\begin{tikzcd}
F\mathcal{X} \arrow[rr] \arrow[dr] & & T_nF\mathcal{X} \arrow[rr] \arrow[dr] & & T_n^2F\mathcal{X} \arrow[rr] \arrow[dr] & & \cdots \\
& \mathcal{Y}_0 \arrow[rr] \arrow[ur] & & \mathcal{Y}_1 \arrow[rr] \arrow[ur] & & \mathcal{Y}_2 \arrow[r] \arrow[ur] & \cdots .
\end{tikzcd}$$

  The colimit of the bottom row is the sequential colimit of Cartesian cubes, which is a condition about finite limits, so the result is itself Cartesian. Hence, P_nF converts strongly co-Cartesian $(n+1)$-cubes to Cartesian ones, so is n-excisive. Using this, we also get a map $P_{n+k}F \to P_nF$ by applying P_{n+k} to $p_nF : F \to P_nF$, then since P_n is n-excisive and hence $(n+k)$-excisive, $P_{n+k}P_nF \simeq P_nF$.

- Let Fun denote the ∞-category of all functors $\mathsf{C} \to \mathsf{D}$, and let Exc^n denote the full subcategory of such functors which are n-excisive. The functor P_n is left-adjoint to the inclusion $\mathsf{Exc}^n \to \mathsf{Fun}$. To show this we have to demonstrate a natural isomorphism $\mathsf{Fun}(F, G) \cong \mathsf{Exc}^n(P_nF, G)$ for an arbitrary functor F and n-excisive functor G. The map $\mathsf{Exc}^n(P_nF, G) \to \mathsf{Fun}(F, G)$ is not so interesting: it is given by precomposition with $F \to P_nF$. The other half is more interesting: a map $F \to G$ induces a square

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & G \arrow[d] \\
P_nF \arrow[r] \arrow[ur, dotted] & P_nG. \arrow[u, dotted, bend right]
\end{tikzcd}$$

The right-hand map is an equivalence since G is already n-excisive, and so we get a composite map $P_nF \to G$ by following the bottom edge and then the homotopy inverse to the right edge. One can check that these two maps are inverses, so give the desired adjunction.[3]

- Finally we have $P_n P_{n+k} \simeq P_n$, since the composite $P_n P_{n+k}$ also satisfies the same left adjoint property to the inclusion $\mathsf{Exc}^n \to \mathsf{Fun}$.

4.4 Simple examples

By assuming the existence of a zero object, the functors Σ and Ω are defined in great generality by pushing out against the two maps to the zero object and pulling back along the two maps from the zero object respectively. This construction coincides with the usual one in the categories of pointed spaces and of spectra. Expanding out the definitions of $\mathcal{X}$ and T_1F, we see that when $F(1) = 1$ we have the formula $T_1F(X) = \Omega F(\Sigma X)$. This allows us to compute two examples right off: taking F to be the identity functor on pointed spaces, we compute

$$P_1 \mathrm{id}_{\mathsf{Spaces}} = \mathrm{colim}_k\, T_1^k \mathrm{id}_{\mathsf{Spaces}} = \mathrm{colim}_k\, \Omega^k \mathrm{id}_{\mathsf{Spaces}} \Sigma^k = \Omega^\infty \Sigma^\infty,$$

sometimes called Q and of immense classical interest. Performing this same computation for spectra yields $P_1 \mathrm{id}_{\mathsf{Spectra}} = \mathrm{colim}_k\, \Omega^k \Sigma^k = \mathrm{colim}_k\, \mathrm{id}_{\mathsf{Spectra}} = \mathrm{id}_{\mathsf{Spectra}}$, meaning that the identity functor on spectra is 1-excisive. In turn, this means that it is k-excisive for all k, so that $P_k \mathrm{id}_{\mathsf{Spectra}} \simeq \mathrm{id}_{\mathsf{Spectra}}$. [4]

More complicated examples of similar flavor abound. For instance, Kuhn [**?**] claims that P_1 of the identity on augmented, commutative $\mathbb{S}$-algebras is $R \mapsto R \vee TAQ(R)$. This is follows from work of Basterra and Mandell [**?**], which draws on Basterra-McCarthy [**?**] and Schwede [**?**]. Generally, one can try to describe what "stabilization" means for some broad class of categories; Schwede [**?**] does this in different language in for simplicial algebraic theories.

The skeptical reader might complain that all these examples are first derivatives, and we ought to be talking about something of higher order to see some genuinely new examples. Unfortunately — but not surprisingly — it turns out to be difficult to compute any further examples with just the technology stated so far. Going back to the analogy with Taylor expansions of functions, we saw two definitions of P_nf: one that looked simple to restate in the language of homotopy functors and one that looked computationally useful — and that they were equivalent was a nontrivial fact. Something similar is going to happen now for us; we have successfully constructed a Taylor tower for any suitable functor F and object X:

$$F(X) \to \cdots \to P_3F(X) \to P_2F(X) \to P_1F(X) \to P_0F(X).$$

Thinking of these things as polynomial approximations of increasing top degree, the "difference" between the nth and $(n-1)$th levels should be exactly one term in the Taylor summation formula. So, let's define D_nF by $D_nF = \mathrm{fib}(P_nF \to P_{n-1}F)$; we quickly see that the functor D_nF is said to be n-homogeneous, meaning that it is n-excisive and is $(n-1)$-reduced, i.e., $P_{n-1}D_nF \simeq 1$. In future talks, we will be principally interested in studying these D_nF; for instance, we'll show that when F is a self-map of the category of spectra, we get a formula $(D_nF)(X) = (C_n \wedge X^{\wedge n})_{h\Sigma_n}$, which is eerily similar to the summand $\frac{f^{(n)}(a)(x-a)^n}{n!}$ in the Taylor formula.[5] Most importantly, we will find out that these D_n are much more readily computable than their cousins P_n.

For a brief moment, let's posit this description of the functors D_n and compute something. Recall that

[3]Here, working in an ∞-categorical setting is to our honest advantage. The inverse $P_nG \to G$ cannot be reliably chosen so that everything strictly commutes, but instead P_n is a left adjoint in the sense of ∞-categories to i_n.

[4]This is an if-and-only-if: the map $\mathrm{id} \to T_1\mathrm{id}$ is an equivalence exactly when the underlying category in question is a stable ∞-category.

[5]This gives new plausibility to one of the above examples: the identity on spectra $X \mapsto (\mathbb{S} \wedge X^{\wedge 1})_{h\Sigma_1}$ looks exactly like a degree 1 polynomial.

because P_n is a left adjoint it commutes with colimits.[6] The seasoned homotopy theorist will recall the Snaith splitting

$$\Sigma^\infty\Omega^\infty\Sigma^\infty X \simeq \bigvee_{j=1}^{\infty}(\Sigma^\infty X^{\wedge j})_{h\Sigma_j}$$

for 0-connected spaces X. This looks awfully similar to what we've been discussing, and we can use formal properties of P_n, along with the fact that $X \mapsto X^{\wedge n}_{h\Sigma_n}$ is n-homogeneous, to compute P_n of this functor:

$$(P_n\Sigma^\infty\Omega^\infty\Sigma^\infty)(X) \simeq P_n\left(\bigvee_{j=1}^{\infty}(\Sigma^\infty(-)^{\wedge j})_{h\Sigma_j}\right)(X) \simeq \left(\bigvee_{j=1}^{\infty}P_n(\Sigma^\infty(-)^{\wedge j})_{h\Sigma_j}\right)(X) \simeq \bigvee_{j=1}^{n}(\Sigma^\infty X^{\wedge j})_{h\Sigma_j}.$$

In turn, $D_n\Sigma^\infty\Omega^\infty\Sigma^\infty$ is the difference between P_n and P_{n-1}:

$$(D_n\Sigma^\infty\Omega^\infty\Sigma^\infty)(X) \simeq (\mathbb{S}\wedge\Sigma^\infty X^{\wedge n})_{h\Sigma_n}.$$

As something to look forward to, once we study the functors D_n more carefully we'll be able to approach this problem from the other direction, concluding with the Snaith splitting.[7] [8]

4.5 Convergence

But we don't actually know these facts about D_n yet, so we'll have to occupy our time with something else. Luckily, this is easy: suppose again that D has a notion of homotopy groups. As algebraic topologists, now that we've drawn a tower of fibrations we should feel an overwhelming compulsion to investigate the associated Bousfield-Kan spectral sequence, with signature

$$E^1_{p,q} = \pi_p D_q F(X) \overset{cond}{\Rightarrow} \pi_p P_\infty F(X), \quad \text{with} \quad d^r_{p,q} : E^r_{p,q} \to E^r_{p-1,q+r}.$$

The first step in analyzing this spectral sequence is to compare the limit $P_\infty F(X) = \lim_k P_k F(X)$ with $F(X)$ itself; in the case that the natural map $F(X) \to P_\infty F(X)$ is an equivalence, the Taylor tower for F is said to converge at X. If $P_\infty F$ converges to F for all inputs, F is said to be entire.

In the case that the tower converges to F at X, we at least get conditional convergence. One way to ensure strong convergence is to force a vanishing line of positive slope into the spectral sequence. To this end, we make two definitions about the behavior of F with respect to connectivity:

- F satisfies property $E_n(c,\kappa)$ when for any strongly co-Cartesian $(n+1)$-cube $\mathcal{X}$ with all 1-vertices $s \in S$ having the map $\mathcal{X}\emptyset \to \mathcal{X}\{s\}$ at least k_s-connected for $k_s \geq \kappa$, then the map $F\mathcal{X}(\emptyset) \to \lim F|_{\mathcal{P}_0 S}$ is $(-c+\sum_s k_s)$-connected.
- Finally, F is said to be ρ-analytic if there exists a d so that F is $E_n(n\rho - d, \rho + d)$ for all $n \geq 1$.

If F is ρ-analytic and X is k-connected for some $k > \rho$, then the map $F(X) \to P_q F(X)$ is at least $(d+k+q(k-\rho))$-connected and hence $D_q F(X)$ is $(d+k+(q-1)(k-\rho))$-connected. Thus, the groups $E^1_{p,q}$ vanish when they satisfy the inequality

$$\begin{aligned} p &\leq d+k+(q-1)(k-\rho) \\ q &\geq \frac{p-d-k}{k-\rho}+1 = p\cdot\left(\frac{1}{k-\rho}\right) - \frac{d+\rho}{k-\rho}. \end{aligned}$$

[6] D_n is defined as the fiber of two functors that commute with sequential colimits and finite limits, so it does too by the standing assumption on D. If D is additionally stable, then fiber and cofiber sequences agree, and we can produce $D_n F$ as the cofiber of $\Omega P_{n+1}F \to \Omega P_n F$. Remarking that P_n commutes with general colimits and Ω does as well, as it's an autoequivalence of D, we see that D_n commutes with general colimits too.

[7] Thinking of $\mathbb{S}$ as the unit for the monoidal structure just like 1 is the unit for multiplication, this gives an amusing comparison between this functor and the exponential function.

[8] Amusingly, $X^{\wedge n}_{h\Sigma_n}$ is sometimes written $D_n X$, called the "nth extended power." This coincidence of notation is almost certainly accidental.

This gives a vanishing line with positive slope, and hence a strongly convergent spectral sequence.[9] If D is $\mathsf{Spectra}$ and E is a connective spectrum, then there is a similar spectral sequence for $E_*P_\infty F(X)$ given by smashing through the tower with E, since $E \wedge X$ is at least as connected as X. For an Eilenberg-Mac Lane spectrum HR, we also get a spectral sequence for $HR^*P_\infty F(X)$, where HR is an ordinary cohomology theory.

4.6 Existence of $\mathcal{Y}$

Let's quickly regurgitate Rezk's proof of the existence of the factorization of a Cartesian cube $\mathcal{Y}$, which works by constructing an n-cube of n-cubes. Let $\mathcal{X}$ be a strongly co-Cartesian n-cube in C, indexed by $\mathcal{P}S$ for a set S with $|S| = n$, and let $F : \mathsf{C} \to \mathsf{D}$ be as before. For any $T \subseteq S$, define $\mathcal{X}_T$ by

$$\mathcal{X}_T(R) = \operatorname{colim}\left(\mathcal{X}(R) \xleftarrow{\text{fold}} \coprod_{t \in T} \mathcal{X}(R) \xrightarrow{\text{cube map}} \coprod_{t \in T} \mathcal{X}(R \cup \{t\})\right).$$

Picking $T = \emptyset$ causes the colimit to collapse, giving $\mathcal{X}_\emptyset = \mathcal{X}$. There is also a natural map $\alpha_T : \mathcal{X}_T \to \mathcal{X} * T$, using the definition

$$X * T = \operatorname{colim}\left(X \xleftarrow{\text{fold}} \coprod_{t \in T} X \to \coprod_{t \in T} 1\right).$$

Putting these two facts together, we factor the map of cubes $(t_{n-1}F)(\mathcal{X}) : F\mathcal{X} \to (T_{n-1}F)\mathcal{X}$ as

$$F(\mathcal{X}(R)) = F(\mathcal{X}_\emptyset(R)) \xrightarrow{\text{univ. property}} \lim_{T \in \mathcal{P}_0 S} F(\mathcal{X}_T(R)) \xrightarrow{\alpha} \lim_{T \in \mathcal{P}_0 S} F(\mathcal{X}(R) * T) \simeq (T_{n-1}F)(\mathcal{X}(R)).$$

When $\mathcal{X}$ is strongly co-Cartesian, the diagram in the colimit defining $\mathcal{X}_T(R)$ picks out a corner of the cube $\mathcal{X}$, and so we get a natural weak equivalence $\mathcal{X}_T(R) \simeq \mathcal{X}(R \cup T)$. The maps $\mathcal{X}(R \cup T) \to \mathcal{X}(R \cup \{t\} \cup T)$ are isomorphisms when $t \in T$, and thus if T is nonempty then a punctured face not containing t of the punctured cube $F \circ \mathcal{X}_T|_{\mathcal{P}_0 S}$ is a duplicate of the face across t, and hence $F \circ \mathcal{X}_T$ is Cartesian. Therefore $\lim_{T \in \mathcal{P}_0 S} F(\mathcal{X}_T(R))$ is a homotopy limit of Cartesian cubes, and so is Cartesian itself, applying a lemma from the previous talk. We take $R \mapsto \lim_{T \in \mathcal{P}_0 S} F(\mathcal{X}_T(R))$ to be our $\mathcal{Y}$.

5 Derivatives are infinite loop spaces (Lennart Meier)

5.1 Setting

Recall that there are two styles, due to Lurie and Goodwillie. In both, we study a functor $F : \mathcal{C} \to \mathcal{D}$, which we assume to be a *homotopy functor*. The first takes place with the assumptions that:

- $\mathcal{C}$ is an ∞-category with finite colimits and a final object;
- $\mathcal{D}$ is a *differentiable ∞-category.*

The latter simply has $\mathcal{C}, \mathcal{D} \in \{\mathtt{Top}, \mathtt{Top}_*, \mathtt{Top}|_Y, \mathtt{Sp}\}$. We will generally use the former, if only to avoid saying "let $\mathcal{C}$ be spaces, or spectra, or...".

