Quantum cohomology and quantum groups

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This is joint work with Varchenko, Rimanyi, and Tarasov. It is based on work of Okounkov and Gepner (different Gepner). Gepner was studying the *fusion algebra*, a feature of WZW field theory, namely the one generated by representations of $SU(n)$ "at level k". This connects to work of Schubert and Schur on Grassmannians, who (in particular) obtained a nice description of $H^*(Gr(k,n+k)) = \mathbb{C}[e_1,\ldots,e_k,\overline{e_1},\ldots,\overline{e_k}]/(1+e_1+\ldots+e_k)(1+\overline{e_1}+\ldots+\overline{e_k}) = 1$. Now, Gepner discovered exactly the same structure in a totally different context in low-energy physics. This talk is constructed by being totally naive about topology and only using a bit of representation theory.

The Lie algebra gl_N has a basis given by the e_{ij}. This has the obvious representation on $\mathbb{C}^N$, and from this we build $V = (\mathbb{C}^N)^{\otimes n}$. This admits a *weight decomposition*: we say that $v \in V$ has weight $(\lambda_1,\ldots,\lambda_N)$ if $e_{ii}v = \lambda_i v$, and then we have

$$V = \bigoplus_{\lambda \in (\mathbb{Z}_+)^N \text{ s.t. } |\lambda|=n} V_\lambda.$$

Now, V has a basis of the form $v_{i_1} \otimes \ldots \otimes v_{i_n}$, and one can use this to check that $\dim V_\lambda = n!/(\lambda_1! \cdots \lambda_N!)$.

Example 1. Take $N = n = 2$. Then we have $V_{2,0} = \langle e_1 \otimes e_1 \rangle$, $V_{1,1} = \langle e_1 \otimes e_2, e_2 \otimes e_1 \rangle$, and $V_{0,2} = \langle e_2 \otimes e_2$.

In order to get ourselves to the WZW model, we consider V-valued functions in the variables $z_1,\ldots,z_n,h$, which we will denote $\mathcal{V} = V \otimes \mathbb{C}[z_1,\ldots,z_n,h]$. There are several interesting objects acting on the space of these functions. In particular, there are two actions of the symmetric group S_n, which we will call S_n^+ and S_n^-; these actions will allow us to define the invariant part of the functions. Precisely, the i^{th} transpositions act as

$$s_i^\pm : f(z_1,\ldots,z_n,h) \mapsto \frac{(z_i - z_{i+1})P^{i,i+1} \pm h}{z_i - z_{i+1}} f(\ldots,z_{i+1},z_i,\ldots) \pm \frac{h}{z_i - z_{i+1}} f(z_1,\ldots,z_n,h).$$

We also have an action of a particular matrix algebra, called the *Bethe algebra*. This admits a few different but equivalent definitions. The one we will use is as the maximal commutative subalgebra in a very noncommutative algebra, the Yangian $Y(gl_N)$. (We could also define it as the maximal commutative subalgebra in the current algebra $gl_N[t]$, a vertex algebra.) We choose this because it will play nicely with the topological side. One might think that it's unlikely to have a noncommutative algebra acting on a cohomology ring, but actually the Yangian will act on a number of flag varieties all taken together, and the action won't preserve any of them.

The Yangian is a quantum group (i.e. a Hopf algebra), generated by symbols

$$T_{ij}(u) = \delta_{ij} + \sum_{s=1}^{\infty} T_{ij}^{(s)} u^{-s}$$

for $i,j = 1,\ldots,N$, with the relations determined by the identity

$$(u-v)[T_{ij}(u), T_{kl}(v)] = T_{kj}(v)T_{il}(u) - T_{kj}(u)T_{il}(v).$$

(This contains the universal envelope of gl_N via $e_{ij} \mapsto T_{ij}^{(1)}$.)

We now define the Bethe algebra $\mathcal{B} \subset Y(gl_N)$ via the *quantum determinant*. If we set $\vec{i} = (i_1,\ldots,i_p)$ and $\vec{j} = (j_1,\ldots,j_p)$, then we define

$$M_{\vec{i}\vec{j}} = \sum_{\sigma \in \Sigma_p} (-1)^{|\sigma|} T_{i,\sigma(1)}(u) \cdots T_{i_p,\sigma(p)}(n+1-p),$$

and then we take $\mathcal{B} = \{M_{\vec{i}\vec{i}}\}$. Without giving more formulas, we'll simply say that $Y(gl_N)$ (and hence $\mathcal{B}$) acts on $\mathcal{V}$ inducing an interesting weight decomposition. Instead, we'll turn to the *topological mirror* of this story, which admits a description using the Schubert calculus.

On the topological side, we have a bunch of flag spaces parametrized by partitions $\lambda = (\lambda_1, \ldots, \lambda_N)$ of n (i.e. $|\lambda| = n$). Recall that a *flag* is a sequence of nested subspaces, and we denote

$$F_\lambda = \{0 = F^0 \subset F^1 \subset \ldots \subset F^N = \mathbb{C}^n : \dim F_{i+1}/F_i = \lambda_i\}.$$

Now, the torus $T = T^n$ acts on F_λ, and the fixedpoint data is given on the left hand side by T^*F_λ.

Let us turn to quantum cohomology. In the standard picture of cup products as intersections, nonintersecting cycles yield zero product. However, Witten altered this to allow nonzero products between cycles that have a particular sort of curve intersecting all of them. The standard example is that $H^*(\mathbb{P}^n) = \mathbb{C}[x]/x^{n+1} = 0$, whereas $QH^*(\mathbb{P}^n) = \mathbb{C}[x, q]/x^{n+1} = q$. Givental wanted to work not just with these formal deformations, but instead assigned a (singular) differential equation $q \cdot \frac{d}{dq} f = x * f$; this not only encodes the quantum cohomology algebra, but also much more.

Theorem 1. *The differential equation on the representation theoretic side is equal to the differential equation on the quantum cohomology side.*

We finally come to the main point. It arises from the naive-looking question: What happens on the topology side when we specialize the value of q? This is already interesting for the cotangent space $T^*\mathbb{P}^1$, which can be recognized as a blowup. It has been proved that the specialization to $q = 1$ of its quantum cohomology pushes down to $\mathbb{P}^1$. Conjecturally, this relationship persists in great generality.