We use $\mathcal{P}(\underline{n+1})$ to denote the powerset of $\{1, \ldots, n+1\}$, and $\mathcal{P}_0(\underline{n+1})$ for $\mathcal{P}(\underline{n+1}) - \{\emptyset\}$.

Recall that F is called *n-excisive* if whenever $\mathcal{P}(\underline{n+1}) \to \mathcal{C}$ is strongly cocartesian, then $\mathcal{P}(\underline{n+1}) \to \mathcal{C} \to \mathcal{D}$ is cartesian.

[9] In particular, this means that the Taylor tower converges to F at X. These properties additionally tell us how fast the convergence is.

One of the main concepts in Goodwillie calculus is the *Taylor tower* of functors and natural transformations

$$F \longrightarrow P_\infty F = \lim(\cdots \to P_n F \to P_{n-1}F \to \cdots \to P_1 F \to P_0 F \simeq F(*))$$

. We will briefly outline where these functors $P_n F$ come from. Now, define a functor $\mathcal{C} \times N(\mathtt{Fin}^{inj}) \to \mathcal{C}$ (where $\mathtt{Fin}^{inf}$ is the category of finite sets and injective maps) by $(X, S) \mapsto C_S(X)$, the S-pointed cone on X, a/k/a the *join* $X * S$. Then, we define a functor $T_n F$ by

$$T_n F(X) = \lim_{\mathcal{P}_0(\underline{n+1})} F(C_\bullet(X));$$

this comes with a natural transformation $F \to T_n F$. We finally define

$$P_n F = \operatorname{colim}(F(X) \to T_n F(X) \to T_n^2 F(X) \to \cdots).$$

The functor F is called *n-reduced* if $P_{n-1}F \simeq *$, and *n-homogeneous* if it is n-excisive and n-reduced. (So 1-reduced means that $F(*) \simeq *$; we often just call this *reduced*.) From the Taylor tower, we can obtain an n-homogeneous functor $D_n F = \operatorname{fib}(P_n F \to P_{n-1}F)$. To check that this is indeed n-homogenous, we check that $P_n D_n F \simeq \operatorname{fib}(P_n P_n F \to P_n P_{n-1} F) \simeq \operatorname{fib}(P_n F \to P_{n-1}F)$ (since P_n is defined as a sequential colimit, and we're assuming these commute with finite limits) so $D_n F$ is n-excisive, and similarly $P_{n-1} D_n F \simeq \operatorname{fib}(P_{n-1} P_n F \to P_{n-1} P_{n-1} F) \simeq \operatorname{fib}(P_{n-1}F \to P_{n-1}F) \simeq *$) so $D_n F$ is n-reduced. We call $D_n F$ the *n-homogeneous approximation of F*.

5.2 The main theorems and their corollaries

One of the goals of this talk will be to show that there is a lift

$$\begin{tikzcd} & \mathtt{Sp}(\mathcal{D}) \arrow[d, "\Omega^\infty"] \\ \mathcal{C} \arrow[ur, dashed] \arrow[r, "D_n F"'] & \mathcal{D}. \end{tikzcd}$$

We will make this more precise later. It will follow from the following theorem.

Theorem 5. *For $F : \mathcal{C} \to \mathcal{D}$, an n-reduced functor, there is a cartesian square in $\mathtt{Fun}_*(\mathcal{C}, \mathcal{D})$,*

$$\begin{tikzcd} P_n F \arrow[r] \arrow[d] & P_{n-1} F \arrow[d] \\ K_n F \arrow[r] & R_n F, \end{tikzcd}$$

where $K_n F \simeq *$ and $R_n F$ is n-homogenous.

Proof sketch of main result from theorem. We sketch the case where $\mathcal{D}$ is pointed. Mapping $* \to P_{n-1}F$ on the right side and taking pullback gives $D_n F \to *$ along the top; the two stacked pullback squares themselves form a pullback square, so we obtain $D_n F \xrightarrow{\sim} \Omega R_n F$. Applying this again shows gives us a double delooping of $D_n F$, and continuing gives us an infinite delooping of $D_n F$. $\square$

Actually, a strenthened version of the theorem is the following.

Theorem 6. *Consider the full subcategory $\mathcal{E} \subset \mathtt{Fun}(\mathcal{P}_0(\underline{2}), \mathtt{Fun}_*(\mathcal{C}, \mathcal{D})$ of corners of functors $H_0 \to H \leftarrow E$ where $H_0 \simeq *$, H is n-homogeneous, and E is $(n-1)$-excisive. Then we have a functor*

$$\lim : \mathtt{Fun}(\mathcal{P}_0(\underline{2}), \mathtt{Fun}_*(\mathcal{C}, \mathcal{D})) \to \mathtt{Fun}_*(\mathcal{C}, \mathcal{D}),$$

and this induces an equivalence $\lim : \mathcal{E} \xrightarrow{\sim} \mathrm{Exc}^n(\mathcal{C}, \mathcal{D})$.

The backwards functor $B : \mathtt{Fun}_*(\mathcal{C}, \mathcal{D}) \to \mathtt{Fun}(\mathcal{P}_0(\underline{2}), \mathtt{Fun}_*(\mathcal{C}, \mathcal{D}))$ takes F to the corner $K_n F \to R_n F \leftarrow P_{n-1} F$.

Corollary 6. *If $\mathcal{D}$ is pointed, then* $\mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$ *is stable!*

(Recall that a pointed ∞-category is called *stable* if it has all finite limits and Ω is an equivalence.)

Proof of corollary from theorem. Let $H_0 \to H \leftarrow E$ be a corner as in the theorem. Consider the functor $E \times_H H_0$. We want to compute P_{n-1} of this:

$$P_{n-1}(E \times_H H_0) \simeq P_{n-1} E \times_{P_{n-1} H} P_{n-1} H_0 \simeq E \times_* * \simeq E.$$

Thus, $E \times_H H_0$ is n-homogeneous iff $E \simeq *$.

Now, recall that above we have $\mathcal{E} \xrightarrow{\sim} \mathrm{Exc}^n(\mathcal{C}, \mathcal{D})$. Let us precompose with the inclusion functor $\mathrm{Homog}^n(\mathcal{C}, \mathcal{D}) \to \mathcal{E}$ by $H \mapsto (* \to H \leftarrow *)$. Then the composition induces an equivalence $\mathrm{Homog}^n(\mathcal{C}, \mathcal{D}) \xrightarrow{\sim} \mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$. But chasing through, this is

$$H \mapsto \lim(* \to H \leftarrow *) \simeq \Omega H.$$

So Ω is indeed an equivalence. $\square$

Here is another corollary.

Corollary 7. *The functor $\Omega^\infty : \mathtt{Sp}(\mathcal{D}) \to \mathcal{D}$ induces an equivalence* $\mathrm{Homog}^n(\mathcal{C}, \mathtt{Sp}(\mathcal{D})) \xrightarrow{\sim} \mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$. *Thus, our desired lift is actually unique (up to equivalence).*

Proof. Write S_*^{fin} for the ategory of finite pointed CW-complexes. Then there is an equivalence $\mathtt{Sp}(\mathcal{D}) \xrightarrow{\sim} \mathrm{Exc}(S_*^{fin}, \mathcal{D})$. (To explain this, recall that excisive functors take pushout squares to pullback squares, so $\{F(S^n)\}_{n \geq 0}$ is an Ω-spectrum.) This sits in a commutative diagram

$$\begin{tikzcd}
\mathtt{Sp}(\mathcal{D}) \arrow[r, "\sim"] \arrow[dr, "\Omega^\infty"'] & \mathrm{Exc}(S_*^{fin}, \mathcal{D}) \arrow[d, "ev_{S^0}"] \\
& \mathcal{D}.
\end{tikzcd}$$

Now, for $K \in S_*^{fin}$, we have $ev_K : \mathrm{Exc}(S_*^{fin}, \mathcal{D}) \to \mathcal{D}$. For $F : \mathcal{C} \to \mathtt{Sp}(\mathcal{D}) \simeq \mathrm{Exc}(S_*^{fin}, \mathcal{D})$, since $P_n F$ is defined as a sequential colimit then P_n commutes with ev_K, i.e. $P_n(\mathrm{ev}_K \circ F) \simeq ev_K P_n F$. So F is n-excisive (resp. n-reduced) iff $ev_K \circ F$ is n-excisive (resp. n-reduced) for all K.

Now we get to proving the result. By general category theory, we have

$$\mathtt{Fun}(\mathcal{C}, \mathtt{Fun}(S_*^{fin}, \mathcal{D})) \simeq \mathtt{Fun}(S_*^{fin}, \mathtt{Fun}(\mathcal{C}, \mathcal{D})),$$

and this restricts to

$$\mathrm{Homog}^n(\mathcal{C}, \mathrm{Exc}(S_*^{fin}, \mathcal{D})) \simeq \mathrm{Exc}(S_*^{fin}, \mathrm{Homog}^n(\mathcal{C}, \mathcal{D})),$$

which is equivalent to

$$\mathrm{Homog}^n(\mathcal{C}, \mathtt{Sp}(\mathcal{D})) \simeq \mathtt{Sp}(\mathrm{Homog}^n(\mathcal{C}, \mathcal{D})).$$

These admit maps

$$\mathrm{Homog}^n(\mathcal{C}, \mathtt{Sp}(\mathcal{D})) \xrightarrow{\Omega^\infty} \mathrm{Homog}^n(\mathcal{C}, \mathcal{D}) \xleftarrow{\Omega^\infty} \mathtt{Sp}(\mathrm{Homog}^n(\mathcal{C}, \mathcal{D}))$$

which commute with the above equivalence, so they are equivalences too. □

This all sort of seems like a miracle: somehow being n-homogenous implies being stable!

5.3 On the proof of the theorem

Recall that we are only going to sketch the proof of the first main theorem (and not its strengthening).

Our main goal is to describe the cartesian square

$$\begin{tikzcd} T_n^i F \arrow[r] \arrow[d] & S_{n-1}^i F \arrow[d] \\ K_{n,i} F \arrow[r] & R_{n,i} F \end{tikzcd}$$

such that:

- $S_{n-1}^i F \simeq T_{n-1}^i F$;
- $K_{n,i} F \simeq *$;
- $R_{n,i} F$ is n-reduced.

So, if $S \subset \mathcal{P}(\underline{N})$, we define $U_S(X) = \lim_S F \circ C_\bullet(X)$. So for instance we can take $S = \mathcal{P}_0(\underline{n+1})$, and then $U_S(X) = T_n F(X)$. Note that if $S' \subset S$, we get a restriction $U_S(X) \to U_{S'}(X)$. Moreover, $U_{S \times S'}(X) \simeq U_S(U_{S'}(X))$.

Now, we set $T_n^i F = U_{(\mathcal{P}_0(\underline{n+1}))^i}$.

We cover the poset $(\mathcal{P}_0(\underline{n+1}))^i$ by two small posets, as follows. Set $\mathcal{B}_n = \mathcal{P}_0(\underline{n+1}) - \{\{n+1\}\}$ and $\mathcal{A}_{n,i} = (\mathcal{P}_0(\underline{n+1}))^i - (\mathcal{P}_0(\underline{n}))^i$. So for example, we have **[DIAGRAM]**. So, $(\mathcal{P}_0(\underline{n+1}))^i = \mathcal{B}_n^i \cup \mathcal{A}_{n,i}$. Then we set

$$K_{n,i} F = U_{\mathcal{A}_{n,i}}, \quad R_{n,i} F = \mathcal{U}_{A_{n,i} \cap \mathcal{B}_n}, \quad S_{n-1} F = U_{\mathcal{B}_n^i}.$$

Now, there is a general fact about hocolims of diagrams obtained from posets that tells us that the above diagram is indeed cartesian. Namely, if S and T are subposets that cover a poset and both are *concave* (i.e. if $s \in S$ and $s' \geq s$ for some $s' \in S \cup T$ then $s' \in S$; similarly for T) then the square

$$\begin{tikzcd} \lim\limits_{S \cup T} F \arrow[r] \arrow[d] & \lim\limits_{S} F \arrow[d] \\ \lim\limits_{T} F \arrow[r] & \lim\limits_{S \cap T} F \end{tikzcd}$$

is cartesian. (This is actually fairly difficult to check for ∞-categories, but in simplicial model categories is essentially boils down to the fact that $NS \cup NT = N(S \cup T)$.)

The other properties we will only verify for the case $n = 2$, $i = 1$.

First, let us verify that $K_{n,i}F \simeq *$. This is easy, because we're taking the holim over a diagram with an initial object (as the indexing category $\mathcal{A}_{2,1}$ has the initial object $\{3\}$), and $C_{\{3\}}X \simeq *$ so $F(C_{\{3\}}X) \simeq *$.

Next, let us verify that $R_{n,i}F$ is n-reduced. We compute that $P_{n-1}R_{n,i}F \simeq P_{n-1}R_{n,i}P_{n-1}F$, so we may assume that F is $(n-1)$-excisive. (For us, $n - 1 = 1$.) Then, $\mathcal{A}_{2,1} \cap \mathcal{B}_2$ is the diagram $\{2, 3\} \to \{1, 2, 3\} \leftarrow \{1, 3\}$, which gives us a cocartesian square

$$\begin{tikzcd}
C_{\{3\}}X \arrow[r] \arrow[d] & C_{\{1,3\}}X \arrow[d] \\
C_{\{1,2\}}X \arrow[r] & C_{\{1,2,3\}}X.
\end{tikzcd}$$

As we are assuming F is 1-excisive, then $F(C_{\{3\}}X) \simeq \lim F(C_{\{1,2\}}X) \to F(C_{\{1,2,3\}}X) \leftarrow C_{\{1,3\}}X)$, but this is just the definition of $R_{2,1}F(X)$. But since F is reduced, this must be contractible. (This generalizes correctly to show that $R_{n,i}F$ is n-reduced.)

Lastly, we must show that $S_{n-1}F \simeq T_{n-1}F$. Recall that in Aaron's talk we saw a more gneral condition than having an initial object for when we can shrink our diagram before taking a limit: if $u : J \to I$ is *homotopy initial*, then $\lim_J u^*D \simeq \lim_I D$. We use this for the inclusion $\mathcal{P}_0(\underline{2}) \to \mathcal{B}_2$, where one can easily check that all "overcategories" are contractible (i.e. have contractible nerves). This is an important technique: once we puncture our cube, we can go down to a lower-dimensional cube. This gives the desired equivalence.

We have no time for the homotopy colimit argument, but we'll just say the idea. If you take the hocolim over the diagram as i grows, we would get the original square we wanted. There's a slight subtlety in actually obtaining the maps, involving some "ups and downs" between the various squares as i varies. (If we call the i^{th} square Q_iF, then we actually only get $Q_iF \to T_nQ_iF \xleftarrow{\sim} Q'_iF \to Q_{i+1}F$, and we need to invert the equivalence.)

5.4 Summary

So, let us summarize what we have done. We have shown that the homogeneous fiber D_nF is canonically deloopable. This means that we can rotate the fiber sequence $D_nF \to P_nF \to P_{n-1}F$ to a fiber sequence $P_nF \to P_{n-1}F \to \Omega^{-1}D_nF$; thus we can obtain P_nF as $\mathrm{fib}(P_{n-1}F \to D_nF)$. These maps are also called *k-invariants*, in analogy with the Postnikov tower.

6 Homogeneous functors and cross-effects (David Carchedi)

6.1 Recollections

Luckily, Lennart covered the first two pages of this talk, so we'll just jump right into things.

For brevity, we will call an ∞-category *good* if it has finite colimits and a terminal object.

Recall the following theorem, which is in some sense the first main theorem of Goodwillie calculus.

Theorem 7. *If $\mathcal{C}$ is good and $\mathcal{D}$ is differentiable, then the inclusion $\mathrm{Exc}^n(\mathcal{C}, \mathcal{D}) \hookrightarrow \mathtt{Fun}(\mathcal{C}, \mathcal{D})$ admits a left-exact left-adjoint P_n.*

We also have a sort of chain rule.

Theorem 8. *Suppose $\mathcal{C}$ and $\mathcal{C}'$ are good and $\mathcal{D}$ and $\mathcal{D}'$ are differentiable, and suppose we have a diagram*

$$\mathcal{C}' \xrightarrow{F'} \mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{D}'.$$

If F' preserves pushouts and the terminal object, then $P_n(F \circ F') \simeq P_n F \circ F'$. Similarly, if G preserves sequential colimits and finite limits, then $P_n(G \circ F) \simeq G \circ P_n(F)$.

6.2 Multivariable calculus

Suppose $\mathcal{C}_1, \ldots, \mathcal{C}_m$ have pushouts and $\mathcal{D}$ has finite limits. Let $\vec{n} = (n_1, \ldots, n_m) \in (\mathbb{Z}_{\geq 0})^m$. We say that a functor $F : \prod_i \mathcal{C}_i \to \mathcal{D}$ is *$\vec{n}$-excisive* if for all $i \in [1, m]$ and for any objects $\{X_j\}_{j \neq i}$, the composite

$$C_i \xrightarrow{\sim} \mathcal{C}_i \times \prod_{j \neq i} \{X_j\} \hookrightarrow \prod_{\alpha} \mathcal{C}_\alpha \xrightarrow{F} \mathcal{C}$$

is n_i-excisive. Similarly we define *$\vec{n}$-reduced* and *$\vec{n}$-homogeneous.* We simply say F is *excisive* if it is $(1, \ldots, 1)$-excisive, and *reduced* if it is $(1, \ldots, 1)$-reduced. If F is reduced and excisive then we call it *multilinear.* These form a full subcategory $\mathrm{Exc}^{\vec{n}}(\mathcal{C}_1, \ldots, \mathcal{C}_m, \mathcal{D}) \subset \mathtt{Fun}(\prod \mathcal{C}_\alpha, \mathcal{D})$.

Remark 11. Let's write this in a kind of silly way. We have the diagram

$$\begin{tikzcd}
\mathrm{Exc}^{\vec{n}}(\mathcal{C}_1, \ldots, \mathcal{C}_m, \mathcal{D}) \arrow[r, hook] \arrow[d, "\wr"] & \mathtt{Fun}(\mathcal{C}_1 \times \prod_{j>1} \mathcal{C}_j, \mathcal{D}) \arrow[d, "\wr"] \\
\mathrm{Exc}^{n_1}(\mathcal{C}_1, \mathrm{Exc}^{\vec{n}'}(\mathcal{C}_2, \ldots, \mathcal{C}_m, \mathcal{D})) \arrow[r, hook] & \mathtt{Fun}(\mathcal{C}_1, \mathtt{Fun}(\prod_{j>1} \mathcal{C}_j, \mathcal{D})).
\end{tikzcd}$$

From this, it follows that this inclusion admits a left-exact left-adjoint $P_{\vec{n}}$. (The proof is by induction.)

This leads us to a string of facts generalizing previous facts we know in the single-variate case. We'll just rattle them off, since we don't have the time to prove them all.

Proposition 13. *If $\mathcal{C}_1, \ldots, \mathcal{C}_m$ have finite colimits and $\mathcal{D}$ has finite limits and $F : \prod \mathcal{C}_\alpha \to \mathcal{D}$ is $\vec{n}$-excisive, then as a functor of one variable, F is $(\sum n_\alpha)$-excisive.*

Corollary 8. *If $\mathcal{C}$ has finite colimits and $F : \mathcal{C}^m \to \mathcal{D}$ is $\vec{n}$-excisive, then $\mathcal{C} \xrightarrow{\Delta} \mathcal{C}^m \xrightarrow{F} \mathcal{D}$ is $(\sum n_\alpha)$-excisive.*

(This follows just because Δ preserves strongly cocartesian cubes.)

We have the following partial converse to the previous proposition.

Proposition 14. *If additionally each $\mathcal{C}_i$ is good and $\mathcal{D}$ is differentiable and $F : \prod \mathcal{C}_\alpha \to \mathcal{D}$ is reduced (in each variable) and m-excisive (as a functor of one variable), then F is $(1, \ldots, 1)$-excisive.*

Corollary 9. *It follows from the above facts that if $\mathcal{C}_1, \ldots, \mathcal{C}_m$ are good and $\mathcal{D}$ is differentiable and $F : \prod \mathcal{C}_\alpha \to \mathcal{D}$ is reduced in each variable, then $P_m(F) \simeq P_{(1,\ldots,1)}(F)$.*

Proof. By definition, F is excisive (as a multivariable functor), so it is m-excisive (as a single-variable functor). So the unit $F \to P_{(1,\ldots,1)}(F)$ of the adjunction must factor uniquely (up to homotopy) through the other unit map $F \to P_m(F)$; this simply follows from adjointness. Thus we have $\alpha : P_m(F) \to P_{(1,\ldots,1)}(F)$.

Now, fix $i \in [1, m]$ and consider the full subcategory $\mathcal{E}_i \subset \prod \mathcal{C}_\alpha$ of objects $\vec{X} = (X_1, \ldots, X_m)$ such that X_i is terminal. Since F preserves pushouts and the terminal object, then $P_m(F \circ j_i) \simeq P_m(F) \circ j_i$. But since F is assumed to be reduced, then $F \circ j_i \simeq 1$ (i.e. is terminal), and hence $P_m(F) \circ j_i \simeq 1$. Thus $P_m(F)$ is $(1, \ldots, 1)$-reduced. Since it is m-excisive, then it is $(1, \ldots, 1)$-excisive.

Now, we can apply the same argument as before. We know $P_m(F)$ is $(1, \ldots, 1)$-excisive, so we have a unique factorization

$$\begin{tikzcd} F \arrow[r] \arrow[dr] & P_m(F) \\ & P_{(1,\ldots,1)}F. \arrow[u, dashed, "\beta"'] \end{tikzcd}$$

But by the uniqueness of α and β, they must be homotopy inverses. $\square$

6.3 Construction of the reduction

Suppose $\mathcal{C}_1, \ldots, \mathcal{C}_m$ have 1 and $\mathcal{D}$ is pointed and has finite limits. Write $S = [m] = \{1, \ldots, m\}$, and define a functor $\alpha : \prod \mathcal{C}_\alpha \times P(S) \to \prod \mathcal{C}_\alpha$ by

$$\alpha(\vec{X}, T)_i = \begin{cases} X_i, & i \notin T \\ 1_i, & i \in T \end{cases}$$

One can easily check that this is indeed functorial in $P(S)$.

Suppose we have a multivariable functor $F : \mathcal{C}_1 \times \cdots \times \mathcal{C}_m \to \mathcal{D}$. For $T \subset S$, define $F^T = F \circ \alpha(\cdot, T)$. Note that we get a canonical map

$$F = F^\emptyset \to \lim_{T \neq \emptyset} F^T.$$

We define the *reduction* $Red(F)$ of F to be the fiber of this map.

Proposition 15. *$Red(F)$ is reduced, and Red : $\mathtt{Fun}(\prod \mathcal{C}_\alpha, \mathcal{D}) \to \mathtt{Fun}_*(\prod \mathcal{C}_\alpha, \mathcal{D})$ is right-adjoint to the inclusion.*

Proof of the first statement. For all $\vec{X}$ such that $X_i = 1_i$ for some i, we have $Red(F)(\vec{X}) = 0$. Note that if $T \subset S$, then the canonical map $F^T(\vec{X}) \to F^{T \cup \{j\}}(\vec{X})$ is an equivalence because $X_i \simeq 1_i$. Now, let $P_{\{j\}}(S) = \{T \subset S : j \in T\}$; this has an inclusion $l : P_{\{j\}}(S) \hookrightarrow P_0(S)$, which has a right adjoint r given by $r(T) = T \cup \{j\}$. Let us notate $F_{\alpha_{\vec{X}}} = F \circ \alpha(\cdot, \vec{X})$. By our adjunction and some yoga, we have

$$\lim_{P_0(S)} r^*(l^* F_{\alpha_{\vec{X}}}) \simeq \lim_{P_{\{j\}}(S)} l^* F_{\alpha_{\vec{X}}}.$$

But this is equivalent to

$$\lim_{P_0(S)} F^{\{j\}} \simeq Ran_{\mathrm{id}}(l^* F_{\alpha_{\vec{X}}})(\{j\}) = F_{\alpha_{\vec{X}}}(\{j\}) = F^{\{j\}}(\vec{X}).$$

Hence, as we have seen,

$$Red(F) = \mathrm{fib}(F^\emptyset(\vec{X}) \to F^{\{j\}}(\vec{X}))$$

is the fiber of an equivalence, so it is trivial. $\square$

We make the following observations.

Remark 12. Suppose $\mathcal{C}_1, \ldots, \mathcal{C}_m$ are good and $\mathcal{D}$ is pointed. For simplicity, denote $\mathcal{C} = \prod \mathcal{C}_\alpha$. For all n, the functor $P_n : \mathtt{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Exc}^n(\mathcal{C}, \mathcal{D}) \hookrightarrow \mathtt{Fun}(\mathcal{C}, \mathcal{D})$ is left-exact. Thus for any functor $F : \mathcal{C} \to \mathcal{D}$, we have that $P_n(Red(F)) \simeq Red(P_n(F))$.

Now let $n = m$. We know that $Red(F)$ is reduced, and by our last corollary, $Red(F) \simeq P_m(F) \simeq P_{(1,\ldots,1)}(Red(F))$. We will use this fact in a moment.

6.4 Cross effects

Suppose $\mathcal{C}$ is good and $\mathcal{D}$ is pointed and has finite limits. We have a functor $q : \mathcal{C}^n \to \mathcal{C}$ given by $q(\vec{X}) = \coprod X_i$. Thus, given a functor $F : \mathcal{C} \to \mathcal{D}$, we have $F \circ q : \mathcal{C}^n \to \mathcal{D}$. We now define the *$n$-fold cross effect* of F to be $cr_n(F) = Red(F \circ q)$.

Proposition 16. *If $\mathcal{C}$ is good and $\mathcal{D}$ is pointed and differentiable, and $F : \mathcal{C} \to \mathcal{D}$ is n-excisive, then for all $m \leq n+1$, $cr_m(F) : \mathcal{C}^m \to \mathcal{D}$ is $(n-m+1, \ldots, n-m+1)$-excisive. In the special case that $m = n+1$, in which case we simply obtain that $cr_{n+1}(F)$ is terminal (i.e. 0-excisive in each variable.)*

Now assume additionally that $\mathcal{D}$ is differentiable. Then,

$$P_{(1,\ldots,1)}(cr_n(F)) = P_{(1,\ldots,1)}(Red(F \circ q)) \simeq P_n(Red(q^* F)) \simeq Red(P_n(q^* F)) \simeq Red(P_n(F) \circ q)$$

(by the chain rule). But this is by definition $cr_n(P_n(F))$. This implies that cr_n is left-exact (since q^* is). Thus, we have that

$$cr_n(D_n(F)) = cr_n(\mathrm{fib}(P_n(F) \to P_{n-1}(F)) \simeq \mathrm{fib}(cr_n P_n(F) \to cr_n P_{n-1}(F)) \simeq \mathrm{fib}(cr_n P_n(F) \to 1) \simeq cr_n P_n(F).$$

Thus,

$$cr_n(P_n F) \simeq P_{(1,\ldots,1)}(cr_n(F)).$$

That is, we can recover the n^{th} cross effect of the n^{th} homogeneous component of F simply from the n^{th} cross effect itself.

6.5 Symmetric functors

Recall that if G is a finite group, we have a functor $EG : BG \to \mathtt{sSet}$ (where BG denotes the delooping of G, i.e. the one-object category determined by G) (landing at the usual G-object EG), and then $\operatorname{colim} EG = N(BG)$. Then, given $K \in \mathtt{sSet}$, the Σ_n-action on K^n is equivalent to a functor $\widetilde{K}^n : B\Sigma_n \to \mathtt{sSet}$ landing at K^n. From this, we can define

$$K^{(n)} = (K^n \times E\Sigma_n)/\Sigma_n = \widetilde{K}^n \otimes_{\Sigma_n} E\Sigma_n \in \mathtt{sSet}.$$

This is a hocolim in $\mathtt{sSet}$ with the Joyal model structure, and the "identity" functor $(\mathtt{sSet}, \text{Joyal}) \to (\mathtt{sSet}, \text{Quillen})$ is left-Quillen; hence this is also a hocolim in $(\mathtt{sSet}, \text{Quillen})$.

Now, suppose $\mathcal{C} \in \mathtt{sSet}$ is an ∞-category. Then $\mathcal{C}^{(n)}$ is also an ∞-category, and if $\widetilde{\mathcal{C}}^{(n)} : B\Sigma_n \to \widehat{\mathtt{Cat}_\infty}$ encodes the action of Σ_n on $\mathcal{C}^n$, then $\mathcal{C}^{(n)} \simeq \lim \widetilde{\mathcal{C}}^{(n)}$. We call $\mathcal{C}^{(n)}$ the n^{th} *extended power* of $\mathcal{C}$. We can finally define a *symmetric n-ary functor* from $\mathcal{C}$ to $\mathcal{D}$ to be a functor $\mathcal{C}^{(n)} \to \mathcal{D}$.

Remark 13. We have

$$\mathtt{Fun}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Hom}_{\mathtt{Cat}_\infty}(\operatorname{colim}(B\Sigma_n \times \mathcal{C}^n \rightrightarrows \mathcal{C}^n)) \simeq \lim(\mathrm{Hom}_{\mathtt{Cat}_\infty}(B\Sigma_n \times \mathcal{C}^n, \leftleftarrows \mathrm{Hom}_{\mathtt{Cat}_\infty}(\mathcal{C}^n, \mathcal{D})) \simeq \lim(\mathtt{Fun}(B\Sigma_n \times \mathcal{C}^n, \mathcal{D}) \leftleftarrows \mathtt{Fu}$$

Thus, explicitly, a functor $F : \mathcal{C}^{(n)} \to \mathcal{D}$ is the same data as a functor $G : \mathcal{C}^n \to \mathcal{D}$ together with certain homotopies encoding the invariance-up-to-homotopy under the symmetric action. (This bridges the gap between Lurie's and Goodwillie's definitions.)

Example 16. If $(\mathcal{C}, \otimes)$ is a symmetric monoidal ∞-category and $\mathcal{C}^n \to \mathcal{C}$ is any functor, then $(c_1, \ldots, c_n) \mapsto c_1 \otimes \cdots \otimes c_n$ descends to a unique symmetric n-ary functor $\otimes : \mathcal{C}^{(n)} \to \mathcal{C}$. In particular, if $\mathcal{C}$ is any ∞-category with coproducts, then we get a functor $\coprod : \mathcal{C}^{(n)} \to \mathcal{C}$.

We introduce a bit more terminology. Write $\theta : \mathcal{C}^n \to \mathcal{C}^{(n)}$ for the evident functor. We say that $F : \mathcal{C}^{(n)} \to \mathcal{C}$ is *$\vec{n}$-excisive* if $\theta^* F$ is; similarly for *$\vec{n}$-reduced*, and *excisive*, and *reduced*. We say that F

is *symmetric multilinear* if $\theta^* F$ is multilinear. We define $Sym\mathtt{Fun}^n(\mathcal{C},\mathcal{D}) = \mathtt{Fun}(\mathcal{C}^{(n)},\mathcal{D})$, and we have an adjoint $Red_{sym} : Sym\mathtt{Fun}^n(\mathcal{C},\mathcal{D}) \to Sym\mathtt{Fun}^n(\mathcal{C},\mathcal{D})$ as before.

If $\mathcal{C}$ has finite coproduct and $\mathcal{D}$ is pointed and has finite limits, we have the *symmetric* n^{th} *cross effect* by

$$cr_{(n)}(F) = Red_{sym}(F \circ \coprod).$$

From the fact that the previous adjunction reduces to the old self-adjunction (involving Red) on $\mathtt{Fun}(\mathcal{C}^n,\mathcal{D})$ via θ^*, it follows that $\theta^* cr_{(n)}(F) = cr_n(F)$.

We now state the main theorem.

Theorem 9. *Suppose $\mathcal{C}$ is good and pointed, and $\mathcal{D}$ is pointed and differentiable. Then we have*

$$cr_{(n)} : \mathrm{Homog}^n(\mathcal{C},\mathcal{D}) \to Sym\mathtt{Fun}^n(\mathcal{C},\mathcal{D}).$$

This functor is fully faithful and essentially surjective; that is, there is an equivalence of ∞-categories between the n-homogeneous functors and the symmetric n-ary functors.

We will at least indicate the inverse functor. The coproduct functor $\coprod : \mathcal{C}^{(n)} \to \mathcal{C}$ induces a functor $\coprod^* : \mathtt{Fun}(\mathcal{C},\mathcal{D}) \to \mathtt{Fun}(\mathcal{C}^{(n)},\mathcal{D})$, which descends to an inverse.

7 $\partial_*(\Sigma^\infty \mathtt{Top}_*(K,-))(*)$ (Karol Szumilo)

In the abstract, we used quasicategories to develop the general theory. This was a good idea, because they encode a lot of information very cleanly. However, today we'll be getting our hands dirty, and our point-set type constructions will require us to choose actual models for our categories.

Here are our conventions:

- $\mathtt{Top}_*$ is the category of based spaces;
- $\mathtt{Sp}$ is the category of spectra;
- K is a based CW-complex of dimension $k < \infty$;
- $F_K = \Sigma^\infty \mathtt{Top}_*(K,-) : \mathtt{Top}_* \to \mathtt{Sp}$.

Unless otherwise stated, all our functors have this signature (and all of our spaces are based).

7.1 Analyticity of F_K

Recall that a functor F is called *ρ-analytic* if there is a q such that for all $n \geq 1$, it satisfies the $E_n(n\rho - q, \rho+1)$ condtion:

- For a strongly cocartesian S-cube X with $|S| = n+1$ such that for all $s \in S$, $X(\emptyset) \to X(s)$ is m_s-connected for $m_s \geq \rho + 1$, then FX is $(q - n\rho + \sum m_s)$-cartesian.

A way to think about this is that analytic functors are not quite excisive, but agree with their excisive approximations to a reasonably high degree.

Proposition 17. *F_K is k-analytic.*

7.2 The Arone tower of F_K

We're going to construct a tower of functors which will turn out to be the Taylor tower of F_K; however, we won't know this a priori, so for now we'll call it the *Arone tower.*

Let J be a small category, and let $\mathtt{Top}_*^J$ and $\mathtt{Top}_{*J}$ be the categories of covariant and contravariant functors from J to $\mathtt{Top}_*$, respectively.

Suppose $\Phi \in \mathtt{Top}_*^J$ and $\Psi \in \mathtt{Top}_{*J}$, and let $X \in \mathtt{Top}_*$. Then a *bitransformation* from Ψ and Φ to X is a collection of maps $\tau_j : \Psi_j \wedge \Phi_j \to X$ for all $j \in J$ such that for all $\alpha \in J(i,j)$, the diagram

$$\begin{tikzcd}
\Psi_j \wedge \Phi_i \arrow[r] \arrow[d] & \Psi_j \wedge \Phi_j \arrow[d, "\tau_j"] \\
\Psi_i \wedge \Phi_i \arrow[r, "\tau_i"'] & X
\end{tikzcd}$$

commutes.

Then, the *tensor product* (or *smash product*) $\Psi \wedge_J \Phi$ is a space equipped with a universal bitransformation from Ψ and Φ such that

$$\begin{tikzcd}
(\Psi, \Phi) \arrow[r] & \Psi \wedge_J \Phi \arrow[d, dashed, "\exists"] & \\
& & X.
\end{tikzcd}$$

Those familiar with coends will observe that explicitly, this is given by

$$\Psi \wedge_J \Phi = \int^{j \in J} \Psi_j \wedge \Phi_j = \mathrm{coeq}\left(\coprod_{i,j} Psi_j \wedge J(i,j) \wedge Phi_i \rightrightarrows \coprod_k \Psi_k \wedge \Phi_k \right) = \left(\coprod_k \Phi_j \wedge \Psi_k \right) \Big/ \sim,$$

where $(x\alpha, y) \sim (x, \alpha y)$.

Example 17. Take $(\mathtt{Top}, \times)$ instead of $(\mathtt{Top}_*, \wedge)$, and take $J = \mathbf{\Delta}$, $\Phi = \Delta^\bullet$, Ψ is a simplicial space. Or, we can replace $\mathtt{Top}$ with $\mathtt{Set}$.

Example 18. If $\Phi = J(j, -)_+$, then $\Psi \wedge_J J(j,-)_+ = \Psi_j$. Similarly, if we take $\Psi = J(-,i)_+$, then $J(-,i)_+ \wedge_J \Phi = \Phi_i$. In particular, if $\Psi = S^0$ then $\Psi \wedge_J \Phi = \mathrm{colim}\, \Phi$.

Example 19. For a fixed functor $\Psi : I \times J^{op} \to \mathtt{Top}_*$, the functor

$$\mathtt{Top}_*^J \xrightarrow{\Psi \wedge_J -} \mathtt{Top}_*^J$$

has a right adjoint $\mathtt{Top}_*^I \to \mathtt{Top}_*^J$ given by $\chi \mapsto \mathtt{Top}_*^I(\Psi -, \chi)$.

Now, the category J that we're really interested in we will denote $\mathcal{E}$; this is the category whose objects are the finite sets of the form $\underline{m} = \{1, \ldots, m\}$ and whose morphisms are surjections. We will use tensor products in order to construct the Arone tower, and the properties above will help us determine its behavior.

For $X \in \mathtt{Top}_*$, we define a functor $X\wedge : \mathcal{E}^{op} \to \mathtt{Top}_*$ by $\underline{m} \mapsto X^{\wedge m}$; given $\alpha \in \mathcal{E}(\underline{m}, \underline{n})$, we get $X^{\wedge n} \to X^{\wedge m}$ by $X_1 \wedge \ldots \wedge X_n \to X_{\alpha(1)} \wedge \ldots \wedge X_{\alpha(n)}$.

For $n \in \mathbb{N}$ and $\underline{m} \in \mathcal{E}$, we define a functor $\mathcal{E}_n(m, -) : \mathcal{E} \to \mathtt{Set}$ by

$$i \mapsto \begin{cases} \mathcal{E}(m,i), & i \leq n \\ \emptyset & i > n. \end{cases}$$

So, we're just truncating the corepresented functor.

For $\Psi \in \mathtt{Top}_{*\mathcal{E}}$, we set $\Psi_n(m) = \Psi \wedge_{\mathcal{E}} \mathcal{E}_n(m,-)_+$, and hence we get $\Psi_n \in \mathtt{Top}_{*\mathcal{E}}$.

If $X, Y \in \mathtt{Top}_*$, we get $Map(X,Y) = \mathtt{Top}_*(X, \Sigma^\infty Y) \in \mathtt{Sp}$.

Now, observe that the truncation functors include into one another:

$$\mathcal{E}_0(m,-) \hookrightarrow \mathcal{E}_1(m,-) \hookrightarrow \cdots \xrightarrow{\text{colim}} \mathcal{E}(m,-)$$

forms a filtration. Applying $\Psi \wedge_J (-)_+$, we get a filtration

$$\Psi_0 \to \Psi_1 \to \cdots \xrightarrow{\text{colim}} \Psi.$$

We should think of this as somehow analogous to the skeletal filtration of a simplicial set.

Let us take the specific case $\Psi = K^\wedge$. If we apply the functor $Map_{\mathcal{E}}(-, X^\wedge)$ for $X \in \mathtt{Top}_*$ (i.e. those maps that commute with the maps from $\mathcal{E}$), we get the tower

$$Map_{\mathcal{E}}(K^\wedge, X^\wedge) \xrightarrow{\lim} \cdots \to Map_{\mathcal{E}}(K_1^\wedge, X_1^\wedge) \to Map_{\mathcal{E}}(K_0^\wedge, X_0^\wedge).$$

We will temporarily denote the levels of this tower by $\widetilde{P}_n F_k X$, and the limit by $\widetilde{P}_\infty F_k X$. Define $\widetilde{D}_n F_k X = \text{hofib}(\widetilde{P}_n F_k X \to \widetilde{P}_{n-1} F_k X)$.

Theorem 10. *For a certain space $K^{(n)}$ (which we'll identify through the course of the proof),*

$$\widetilde{D}_n F_k X \simeq Map(K^{(n)}, X^{\wedge n})^{h\Sigma_n} \simeq (Map(K^{(n)}, \mathbb{S}) \wedge X^{\wedge n})_{h\Sigma_n}.$$

Moreover, if X is i-connected, then $\widetilde{D}_n F_k X$ is $((1+i-k)n-1)$-connected.

Proof. For all $m \in \mathcal{E}$, there is a pushout square in $\mathtt{Set}^{\mathcal{E}}$ given by

$$\begin{tikzcd}
\mathcal{E}_{n-1}(n,-) \times_{\Sigma_n} \mathcal{E}(m,n) \arrow[r] \arrow[d] & \mathcal{E}_{n-1}(m,-) \arrow[d] \\
\mathcal{E}_n(n,-) \times_{\Sigma_n} \mathcal{E}(m,n) \arrow[r] & \mathcal{E}_n(m,-).
\end{tikzcd}$$

Notice that the rows are the same, and in fact the vertical maps are the inclusions we identified above. We can check this objectwise, and it is basically trivial to check, because:

- below n, the vertical arrows are the identity;
- above n, everything is empty;
- at n, the top row is empty and the bottom row takes the form

  $$\mathcal{E}_n(n,n) \times_{\Sigma_n} \mathcal{E}(m,n) \to \mathcal{E}_n(m,n);$$

  but we can identify $\mathcal{E}_n(n,n) = \Sigma_n$, and then we see that the map is an isomorphism.

We previously smashed the functors in the diagram with $K^\wedge$; now we're going to apply it to the whole diagram. This will give us another pushout square (since we're hitting a pushout square with a left adjoint) in $\mathtt{Top}_{*\mathcal{E}}$, namely

$$\begin{tikzcd}
K_{n-1}^{sm}(n) \wedge_{\Sigma_n} \mathcal{E}(-,n)_+ \arrow[r] \arrow[d] & K_{n-1}^\wedge \arrow[d] \\
K_n^\wedge(n) \wedge_{\Sigma_n} \mathcal{E}(-,n)_+ \arrow[r] & K_n^\wedge.
\end{tikzcd}$$

Now, we apply $Map_{\mathcal{E}}(-, X^{\wedge})$ to get a pullback square

$$\begin{tikzcd}
Map_{\mathcal{E}}(K_n^{\wedge}, X^{\wedge}) \arrow[r] \arrow[d] & Map_{\mathcal{E}}(K_n^{\wedge}(n) \wedge_{\Sigma_n} \mathcal{E}(-, n)_+, X^{\wedge}) \arrow[d] \\
Map_{\mathcal{E}}(K_{n-1}^{\wedge}, X^{\wedge}) \arrow[r] & Map_{\mathcal{E}}(K_{n-1}^{\wedge}(n) \wedge_{\Sigma_n} \mathcal{E}(-, n)_+, X^{\wedge}).
\end{tikzcd}$$

Now, the left vertical arrow is just $\widetilde{D}_n F_k X \to \widetilde{D}_{n-1} F_k X$. Moreover, we can identify the right vertical arrow via left Kan extensions as $Map_{\Sigma_n}(K_n^{\wedge}(n), X^{\wedge n}) \to Map_{\Sigma_n}(K_{n-1}^{\wedge}(n), X^{\wedge n-1})$.

Now, the fiber of $\widetilde{P}_n F_k X \to \widetilde{P}_{n-1} F_k X$ is $Map_{\Sigma_n}(K_n^{\wedge}(n) K_{n-1}^{\wedge}(n), X^{\wedge n})$. We'd like it also to be the *homotopy* fiber, which will be true if $K_{n-1}^{\wedge}(n) \to K_n^{\wedge}(n)$ is a Σ_n-cofibration. Indeed, this is

$$\begin{tikzcd}
K_{n-1}^{\wedge} = K^{\wedge} \wedge_{\mathcal{E}} \mathcal{E}_{n-1}(n, -)_+ = K^{\wedge n-1} \wedge_{\Sigma_{n-1}} \mathcal{E}(n, n-1)_+ = \{x_1 \wedge \ldots \wedge x_n \in K^{\wedge n} : \exists i \neq j \text{ s.t. } x_i = x_j\} =: \Delta_n K \arrow[d] \\
K_n^{\wedge}(n) = K^{\wedge} \wedge_{\mathcal{E}} \mathcal{E}_n(n, -)_+ = K^{\wedge} \wedge_{\mathcal{E}} \mathcal{E}(n, -)_+ = K^{\wedge n}.
\end{tikzcd}$$

The object in the top right is called the *fat diagonal*, and $(K^{\wedge n}, \Delta_n K)$ is indeed a relative Σ_n-CW-complex, obtained by freely attaching Σ_n-cells. Thus, we take $K^{(n)} = K^{\wedge n}/\Delta_n K$.

This completes the first half of the theorem. Now, it follows that $\widetilde{D}_n F_k X \simeq Map(K^{(n)}, X^{\wedge n})^{h\Sigma_n}$. By equivariant Spanier-Whitehead duality and the Adams isomorphism, this is equivalent to $Map(K^{(n)}, \mathbb{S}) \wedge X^{\wedge n})_{h\Sigma_n}$.

Lastly, we must show that $\widetilde{D}_n F_k X$ has the claimed connectivity. If X is i-connected, we can take its bottom cell to be in dimension $i+1$. Thus, the bottom cell of $X^{\wedge n}$ is in dimension $i(n+1)$. As $\dim K = k$, then $\dim K^{(n)} = kn$, so $Map(K^{(n)}, \mathbb{S})$ has bottom cell in dimension $-kn$. So, the smash product $Map(K^{(n)}, \mathbb{S}) \wedge X^{\wedge n}$ has its bottom cell in dimension $(1+i-k)n$, so it is $((1+i-k)n-1)$-connected. Therefore, so is its homotopy-fixed-points $\widetilde{D}_n F_k X$.

□

Corollary 10. *The tower $\{\widetilde{P}_n F_k\}_n$ is the Taylor tower of $\widetilde{P}_\infty F_k$, and $\widetilde{P}_\infty F_k$ is k-analytic with n^{th} differential given by $Map(K^{(n)}, \mathbb{S})$.*

Proof. The last statement follows immediately if we know that this is indeed the Taylor tower. So first of all, $\widetilde{D}_n F_k$ is n-homogeneous by our explicit formula, and moreover $\widetilde{P}_\infty F_k = \operatorname{holim}_n \widetilde{P}_n F_k$. Since the maps in this system are fibrations, it follows that $\widetilde{P}_n F_k$ is the n-excisive approximation to $\widetilde{P}_\infty F_k$.

So it just remains to show that the functor $\widetilde{P}_\infty F_k$ is analytic. But the map $\widetilde{q}_n : \widetilde{P}_\infty F_k X \to \widetilde{P}_n F_k X$ is the homotopy limit of the tower

$$\cdots \widetilde{P}_{n+1} F_k X \xrightarrow{\widetilde{P}_{n+1}} \widetilde{P}_n F_k X,$$

so $\widetilde{P}_{n+1}$ is $(1+i-k)(n+1)$-connected, hence so is $\widetilde{q}_n$. Thus, $\widetilde{P}_\infty F_k$ is k-analytic. □

7.3 The comparison of the Taylor and Arone towers

So far, all we know is that we've constructed the Taylor tower of $\widetilde{P}_\infty F_k$. We'd like to show that this is actually the Taylor tower of F_k. So for each n, consider the map $\mathtt{Top}_*(K, X) \to \mathtt{Top}_*(K^{\wedge n}, X^{\wedge n})$ given by $f \mapsto f^{\wedge n}$. Apply Σ^∞ to this. Then we have the composition

$$\Sigma^\infty \mathtt{Top}_*(K, X) \to \Sigma^\infty \mathtt{Top}_*(K^{\wedge n}, X^{\wedge n}) \to Map(K^{\wedge n}, X^{\wedge n}).$$

These assemble into a map $e_{K,X} : F_K X = \Sigma^\infty \mathtt{Top}_*(K,X) \to Map_{\mathcal{E}}(K^\wedge, X^\wedge) = \widetilde{P}_\infty F_K X$.

Theorem 11. *$e_{K,X}$ is an equivalence for k-connected X.*

In the course of the proof, we will use the following standard fact.

Proposition 18. *If $\tau : G \to J$ is an transformation of n-excisive functors such that $P_{n-1}\tau$ and $cr_n\tau$ (or just $D_n\tau$) are equivalences, then so is τ itself.*

We will also need the following proposition.

Proposition 19.

1. *If $\tau : H \to \widetilde{H}$ is a transformation of n-homogeneous functors such that $\tau\Sigma : H\Sigma \to \widetilde{H}\Sigma$ is an equivalence, then so is τ.*
2. *If $\tau : G \to J$ is a map of k-analytic functors such that $\tau\Sigma^m : G\Sigma^m \to J\Sigma^m$ is an equivalence on connected spaces, then τ is an equivalence on k-connected spaces.*

The "moral" reason for these, drawing on classical calculus, is that if two analytic functions with the same radius of convergence agree in some small neighbourhood of the origin, then they agree within their radius of convergence. (Moreover, the class of m-fold suspensions of connected spaces behaves as if it were "dense" in the class of m-connected spaces.)

Proof of latter proposition.

1. We compute that
$$cr_n H(Y_1, \dots, Y_n) \simeq |Omega^n cr_n H(\Sigma Y_1, \dots, \Sigma Y_n) \simeq \Omega^n cr_n(H\Sigma)(Y_1, \dots, Y_n),$$
so $cr_n\tau$ is an equivalence and hence so is τ.
2. By the cain rule, $P_n(G\Sigma^m) = (P_n G)\Sigma^m$, so $D_n(G\Sigma^m) \simeq (D_n G)\Sigma^m$. Now, if $\tau\Sigma^m$ is an equivalence on connected spaces, then so is $D_n(\tau\Sigma^m) \simeq (D_n\tau)\Sigma^m$, and hence so is $D_n\tau$. By induction, so is $P_n\tau$. The Taylor towers converge to G and J on k-connected spaces, and so τ is an equivalence on k-connected spaces.

□

Sketch of proof of theorem. We will sketch a proof in the simplicial case when $K = S^k$.

We will consider another sort of approximation to our functor. Namely, write $C(k,n)$ for the configuration space of n points in a k-cube. For $Y \in \mathtt{Top}_*$, then we have $C_k(Y) \xrightarrow{\sim} \Omega^k\Sigma^k Y$. (This is a theorem of May, one of the first steps in his operadic approach to the detection of loopspaces.)

Now, we have $\Sigma^\infty C(k,n)_+ \to Map((S^k)^{(n)}, S^{kn})$. We have the filtration $F_n C_k Y \subset \cdots \subset C_k Y$, and we have an equivalence $F_n C_k Y \xrightarrow{\sim} \widetilde{P}_n F_{S^k}\Sigma^k Y$; by our previous reduction, this will suffice. □

7.4 Summary of consequences

There are lots of nice things that come out of this theorem. For instance:

- $Map((S^k)^\wedge_n, (-)^\wedge)$ converges to F_{S^k} on k-connected spaces.
- The previous argument amounts to a sort of Snaith-type splitting. Thus, on spaces of the form $\Sigma^k Y$, the tower splits. This implies that
$$\Sigma^\infty\Omega^k\Sigma^k Y \simeq \prod_n \Sigma^\infty(C(k,n)_+ \wedge_{\Sigma_n} Y^{\wedge n}).$$
(This becomes the usual Snaith splitting when we pass to the limit $k \to \infty$.)
- The proof of the theorem also generalizes to other spaces than $K = S^k$. We could replace K by a compact parallelizable manifold with boundary, and we get similar results.

8 $\partial_*(\mathbf{Id}_{\mathtt{Top}_*})(*)$ (Irakli Patchkoria)

We will attempt to sketch the derivatives of $\mathrm{Id} : \mathtt{Top}_* \to \mathtt{Top}*$.

8.1 Background and overview

Recall that given a functor $F : \mathtt{Top}_* \to \mathtt{Top}*$, its n^{th} *cross effect* is by definition the functor

$$cr_n F(X_1, \ldots, X_n) = \mathrm{hofib}\left(F\left(\bigvee_{i=1}^{n} X_i\right) \to \mathrm{holim}_{\emptyset \neq Y \subset \underline{n}}\left(F\left(\bigvee_{i \notin Y} X_i\right)\right)\right).$$

Via this functor, n-homogeneous functors are equivalent to n-variate multilinear functors.

Recall that we have the fiber sequence $D_n F \to P_n F \to P_{n-1} F$. We have previously identified

$$D_n(X) \simeq \Omega^\infty (C_n \wedge \Sigma^\infty X^{\wedge n})_{h\Sigma_n},$$

for C_n a Σ_n-spectrum, called the n^{th} *derivative* of F. Recall that

$$P_{(1,\ldots,1)} cr_n F(X_1, \ldots, X_n) \simeq \Omega^\infty((C_n \wedge X_1 \wedge \ldots \wedge X_n)$$

is the n^{th} *differential*, and we denote this $\partial_n F(*)$.

The strategy of the computation is as follows. Define $Q : \mathtt{Top}_* \to \mathtt{Top}_*$ in the standard way, $Q = \Omega^\infty \Sigma^\infty$. We will define certain functorial cosimplicial objects $Q^{\bullet+1} X$ and obtain $X \simeq Tot(Q^{\bullet+1} X)$ if X is simply-connected. Then, we will have that

$$(P_n \mathrm{Id})(X) = Tot(P_n Q^{\bullet+1} X),$$

and similarly for the D_n.

However, a few warnings are in order.

- First of all, P_n does not in general commute with infinite inverse limits. Thus, we cannot simply apply P_n to the equivalence $\mathrm{Id} \simeq Tot(Q^{\bullet+1})$; this must be justified.
- Recall the Snaith splitting $\Sigma^\infty QX \simeq \bigvee_{i=1}^\infty X^{\wedge i}_{h\Sigma_i}$. We actually have that $P_n Q^2 X \simeq Q(\prod_{i=1}^n X^{\wedge i}_{h\Sigma_i})$. We can try to mimic this at higher levels, but the naive approach will fail. We have that the "Snaith models" are equivalent to the levels of the cosimplicial object $P_n Q^{\bullet+1} X$, but it's not feasible to write out the structure maps on the "Snaith model" side.

The original computations are due to Arone-Kankaanrinta.

Theorem 12 (Johnson)**.** *For a particular finite Σ_n-CW-complex K_n such that $K_n \simeq \bigvee_{(n-1)!} S^{n-1}$, we have $D_n X \simeq \Omega^\infty Map_*(K_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}$.*

Note that we have the equivalence $\Omega^\infty Map_*(K_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n} \simeq \Omega^\infty (Map_*(K_n, \mathbb{S}) \wedge \Sigma^\infty X^{\wedge n})_{h\Sigma_n}$.

Idea of proof. Let $A = \{x_1, \ldots, x_n\}$ be a finite set. Consider the free Lie algebra L generated by A. This has a preferred basis, called the *Hall basis*, consisting of Lie monomials, given by

$$\overline{B}_1 = \{\mathrm{ad}^i(x_1)(x) : x \in A, x \neq x_1, i \geq 0\}, \quad \overline{B}_2 = \{\mathrm{ad}^i(x_2)(x) : x \in \overline{B}_1, i \geq 0\}, \quad \ldots.$$

Then we set $B_1 = \{x_1\} \cup \overline{B}_1$, $B_2 = \{x_2\} \cup \overline{B}_2$, etc., and we set $B = \bigcup_m B_m$.

We order the elements of B as follows. $\omega_i(x_1, \ldots, x_n)$ is the "i^{th} basic product", i.e. 0^{th} is $\{x_1, x_2\}$, 1^{st} contains e.g. x_1, x_2, and $[x_1, x_2]$, 2^{nd} contains $[x_1, [x_2, x_3]]$, etc.

Now, we return to topology via the following theorem.

Theorem 13 (Hilton-Milnor)**.** *We have*

$$\Omega\Sigma(X_1 \vee \ldots \vee X_n) \simeq \prod_{i=1}^{\infty} \Omega\Sigma\omega_i(x_1, \ldots, x_n),$$

with $[-,-]$ given by smash product.

This will motivate why K_n should be the indicated wedge of spheres. Namely, we start by applying this to the cross effect

$$cr_n(\Omega\Sigma)(X_1, \ldots, X_n) \simeq \prod_i \Omega\Sigma\omega_i(X_1, \ldots, X_n),$$

where i runs over all basic products that contain all the X_i.

So in a certain range, we can express the homotopy groups of the left side via this equivalence. Explicitly, if our spaces are all k-connected, then for $m \in [0, (n+1)(k+1) - 1,$

$$\pi_m(cr_n(\Omega\Sigma)(X_1, \ldots, X_n)) \cong \pi_m(\prod_{(n-1)!} \Omega\Sigma(\bigwedge_{i=1}^{n} X_i)).$$

By the Freudenthal suspension theorem, we can rewrite this as

$$\pi_m(\prod_{(n-1)!} \Omega^n S^n(\bigwedge_{i=1}^{n} X_i)) \cong \pi_m Map((\Sigma(\bigvee_{(n-1)!} S^{n-1}), \Sigma X_1 \wedge \ldots \wedge \Sigma X_n)).$$

This tells us that in our specified range, $cr_n(\Omega\Sigma)(X_1, \ldots, X_n)$ agrees up to order n with $Map(\Sigma(\bigvee_{(n-1)!} S^{n-1}), \Sigma X_1 \wedge \ldots \wedge \Sigma X_n)$ (in the sense of homotopy groups).

Similarly, using Johnson's theorem, we can identify the derivatives as the n^{th} homogeneous parts of the Taylor tower, and hence $cr_n(\mathrm{Id})$ and $Map_*(K_n, X_1 \wedge \ldots \wedge X_n)$ *agree to order* n (in the sense of Goodwillie calculus). Hence for the same range of m, these have isomorphic π_m. So,

$$cr_n(\Omega\Sigma(X_1, \ldots, X_n)) = \Omega cr_n(\mathrm{Id})(\Sigma X_1, \ldots, \Sigma X_n)$$

by definition, so $cr_n(\Omega\Sigma)(X_1, \ldots, X_n)$ and $Map(\Sigma K_n, \Sigma X_1 \wedge \ldots \wedge \Sigma X_n)$ agree up to order n. Thus,

$$\pi_m Map(\Sigma K_n, \Sigma X_1 \wedge \ldots \wedge \Sigma X_n) \cong \pi_m Map(\bigvee_{(n-1)!} S^n, \Sigma X_1 \wedge \ldots \wedge \Sigma X_n)$$

for the same range of m.

Johnson continues the argument along these lines; we will not. □

Let us say a word about why we can expect to compute the derivatives of the identity from this tower of iterated cubes. We have this cosimplicial object $Q^{\bullet+1}X$, and the goal is to show that $P_n\mathrm{Id}(X) \simeq Tot(P_n Q^{\bullet+1}X)$. The associated Tot tower of $Q^{\bullet+1}X$ looks like

$$* \leftarrow E_1(X) \leftarrow E_2(X) \leftarrow \cdots \xleftarrow{\lim} E_i(X) = Tot(Q^{\bullet+1}(X))$$

with hofibers $\Omega^{n-1}F_n(X)$. Note that $F_n(X) = \mathrm{hofib}((s^0, \ldots, s^{n-1}) : Q^{n+1}X \to \prod Q^n X)$. Now, there's a general fact that if X is c-connected, then $\Omega^{i-1}F_i(X)$ is ic-connected. In other words, the functor $\Omega^{i-1}F_i$ and the constant functor $*$ agree up to order $(i-1)$.

Once we know this, we can apply the following lemma to conclude that $P_n\Omega^{i-1}F_i(X) \simeq *$ if $n < i$.

Lemma 4 (Goodwillie)**.** *If $u : F \to G$ is a transformation such that F and G agree up to order n via u, then $P_nF \xrightarrow{\simeq} P_nG$.*

Now, we can finally justify our main claim. The above fact implies that the Tot tower of $P_n Q^{\bullet+1}X$ is eventually constant (since the hofibers are eventually contractible), and so $Tot(P_n Q^{\bullet+1}X) \simeq P_n E_n(X)$. Moreover, the map $Tot(Q^{\bullet+1}(-)) \to E_n(-)$ is an equivalence up to order n, and so $P_n(Tot(Q^{\bullet+1}X)) \simeq P_n(E_n(X))$. In the simply-connected case, $P_n(Tot(Q^{\bullet+1}X)) \simeq (P_n \mathrm{Id})(X)$.

The exact same argument works for D_n.

8.2 Sketch of constructions

Let us introduce some notation.

- Let $(O_n^k)^{op}$ be the category whose objects are the sequences $\underline{i_k} \twoheadrightarrow \underline{i_{k-1}} \twoheadrightarrow \cdots \twoheadrightarrow \underline{i_1} \twoheadrightarrow \underline{i_0} = 1$ (where $\underline{m} = \{1, \ldots, m\}$) where $i_k \leq n$, and whose morphisms are maps of sequences that are degreewise surjections (along with some technical conditions that are important for the actual proof but that in our sketch we won't need to consider). It will become clear later why we've introduced this, but we'll say for now that $O_n^\bullet$ is a simplicial category (i.e. a simplicial object in $\mathtt{Cat}$) with

$$d_0(\underline{i_k} \twoheadrightarrow \cdots \twoheadrightarrow \underline{i_0}) = \begin{cases} \underline{i_k} \twoheadrightarrow \cdots \twoheadrightarrow \underline{i_1}, & i_1 = 1 \\ \emptyset, & \text{otherwise.} \end{cases}$$

  (The other d_j are defined by omitting $\underline{i_j}$.)

- Now, suppose we have a sequence of spaces $(X_k, \ldots, X_0)$. Then we define a functor $F_{X_k,\ldots,X_0} : O_n^k \to \mathtt{Top}_*$ by

$$\alpha = (\underline{i_k} \twoheadrightarrow \cdots \twoheadrightarrow \underline{i_0}) \mapsto \bigwedge_{j=0}^{k} X_j^{i_j}.$$

  That this is a functor follows from the fact that the morphisms in O_n^k are degreewise surjections.

- Let $\vec{l} = (l_k, \ldots, l_0) \in \mathbb{N}^k$. Then we define functors $S^{\vec{l}}, S_X^{\vec{l}} : O_n^k \to \mathtt{Top}_*$ by $S^{\vec{l}} = F_{S^{l_k},\ldots,S^{l_0}}$ and $S_X^{\vec{l}} = F_{S^{l_k}\wedge X, S^{l_{k-1}},\ldots,S^{l_0}}$.
- Let Σ be the category of finite sets and injections.
- Define a functor $\Omega S(-)(X)$ by $\Omega S(\vec{l})(X) = \Omega^{l_k} S^{l_k}(\cdots \Omega^{l_0} S^{l_0} X)$.

The crucial observation is that $Nat_{O_n^k}(S^{\vec{l}}, s_X^{\vec{l}})$ is actually a functor over Σ^{k+1}. Hence $\Omega S(-)(X)$ is also a functor over Σ^{k+1}. Moreover, we have a particular natural transformation $\rho_n : \Omega S(\vec{l})(X) \to Nat_{O_n^k}(S^{\vec{l}}, S_X^{\vec{l}})$ defined by induction. Passing to hocolims, we get

$$Q^{k+1}X \to \mathrm{hocolim}_{\Sigma^{k+1}}\, Nat_{O_n^k}(S^{\vec{l}}, S_X^{\vec{l}}) + : \widetilde{P}_n Q^{\bullet+1}X.$$

Theorem 14. *$Q^{\bullet+1}X \to \widetilde{P}_n Q^{\bullet+1}X$ is the n-polynomial approximation.*

Proof sketch. We define the full subcategory ${}^nO_n^{k-1} \subset O_n^k$ to be those sequences with $i_k = n$. This is actually a groupoid (by the technical condition above that we didn't spell out). Thus for an object α, we have an automorphism group $\Sigma_\alpha = \mathrm{Aut}(\alpha)$, and in fact $\Sigma_\alpha \subset \Sigma_n$. Then, we have an action of Σ_α on $S^{\vec{l}}(\alpha) = \bigwedge_{j=0}^{k} S^{l_k i_j}$.

There is a pullback square

$$
\begin{tikzcd}
Nat_{O_n^k}(S^{\vec{l}}, S_X^{\vec{l}}) \arrow[r] \arrow[d] & \prod_{\alpha \in \pi_0({}^nO_n^{k-1})} Map(S^{\vec{l}}(\alpha), S_X^{\vec{l}}(\alpha))^{\Sigma_\alpha} \arrow[d] \\
Nat_{O_{n-1}^k}(S^{\vec{l}}, S_X^{\vec{l}}) \arrow[r] & \prod_{\alpha \in \pi_0({}^nO_n^{k-1})} Map(\Delta^{\vec{l}}(\alpha), S_X^{\vec{l}}(\alpha))^{\Sigma_\alpha};
\end{tikzcd}
$$

note that the fat diagonal $\Delta^{\vec{l}}(\alpha) \subset S^{\vec{l}}(\alpha)$ is precisely the locus on which the Σ_α-action isn't free. The fact that this is a pullback square follows from the fact that this inclusion is a Σ_α-cofibration. $\square$

Thus, we have a cofibration sequence

$$\prod_{\alpha \in \pi_0({}^nO_n^{k-1})} Map(S^{\vec{l}}(\alpha)/\Delta^{\vec{l}}(\alpha), S_X^{\vec{l}}(\alpha))^{\Sigma_\alpha} \to Nat_{O_n^k}(S^{\vec{l}}, S_X^{\vec{l}}) \to Nat_{O_{n-1}^k}(S^{\vec{l}}, S_X^{\vec{l}}).$$

One can show that:

- the hocolims of the fibers are n-homogeneous;
- the hocolims of the bases are $P_n\mathrm{Id}(X)$;
- at $n = 1$, this identifies QX.

Thus the hocolim of the fibers are D_nX. If we set $K_n = |T_n^\bullet|$, where

$$T_n^k = surj(\underline{n}, \underline{i_{k-1}}) \times_{\Sigma_{k-1}} \cdots \times_{\Sigma_1} surj(\underline{i_1}, \underline{i_0})$$

form the levels of a Σ_n-simplicial set, then K_n itself carries a Σ_n-action.

So all in all,

$$Map(S^{\vec{l}}(\alpha)/\Delta^{\vec{l}}(\alpha), S_X^{\vec{l}}(\alpha))^{\Sigma_\alpha} \cong Map(S^{\vec{l}}(\alpha), S_X^{\vec{l}}(\alpha) \wedge E\Sigma_{n+})^{\Sigma_\alpha}$$

by basic equivariant homotopy theory (considering $\Sigma_\alpha \subseteq \Sigma_n$, so that $E\Sigma_n$ is a model for $E\Sigma_\alpha$). Thus, we have an equivalence

$$\prod_{\alpha \in \pi_0({}^nO_n^{k-1})} Map(S^{\vec{l}}(\alpha), S_X^{\vec{l}}(\alpha) \wedge E\Sigma_{n+})^{\Sigma_\alpha} \simeq Nat_{{}^nO_n^{k-1}}(S^{\vec{l}}, S_X^{\vec{l}} \wedge E\Sigma_{n+1}),$$

from which we deduce that $O_n^\bullet$ has a simplicial structure, so ${}^nO_n^{\bullet-1}$ has a simplicial structure as well, so that

$$Nat_{{}^nO_n^{\bullet-1}}(S^{\vec{l}}, S_X^{\vec{l}} \wedge E\Sigma_{n+})$$

is a cosimplicial object. Then,

$$Tot\left(\mathrm{hocolim}_{I^{\bullet+1}} Nat_{{}^nO_n^{\bullet-1}}(S^{\vec{l}}, S_X^{\vec{l}} \wedge E\Sigma_{n+})\right) \simeq Tot\left(\mathrm{hocolim}_I Nat_{{}^nO_n^{\bullet-1}}(S^{\underline{m}\cdot n}, S_X^{\underline{m}\cdot n} \wedge E\Sigma_{n+1})\right),$$

where $S^{\underline{m}\cdot n} : {}^n O_n^{\bullet-1} \to \mathtt{Top}_*$ via $\alpha \mapsto S^{mn}$. (This follows from an argument in Bockstedt's paper.) Then we can rewrite these as

$$Tot(Nat_{{}^nO_n^{\bullet-1}}(S^{\infty\cdot n}, S_X^{\infty\cdot n} \wedge E\Sigma_{n+})) \simeq Tot(Map(T_n^\bullet, Map(S^{\infty n}, S_X^{\infty n} \wedge E\Sigma_{n+}))^{\Sigma_n})$$
$$\simeq Map(K_n, Q^{\Sigma_n}(\Sigma^\infty X^{\wedge n} \wedge E\Sigma_{n+1})^{\Sigma_n}$$

where Q^{Σ_n} is the Q-functor for the natural Σ_n-universe. For this natural Σ_n-universe, the Adams isomorphism tells us that this is equivalent to

$$\Omega^\infty Map(K_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}.$$

9 TC and the cyclotomic trace (Steffen Sagave)

The aim of this talk is to explain the cyclotomic trace map $K(A) \to TC(A)$ from the algebraic K-theory of a symmetric ring spectrum A to its topological cyclic homology. In particular, we'll have to define these objects too. We'll also have to explain why this is related to the Goodwillie calculus, but Jeremiah will handle that in the next talk. The idea is that the difference between these functors is "locally constant", meaning that to compute $K(A)$ it's often sufficient to compute $TC(A)$ and their difference. This is a big deal.

We will assume that our ring spectra are connective; the definitions work for nonconnective spectra, but they capture only information about the connective cover anyhow.

9.1 TC from THH

$TC(A)$ is built from $THH(A)$, the topological Hochschild homology, so we start there. The easiest definition to give is as follows: $THH(A) = \mathrm{Tor}^{A\wedge A^{op}}(A, A) = A \wedge^{\mathbb{L}}_{A\wedge A^{op}} A$. Unfortunately this hides the cyclic structure we'll need to define TC, so this is not good enough. Using the bar construction, we can expand this out to $THH(A) = |B^{cy}_* A| = |[q] \mapsto A^{\wedge(q+1)}|$, the cyclic bar construction on A. The boundary maps are

$$d_i(a_0 \wedge \cdots \wedge a_q) = \begin{cases} \cdots \wedge a_i a_{i+1} \wedge \cdots, & 0 \le i \le q-1, \\ a_q a_0 \wedge a_1 \wedge \cdots, & i = q. \end{cases}$$

The definition looks like it only uses associative structure, but we need commutativity to have a good enough smash product. The degeneracy maps insert 1s. Finally, there is a cyclic operator, which permutes the factors cyclically: $t_q(a_0 \wedge \cdots \wedge a_q) = a_q \wedge a_0 \wedge \cdots \wedge a_{q-1}$. This gives a factorization $\Delta^{op} \to \Lambda^{op} \to Sp^\Sigma$, so an action of S^1 on $|B^{cy}(A)|$. Another way to see this is $B^{cy}(A) = A \otimes S^1$ as soon as A is commutative. (N.B.: If A is E_n then $THH(A)$ is E_{n-1}.)

These definitions are nice because we can compute with them, but they don't give TC. Later on we'll define a spectrum $TH(A)$ with $TH(A) \simeq THH(A)$ and with structure maps F_r a Frobenius and R_r a restriction, with $F_r, R_r : TH(A)^{C_{nr}} \to TH(A)^{C_n}$. (Note that these are *actual* fixed points, not geometric or homotopy or anything.) F_r will turn out to be easy to define and R_r will be hard; the Frobenius just comes out of fixed points machinery, but not every spectrum with an S^1-action supports R_r guys. These maps are compatible in the following sense: they assemble to a functor $I \to Sp$, where the objects of I are the positive naturals (and the functor sends n to $TH(A)^{C_n}$) and the morphisms are given by $F_r, R_r : nr \to N$ with $F_r F_s = F_{rs}$, $R_r R_s = R_{rs}$, $F_r R_s = R_s F_r$, and $F_1 = R_1 = id$. With this in hand, we set $TC(A) = \mathrm{holim}_I\, TH(A)^{C_n}$. There is also a primary version of this, with a map $TC(A) \to TC(A, p) = \mathrm{holim}_{I_p}\, TH(A)^{C_n}$, where $I_p \subseteq I$ is the full subcategory on p^js. So, in principle, we'll be done when we define TH, but rather than doing that we'll take another detour.

We won't be talking about computations, because they're obscenely difficult. The easiest case of $H\mathbb{F}_p$ is already much more than just algebra; Justin said it took him about an hour to work out, using a Bökstedt spectral sequence and on and on. $THH(\mathbb{F}_p)$ is a module over $H\mathbb{F}_p$ so a wedge of E-M spectra, but TC gives you image-of-J things away from p in addition to some p-adic data at p.

Take $T = S^1$ a torus. $TH(A)^{C_n}$ has a T-action by restricting along $\rho_n : T \to T/C_n$. Our maps R_r will be T-equivariant, but the Frobenius map will be twisted up in the following way: $F_r(z^r \cdot -) = zF_r(-)$. We have a kind of twisted product then $I \ltimes T$ with the same objects but whose morphisms are tuples $(r, s, z) \in N \times N \times T : m \to n$, with $m = rns$, and then $(r_1, s_1, z_1)(r_2, s_2, z_2) = (r_1 r_2, s_1 s_2, z_1^{r_2} z_2)$. $n \mapsto TH(A)^{C_n}$ is a

$I \ltimes T$-diagram. There's a diagram

$$\begin{tikzcd} I \ltimes T & I_p \ltimes T \arrow[l] & T \arrow[l] \\ U \arrow[u] & I_p \arrow[l] \arrow[u] & \{1\}, \arrow[l] \arrow[u] \end{tikzcd}$$

becoming

$$\begin{tikzcd} \underline{TC}(A) \arrow[r] \arrow[d] & \underline{TC}(A,p) \arrow[r] \arrow[d] & TH^{hS^1}(A) \arrow[d] \\ TC(A) \arrow[r] & TC(A,p) \arrow[r] & TH(A), \end{tikzcd}$$

so an interesting integral version of TC. In particular, we will construct our cyclotomic trace map $K(A) \to \underline{TC}(A)$ with values in this global topological cyclic homology. The first vertical map is an equivalence after profinite completion, and the second after p-completion.

9.2 A model for THH

So, OK, we need TH now. Our main tool for this are "epicyclic spaces", which involve some fancy subdivision called "edgewise," which is not the usual sort. We define $\sqcup_r : \Delta \to \Delta$ by $[k] \mapsto [k]^{\sqcup r} = [k] \sqcup \cdots \sqcup [k] = [r(k+1)-1]$. The r-fold subdivision functor $Sd_r : sSp \to sSp$ by pulling back along $\sqcup_r$. $Sd_2\Delta^2$ looks like a triforce. We also have a map $D_r : \Delta^k \to \Delta^{\sqcup_r[k]}$ by $v \mapsto (v/r, \ldots, v/r)$; this induces a homeomorphism $|Sd_r X_*| \to |X_*|$. When X is cyclic, D_r is T-equivariant, but additionally $Sd_r X$ has a simplicial C_r-action, and the two actions on the geometric realization coincide. So it simplicial-ifies part of the T-action. A generator of C_r acts by $t^{k+1}_{r(k+1)-1}$.

Definition 20 (Goodwillie)**.** An epicyclic space is a cyclic space X together with cyclic maps $R_r : (Sd_r X)^{C_r} \to X$ for $r \geq 1$ such that $R_1 = 1$ and $R_{rs} = R_r(Sd_r R_s)^{C_r}$.

If X is an epicyclic space, then one can consider $n \mapsto \rho_n^*|X_*|^{C_n}$, a functor $I \ltimes T$. For R_r we set $R_r = \rho_n^*(\rho_r^*|X|^{C_r} \xleftarrow{\simeq} \rho_r^*|sd_r X|^{C_r} \simeq |sq_r X_*^{C_r}| \xrightarrow{|R_r|} |X|)^{C_n}$, inheriting up the epicyclic structure map. The Frobenius map is the inclusion of fixed points: $F_s : \rho_{sn}^* X^{C_{sn}} \to \rho_n^* X^{C_n}$.

This refines our task: we want an epicyclic model $TH(A)$ of $THH(A)$. Segal says that for any category C with coproducts, we can define $C\langle - \rangle$ with $\Gamma^{op} \to Cat$, where $\Gamma^{op} \simeq FiniteSets$. Take A to be well-based and semistable, and let $\mathcal{F}_A \subseteq Mod_A$ be the full subcategory on $A^{\wedge n}$. This has coproducts, so gives a functor $F_A\langle - \rangle : \Gamma^{op} \to Sp^\Sigma - Cat$. Composing this with $Q_I = (E \mapsto \mathrm{hocolim}_I\, \Omega^n E_n)$ gives a topological category $S \mapsto Q_I F_A\langle S \rangle$, which has a subcategory of weak equivalences wF_A. Then we define $K(A) = BwF_A\langle S \rangle$. There are a lot of different definitions of $K(A)$, most of them less complicated than this, but this gives us the Γ-structure we'll need to define the cyclotomic trace map.

Now let C be a spectral Γ-category. Then we can perform the construction $V_K[C] = \coprod_{c_0,\ldots,c_K} C(c_k, c_0) \bar{\wedge} \cdots \bar{\wedge} C(c_{k-1}, c_k)$, where $\bar{\wedge}$ is a sort of external smash product, making this a "$(k+1)$-multisymmetric spectrum", like a bispectrum but with even more directions. Then $TH_*(F_A\langle - \rangle) = [k] \mapsto Q_{I^{k+1}} V_K[F_A\langle - \rangle]$. Lastly, we define $TH(A) = |TH_*(F_A\langle S \rangle)|$, which has the desired epicyclic structure.

So we have TH and hence TC and also K; now we need the map. Let C be a topological category, and consider $B^{cy}C$, which has an epicyclic structure with structure maps isomorphisms. We define $K^{cy}(A) = B^{cy}(wF_A)(S)$. This fits into an $I \ltimes T$-diagram where the R_r maps are isomorphisms, which means we can take the homotopy limit just over $N \ltimes T$ to compute $K^{cy}(A)^{h(I\ltimes T)} \simeq K^{cy}(A)^{h(N\ltimes T)}$, where $N \ltimes T \subseteq I \ltimes T$ only has F_s and T living in it.

The inclusion of the weak equivalences gives a map of Γ-epicyclic spaces $B^{cy}(wF_A\langle -\rangle) \to TH_*(F_A\langle -\rangle)$ (though this is a bit complicated to write down and uses the existence of monoidal structures to move from Q_I to $Q_{I^{k+1}}$). Taking homotopy limits over $I \ltimes T$ gives $K^{cy}(A)^{h(N\ltimes T)} \to \underline{TC}(A)$. Now we take $S \in \Gamma^{op}$ and $C = wF_A\langle S\rangle$, then form the homotopy pullback of $B^{cy}(C)^{h(N\ltimes T)} \to Maps(BN, BC) \leftarrow BC$, called $K'(A)(S)$. When C is grouplike, the map involving B^{cy} an equivalence (hard theorem), meaning the the map opposite it in the square is too, which altogether gives $K(A) \xleftarrow{\simeq} K'(A) \to K^{cy}(A)^{h(N\ltimes T)} \to \underline{TC}(A)$. The composite is the cyclotomic trace map.

The main property of the cyclotomic trace map is that a map $A \to B$ which is surjective on π_0 with nilpotent kernel induces a pullback square with edges $K(A) \to K(B) \to TC(B)$ and $K(A) \to TC(A) \to TC(B)$. This is useful for people who want to compute the K-theory of things, mostly.

10 The comparison between K and TC (Jeremiah Heller)

Here is the advertised theorem:

Theorem 15. *Let $B \to A$ be a map of connective symmetric ring spectra such that the induced map on π_0 is surjective with nilpotent kernel. Then the square (*)*

$$\begin{tikzcd} K(B) \arrow[r, "tc"] \arrow[d] & TC(B) \arrow[d] \\ K(A) \arrow[r, "tc"'] & TC(A) \end{tikzcd}$$

is homotopy Cartesian.

(Notes: the horizontal maps are the trace maps from the previous talks, and TC was denoted $\underline{TC}$ in the previous talk. If you don't use the global TC here, you don't get an integral statement, and so you have to say things about completions.) Another way to say this is that the homotopy fibers of $K(B) \to K(A)$ and $TC(B) \to TC(A)$ agree, i.e. K-theory and TC have the same infinitesimal behavior, i.e. their difference is locally constant.

The goal of this talk is to outline a proof of this theorem, due to Dundas-Goodwillie-McCarthy. This happens in three steps:

1. The square (*) is homotopy Cartesian when the map $f : B \to A$ of simplicial rings is a split square-zero extension of simplicial rings. (Note: The Eilenberg-Mac Lane functor sending simplicial rings to connective ring spectra factors through monoids in Γ-spaces and in connective FSP (monoids in simplicial functors).)
2. The square (*) is homotopy Cartesian when $f : B \to A$ is a map of simplicial rings with the above condition on π_0.
3. Finally, we extend to all connective ring spectra, which means an approximation result about arbitrary connective ring spectra and modules over $H\mathbb{Z}$.

10.1 Outline of step 1

First, a useful fact: start with a map $f : B \to A$ of simplicial rings, which is surjective with nilpotent kernel. Let $K(f)$ denote the homotopy fiber of $K(B) \to K(A)$; the useful fact is that $K(f) \simeq \|d \mapsto K(f_d)\|$.

Similarly, we can define $TC(f)$ analogously, and we again have $TC(f) \simeq \|d \mapsto TC(f_d)\|$. A consequence of this useful fact is that we may assume A to be discrete.

Let P be an A-bimodule. We define a new ring $A \ltimes P$ (called "A fish P" :)) which as a group is $A \ltimes P = A \oplus P$ and multiplication given by $(a_1, p_1)(a_2, p_2) = (a_1a_2, a_1p_2 + p_1a_2)$. This has a surjective map $A \ltimes P \xrightarrow{p} A$ and it's a split square-zero extension of A by P (i.e., $\ker p = P$). (Note: we've assumed A is discrete, but P is allowed — encouraged! — to be simplicial.)

Recall that if $\mathcal{X}$ is an s-cube, then its iterated homotopy fiber is the homotopy fiber of the map $\mathcal{X}_\emptyset \to \operatorname{holim}_{P_0S} \mathcal{X}$. We define $F_A(P)$ to be the iterated homotopy fiber of the following 2-cube given by applying (*) to $A \ltimes P \to A$. F_A is then a functor from A-bimodules to spectra; the goal is to show that this is "highly connected," i.e., $F_A(P) \simeq *$.

Lemma 5. *If P is k-connected, then F_AP is $2k$-connected.*

Here's the idea of the proof: write $\tilde{F}(A \ltimes P) = \text{hofib } F(A \ltimes P) \to F(A)$, and consider the "diagram"

$$\begin{tikzcd}
\widetilde{K}(A \ltimes P) \arrow[r] \arrow[d] & \widetilde{TC}(A \ltimes P) \arrow[dl] \\
\widetilde{THH}(A \ltimes P) \arrow[d, dashed, "\alpha"] & \\
S^1 \wedge THH(A; P) &
\end{tikzcd}$$

("diagram" is in quotes because α is a zig-zag). The object $THH(A; P)_q$ is defined by $P \wedge A^{\wedge q}$, which doesn't have a cyclic structure; for $P = A$, $THH(A; A) = THH(A)$. The map α arises from understanding the decomposition of $THH(A \ltimes P) = \prod_{j \geq 0} THH^{(j)}(A; P)$, which is approximately $THH(A) \times (S^1 \wedge THH(A : P))$. One then shows that $\widetilde{K}(A \ltimes P) \to S^1 \wedge THH(A; P)$ and $\widetilde{TC}(A \ltimes P) \to S^1 \wedge THH(A; P)$ are sufficiently connected — $(2k + 1)$-connected, specifically.

Now consider the Cartesian square

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & BP,
\end{tikzcd}$$

which gives a map $\eta : F_A(P) \to \Omega F_A BP$. There is then another lemma: if $\Omega F_A BP$ is k-connected, then F_AP is also k-connected. This finishes step 1, since the connectivity of F_AP is equal to the connectivity of $\Omega^r F_A(B^r P)$ for all r, and by the previous lemma we can make this arbitrarily highly connected.

To prove this, we consider the cobar construction model for loopspaces. This has the advantage that it comes with a coaugmentation, i.e., a coaugmented cosimplicial object (which is really just a cube, expanded out). The remainder of the proof is then book-keeping on the resulting cube.

Now recall that given a surjection $p : E \twoheadrightarrow P$ of simplicial A-bimodules, we write $\omega(0, P, E)$ for the cosimplicial-simplicial A-bimodule with $q \mapsto \omega(0,P,E)_q = P^q \times E$ and maps

$$d^i(x_1,\dots,x_q,e) = \begin{cases} (0,x_1,\dots,x_q,e) & i=0, \\ (x_1,\dots,x_i,x_i,\dots,x_q,e) & 0<i<q, \\ (x_1,\dots,x_q,pe,e) & i=q. \end{cases}$$

For degeneracies we remove appropriate factors. If E is contractible, then $\omega(0,P,E) \simeq \Omega P$; moreover, if $I = \ker p$ then $I = \omega(0,0,I) \subseteq \omega(0,P,E)$, a coaugmentation modeling $I \simeq \Omega P$.

We apply this fact to the case $BP = EP/P$. The augmented cosimplicial simplicial A-bimodule $P \to \omega(0,BP,EP)$ gives an n-cube W^n as follows: there is a composite $\mathcal{P}\underline{n} \subseteq \mathcal{P} \subseteq Ord \subseteq \Delta \cup \{\emptyset\} \to A-Bimodules$ sending $\emptyset$ to P and q to $\omega(0,BP,EP)_q$. Define $\mathcal{X} = F_A \circ W^n$, and write F_j for the iterated homotopy fiber of $\mathcal{X}|_{\mathcal{P}\underline{j}}$. As an example, consider $j=2$: $\mathcal{X}|_{\mathcal{P}\underline{2}}$ equals

$$\begin{tikzcd} X_\emptyset \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & X_{1,2} \end{tikzcd}$$

equals

$$\begin{tikzcd} F_A(P) \arrow[r] \arrow[d] & F_A(EP) \arrow[d] \\ F_A(EP) \arrow[r] & FA(EPxBP); \end{tikzcd}$$

then $F_0 = F_A(P)$, $F_1 = F_A(P)$, and $F_2 = \mathrm{hofib}(F_A(P) \to \Omega F_A BP$.

We need two more pieces of notation: write $\overline{Pj}$ for those subjects in Pj which actually contain j, and let Φ_j be the iterated homotopy fiber of $\mathcal{X}|_{\overline{Pj}}$. As a result we have homotopy fiber sequences $F_j \to F_{j-1} \to \Phi_j$ (for $j \geq 2$).

We've reduced the situation to wanting to show two things: first that Φ_j has connectivity at least that of $\Omega F_A BP$ for all j, and then that by choosing n large enough, F_{n+1} has connectivity larger than that of $\Omega F_A BP$. This can be done, and it finishes step 1 for real.

10.2 Step 2

Let $f : B \to A$ be a map of simplicial rings which is degree-wise surjective and has degree-wise nilpotent kernel; we want to show that (*) is homotopy Cartesian.

Begin by invoking the useful fact again, we can consider the case where $B \to A$ is discrete; we write $I = \ker f$, so that there's some n with $I^n = 0$. We then consider the sequence $B/I^n \to B/I^{n-1} \to \cdots \to B/I = A$. At each stage we have a square-zero kernel, so we reduce to the case $I^2 = 0$. The idea now is to take a free resolution of A: let $F \xrightarrow{\simeq} A$ be a degree-wise free resolution, and consider the pullback of the corner $B \to A \leftarrow F$, called P, which is weakly equivalent to B. We reduce then to showing (*) for $P \to F$, and again we can consider the discrete case — but F is free, so $P \to F$ is split, and so we invoke step 1.

Now we consider $B \to A$ which has only the surjection and square-zero conditions on π_0. We can draw a square

$$\begin{tikzcd} B \arrow[r] \arrow[d] & A \arrow[d] \\ \pi_0 B \arrow[r] & \pi_0 A. \end{tikzcd}$$

We know the theorem for the bottom map, so it's enough to show the theorem for $A \to \pi_0 A$ (i.e., for 1-connected maps). This is done by the following diagram, due to Goodwillie:

$$\begin{tikzcd} B \arrow[d, "f"'] & B' \arrow[l, two heads] \arrow[d] \\ A & A', \arrow[l, two heads, "h"] \end{tikzcd}$$

where g and h are both surjective, g is $(k+1)$-connected when $B \to A$ is k-connected, and h is a square-zero extension.

10.3 Step 3

Finally we are supposed to deduce the theorem for general connective ring spectra from simplicial rings, by showing that "simplicial rings are dense in connective ring spectra, and K and TC are continuous." We have some facts to guide us:

A connective $H\mathbb{Z}$-algebra is stably equivalent to a simplicial ring.

There are maps to and from connective $H\mathbb{Z}$-algebras and connective $\mathbb{S}$-algebras (leftward is $\mathbb{Z}$, rightward is U); this gives a monad in connective $\mathbb{S}$-algebras $U\mathbb{Z}$, and so an augmented cosimplicial object in $\mathbb{S}$-algebras $A \mapsto U\mathbb{Z}^* A$. This gives a cube $s \mapsto A_s = U\mathbb{Z}^{|s|} A$. Analyzing this cube shows that $K(A) = \operatorname{holim}_{P_0 S} K(A_S)$. For $S \neq \emptyset$ these are all $H\mathbb{Z}$-algebras, and TC is also obtained in this way.

11 $\partial_* A(-)$ (Wolfgang Steimle)

We'll write $A(X)$ for the algebraic K-theory of X and $P(X)$ for the stable pseudo-isotopy of X. If M is a smooth manifold with boundary, then a *pseudoisotopy* of M is a diffeomorphism $h : M \times I \to M \times I$ such that $h|_{M\times 0} = h|_{\partial M \times I} = \mathrm{id}$. Why is this P a functor? Well, if $M \hookrightarrow N$ is a codimension 0 embedding, then we can extend by id. We can also build $P(M) \to P(I \times M)$ by the picture following the original pseudoisotopy $\psi(m, s)$ along $M \times 1/2$ and following a slanted pseudoisotopy elsewhere. (Sorry, this really needs a picture.) This gives us a notion of stabilization, and we set

$$P(X) = \lim_{n} \lim_{\{M^n \to X,\ M \text{ parallelizable}\}} P(M).$$

This gives a definition for arbitrary homotopy types. [Note: This looks weird for X nonparallelizable, but we want homotopy invariance and things are stably parallelizable — consider the tangent bundle.]

Theorem 16 (Waldhausen)**.** $A(X) \cong Q(X) \times Wh^d(X)$.

Theorem 17 (Waldhausen, Jahren, Rognes)**.** $\Omega^2 Wh^d(X) \cong P(X)$.

These connect A to things we care about.

11.1 Derivatives

Let $D_1 A(Y \to X)$ denote the linear term in the postnikov approximation $Top_{/X} \to Top$, $(Y \to X) \mapsto A(Y)$. Then $\partial_x A(X) = D_1 A(X_+ \to X)$ sending $*$ to x. We also define the parametrized loopspace $\mathcal{L}(Y \to X) = Y \times_X \mathcal{L}(X)$, for $\mathcal{L}(X)$ the free loopspace. Remark: $\Sigma^\infty_+ \mathcal{L}(Y \to X)$ is excisive as a functor $Top_{/X} \to Sp$, but it is not reduced. So, we reduce it: $red(\Sigma^\infty_+ \mathcal{L}(Y \to X)) = \mathrm{hofib}(\Sigma^\infty_+ \mathcal{L}(Y \to X) \to \Sigma^\infty_+ \mathcal{L}(X))$. Then, the main theorem:

Theorem 18 (Goodwillie)**.** *$D_1 A(Y \to X) \simeq red(\Sigma^\infty_+ \mathcal{L}(Y \to X))$, and $\partial_x A(X) = \Sigma^\infty_+ \Omega_x X$.*

We can sketch that the second part follows from the first, at least: consider $D_1 A(X_+ \to X) \simeq red\Sigma^\infty_+ \mathcal{L}(X_+ \to X) = red(\Sigma^\infty_+ \mathcal{L}(X) \times \Sigma^\infty_+ \mathcal{L}(* \to X))$. We can also state this in various other forms: $D_1 Wh^d(Y \to X) = red\Sigma^\infty \mathcal{L}(Y \to X)/Y$, and $D_1 P(Y \to X) \simeq red\Sigma^{\infty - 2} \mathcal{L}(Y \to X)/Y$. This last form is the version that we'll approach.

There's a natural transformation $\tau : P(X) \to \Omega^2 Q(\mathcal{L}(X)/X)$. For $f : Y \to X$, there is a square $P(Y) \to P(X) \to \Omega^2 Q\mathcal{L}(X)/X$ and $P(Y) \to \Omega^2 Q\mathcal{L}(Y \to X)/Y \to \Omega^2 Q\mathcal{L}(X)/X$. If f is k-connected, then we'll want to show that this square is $2k$-Cartesian. We have maps $D_1 P(Y \to X) \leftarrow hofib(P(Y) \to P(X)) \to red\Omega^2 Q\mathcal{L}(Y \to X)/Y$; the left map is $2k$-connected because P is analytic, and then both of these targets are linear, so we get the square's connectivity result we wanted. [[??]]

Now, we apply π_* to τ to get $\tau : \pi_* P(M) \to \pi_{*+2} Q\mathcal{L}(M)/M = \Omega^{fr}_{*+2}(\mathcal{L}M, M)$. Now select a pseudo-isotopy $h = (h_1, h_2) : M \times I \to M \times I$, and define $\Delta = \{(s,t) : s \leq t\}$. We consider regular crossing points, which are those (m, t_1, t_2) with $t_1 < t_2$, $h(m, t_1) = (m', t_1')$, $h(m, t_2) = (m', t_2')$, and $t_2' < t_1'$. We also consider infinitesimal crossing points, which are those with $h_1'(m,s) = 0$ and $h_2'(m,s) < 0$, i.e., the limit of a regular crossing.

Now select a map $\phi : (D^i, S^{i-1}) \to P(M)$, and define $W = \{(z, m, s, t) \in D^i \times M \times \Delta \mid (m,s,t) \text{ is a crossing of } \phi(z).\}$. If we put ϕ into general position, then W is a submanifold of codimension m ($= \dim M$), and it is even stably framed. We then build

$$\begin{tikzcd}
D^1 \times M \times \Delta \arrow[r, "\phi_{left} \times \phi_{right}"] & M \times M \\
W \arrow[u] \arrow[r] & M. \arrow[u, "\Delta"']
\end{tikzcd}$$

This gives a map $W \to \mathcal{L}M$ restricting to $\partial W \to M$, the constant loops, i.e., $(W, \partial W) \in \Omega^{fr}_?(\mathcal{L}M, M)$.

Now suppose $N \subseteq M$ is a submanifold. A pseudoisotopy embedding is a map $N \times I \xrightarrow{h} M \times I$ with $h|_{N \times 0}$ and $h|_{\partial N \times I}$ the inclusions and with $h|_{N \times 1} \subseteq M \times 1$. We write $PE(N, M)$ for the simplicial set of such embeddings. Now take $N \subseteq M$ of codimension zero; then the sequence $P(N) \to P(M) \to PE(\overline{M \setminus N}, M)$ is a fibration sequence. If M decomposes as N and a handle H, then this gives $PE(coreH, M)$. We use this to extend our square:

$$\begin{tikzcd}
P(N) \arrow[r] \arrow[d, "\tau"] & P(M) \arrow[r] \arrow[d, "\tau"] & PE(x, M) \arrow[d, "\tau"] \\
\Omega^2 Q\mathcal{L}(N)/N \arrow[r] & \Omega^2 Q\mathcal{L}(M)/M \arrow[r] & \Omega^2 Q(\Sigma^m \Omega_x M).
\end{tikzcd}$$

The sequence on top is a fiber sequence by definition; the one on bottom comes from using the decomposition $M = N \cup_{\partial N} H$ for a handle H with core x. Once again, this last map turns out to be $2k$-connected. [Note: a PE without crossings can be homotoped to one with pointwise constant speed, i.e., it's level-preserving.

The space of level-preserving guys is in turn contractible. This is the main idea of the connectivity proof for this new τ.]

11.2 Higher derivatives

There turn out to be easiest to study using algebraic K-theory proper, or with THH. Recall that $A(X) = K(S[\Omega X])$, where the Kan loop group ΩX is the topological group with $B\Omega X = X$. We built a trace map $A(X) \xrightarrow{\mathrm{tr}} THH(S[\Omega X])$, where $THH(S[\Omega X])$ is the cyclic realization of $n \mapsto S[\Omega X] \wedge \cdots \wedge S[\Omega X] \simeq S[\Omega X \times \cdots \times \Omega X]$. Hence, $THH(S[\Omega X]) \simeq S \wedge B^{cy}(\Omega X)_+$. There is a fiber diagram

$$\begin{tikzcd}
G \arrow[r] \arrow[d, "\wr"] & B^{cy}G \arrow[r] \arrow[d, "\wr"] & BG \arrow[d, equal] \\
\Omega BG \arrow[r] & \mathcal{L}BG \arrow[r] & BG
\end{tikzcd}$$

with all vertical arrows homotopy equivalences, and hence $S \wedge B^{cy}(\Omega X)_+ \simeq S \wedge \mathcal{L}(X)_+$. (All this is due to Waldhausen).

Now we use the trace map $A(X) \xrightarrow{\mathrm{tr}} Q_+\mathcal{L}X$ rather than $P(X) \to \Omega^2 Q\mathcal{L}(X)/X$. Fact: there is a factorization of the trace $A(X) \to Q_+\mathcal{L}X^{hS^1} \to Q_+\mathcal{L}X$. In turn, when we study the derivatives of the trace, the derivatives also factor through these fixed points, and using models for the fixed points turns the interesting map in the factorization into $\Sigma^\infty_+\Omega_x X \to maps(S^1_+, \Sigma^\infty_+\Omega_x X)$. There is a family affairs theorem:

Theorem 19 (Bokstedt, Carlsson, G?, Goodwillie, Hsing, Madsen)**.** *The uninteresting map in the factorization is a weak equivalence.*

A similar description holds for higher derivatives:

Theorem 20 (Goodwillie)**.** $\partial_{x_1,\ldots,x_n}A(X) = D_1A(X_+ \to X, \ldots, X_+ \to X)$ *by* $* \mapsto x_1, \ldots, * \mapsto x_n$*. Then* $\partial_{x_1,\ldots,x_n}A(X) \to (\partial_{x_1,\ldots,x_n}\Sigma^\infty_+\mathcal{L}X)^{hS^1}$ *is a homotopy equivalence.*

[By corollary, you can calculate this formula at a point pretty easily, but he wrote it illegibly on the board.]

12 Interactions with chromatic homotopy theory (Markus Szymik)

We're interested ina result of Kuhn, from his paper *Tate cohomology and periodic localization of polynomial functors*. We'll want a homotopy functor $F : Sp \to Sp$, and the best example to keep in mind is $F = \Sigma^\infty\Omega^\infty$ since we're going to deduce something about all F from this particular F. Of course, we have a Goodwillie tower for F, and Kuhn's main result is that the maps $p_n : P_{n+1}F \to P_nF$ admit natural homotopy sections after any periodic localization (we'll abbreviate to just saying "locally".) In the end, this means F splits as a product of its derivatives: $F \simeq \prod_n D_nF$.

12.1 Periodic localizations

Fix a prime p; we'll use $p = 2$, but odd primes will follow identically. Let $n > 0$ be an integer. For each such pair we define $K(n)$ with $\pi_*K(n) = \mathbb{F}_p[v_n^\pm]$ and $|v_n| = 2(p^n - 1)$. We also receive Bousfield localization $L_{K(n)}$ with respect to this functor. When $n = 1$, $K(1)$ is a minimal summand of mod 2 K-theory. We can also define $T(n)$ to be the telescope of a v_n-self-map for a finite complex F of type n ($K(m)F$ vanishes for $m < n$ and does not vanish for $m = n$). For example, $n = 1$ has $F = M(2) = S/2$, and Adams shows there is a map inducing $v_1^4 : \Sigma^8 M(2) \to M(2)$, so $T(1) = v_1^{-4} M(2)$. We can then also define $L_{T(n)}$, the Bousfield localization again $T(n)$; this is independent of choice of F. There is a map $L_{T(n)} \to L_{K(n)} L_{T(n)} \simeq L_{K(n)}$. When $n = 1$ this is an equivalence, but for $n > 1$ this is known as the telescope conjecture, largely believed to be false. For us, periodic localization L will mean either of those concepts.

12.2 The Bousfield-Kuhn functors

There are functor $\Phi : \mathit{Spaces} \to \mathit{Spectra}$ such that $L = \Phi\Omega^\infty$. (This is due to Bousfield for $n = 1$ and to Kuhn for $n > 1$.) By corollary, the evaluation map $\Sigma^\infty\Omega^\infty \to id$ has a section locally; to see this, use the adjunction to get $\Omega^\infty \to \Omega^\infty\Sigma^\infty\Omega^\infty \to \Omega^\infty$ factoring the identity. Then apply Φ to get $L \to L\Sigma^\infty\Omega^\infty \to L$, again factoring the identity. For the tower of $\Sigma^\infty\Omega^\infty$, we get a map $P_1 = \mathrm{id} \to \Sigma^\infty\Omega^\infty \to P_p$, a section of the map $P_p \to P_1$. Because we have a section, this means that the connecting map $\delta : id \to \Sigma D_p$ is zero. Now, recall what the Ds look like: $D_d X = (X^{\wedge d})_{h\Sigma_d}$. Let's evaluate this tower on the spheres S^{-k}, giving $\Sigma^k LidS^k \xrightarrow{0} \Sigma^{k+1} LD_p S^{-k}$, and passing to the limit gives $LS = \lim_k \Sigma^k LidS^{-k}$ mapping by $0 = \lim L\delta$ to $\lim \Sigma^{k+1} LD_p S^{-k}$. These spectra turn out ot be instances of Tate spectra.

12.3 Tate spectra

Take G to be a finite group and M to be a G-spectrum (e.g., a change of universe from a naive spectrum with trivial G-action). Then we have the diagram of cofiber sequences

$$\begin{tikzcd}
M \wedge EG_+ \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M \wedge \widetilde{EG} \arrow[d] \\
F(EG_+, M) \wedge EG_+ \arrow[r] & F(EG_+, M) \arrow[r] & F(EG_+, M) \wedge \widetilde{EG}.
\end{tikzcd}$$

We define $T_G(M) = (F(EG_+, M) \wedge \widetilde{EG})^G$. For example, if G acts freely on $S(V)$ and M is trivial, then $T_G M = \lim(BG^{-kV} \wedge \Sigma M)$, because $EG = S(\infty V)$ and $\widetilde{EG} = S^{\infty V}$. As a sub-example, if $G = C_2$, then $T_{C_2} M = \lim \mathbb{R}P^\infty_{-k} \wedge \Sigma M$ and $T_{C_2} S = \lim \Sigma^{k+1} DS^{-k}$ (see Greenlees and May from the 90s).

If you're careful about tracking localizations, you can prove Lem: $LT_{C_2}(LS) = \lim \Sigma^{k+1} LDS^{-k}$. Thm: For all finite G and all LS-modules M, $LT_G M = pt$. It's hard to say why this theorem is interesting, but it has a long history with Bousfield, Hovey, Mahowald, Sadofsky, ..., all being kicked around in various forms and strengths. In the proof, you reduce to the case $G = C_p$ and $M = LS$. The Tate spectra are all ring spectra, so we will show the unit is zero.

$$\begin{tikzcd}
S \arrow[r] \arrow[d, equal] & T_{C_2} S \arrow[r] \arrow[d] & LT_{C_2} LS \arrow[d] \\
S \arrow[r, "(1)"] & \lim \Sigma^{k+1} DS^{-k} \arrow[r] & \lim \Sigma^{k+1} LDS^{-k} \\
 & S \arrow[r] \arrow[u, "(2)", "\lim \delta"'] & LS. \arrow[u, "\lim L\delta = 0"']
\end{tikzcd}$$

If we can say enough about maps (1) and (2), then we can produce a dashed factorization crossing their sources, which will make eta wind all the way around the diagram and through the zero map. The map (1) is analyzed by Lin at $p = 2$ and by Gua? for $p > 2$, where they find it is p-completion. So, we need to check that $\lim \delta$ is an equivalence in mod 2 homology, since its source can be replaced by the 2-adic sphere. Kuhn checks that this is so, and we are done.

Now to the proof of the main theorem! We have the diagram of fiber sequences

$$\begin{tikzcd}
D_d F \arrow[r] & P_d F \arrow[r] & P_{d-1} F \\
(\Delta_d F)_{h\Sigma_d} \arrow[r] \arrow[u] & (\Delta_d F)^{h\Sigma_d} \arrow[r] \arrow[u] & T_{\Sigma_d}(\Delta_d F). \arrow[u]
\end{tikzcd}$$

This starts with Goodwillie's identification of $D_d F$ and then continues with McCarthy's dual calculus to describe the rest of the bottom row. Since the left vertical map is a weak equivalence, we see that we're done when we can move into a localization situation – we need F to be local, not just apply localization eveywhere. That's the content of the following lemma: if $F \to G$ is a local equivalence, then so are the $P_d F \to P_d G$ and the $D_d F \to D_d G$ maps. This follows from examining the constructions of the P_d and D_d guys. So then $F \to LF$ is what we want, and we can apply the lemma to finish the argument.