Recent developments in chromatic stable homotopy theory

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This talk is based on various joint work with Goerss, Mahowald, Rezk, and Karamanov. We will fix a prime p for once and for all.

We begin with a rapid introduction to chromatic stable homotopy theory. There is a *height filtration* in the theory of (1-dimensional commutative) formal groups; this corresponds to the *chromatic filtration* in stable homotopy theory. In particular, there is a sequence of Bousfield localization functors $L_n = L_{K(0)\vee\ldots\vee K(n)}$, where the $K(i)$ are the *Morava K-theories*, which are complex-orientable cohomology theories with $K(n)_* = \mathbb{F}_p[v_n^{\pm1}]$ (where $|v_n| = 2(p^n-1)$). This yields the *chromatic tower* $\cdots \to L_nX \to L_{n-1}X \to \cdots \to L_0X$; this admits a map from X, and the *chromatic convergence theorem* says that under mild assumptions, the limit is the p-localization of X (and the tower is actually *pro-isomorphic* to the constant tower). This tower is inductively built up from the *chromatic square*, which is the pullback diagram

$$\begin{array}{ccc} L_nX & \longrightarrow & L_{K(n)}X \\ \downarrow & & \downarrow \\ L_{n-1}X & \longrightarrow & L_{n-1}L_{K(n)}X. \end{array}$$

So, in theory if we understand the $L_{K(n)}X$ then we can understand X. When X is a finite spectrum, there is a spectral sequence $E_2^{s,t} = H^s_{cts}(\mathbb{G}_n, (E_n)_tX) \Rightarrow \pi_{t-s}L_{K(n)}X$; this is a consequence of a theorem of Devinatz and Hopkins, that $(E_n)^{h\mathbb{G}_n} \simeq L_{K(n)}S^0$. (Then the Adams-Novikov spectral sequence becomes a descent spectral sequence.)

Let us quickly describe the bottom two cases. First, at $n = 0$ we have $K(0) = H\mathbb{Q}$; this case is rather trivial. (This is because $\mathbb{Q} = \mathbb{Z}_{(p)}[p^{\pm1}]$.) Next, at $n = 1$, let us write $KU\mathbb{Z}_p$ for p-completed KU; this admits an action by $\mathbb{Z}_p$ via the Adams operations. The units $\mathbb{Z}_p^\times$ are topologically cyclic, and if μ is any generator, then (by work of Adams, Baird, Bousfield, and Ravenel) there is a fiber sequence $L_{K(1)}S^0 \to (KU\mathbb{Z}_p)^{h\mu} \xrightarrow{\psi^{p+1}-\mathrm{id}} (KU\mathbb{Z}_p)^{h\mu}$.

The case $n = 2$ is the edge of our knowledge. Shimomura and collaborators have given extensive calculations for $p \geq 3$ and partial results for $p = 2$; however, there seems to be approximately nobody who really understands this work. (There is a recent paper of Behrens which simply reinterprets Shimomura's computations for $p > 3$.)

The cohomological properties of the ("full") Morava stabilizer groups $\mathbb{G}_n$ essentially govern the $K(n)$-local stable homotopy category. This decomposes as $\mathbb{G}_n = \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n} : \mathbb{F}_p)$. Here, $\mathbb{S}_n = \mathrm{Aut}_{\mathbb{F}_{p^n}}(F_n)$ is the ("classical") Morava stabilizer group, the automorphisms of the Honda formal group of height n over $\mathbb{F}_p$. Now, the associated complex-orientable theory is the *Morava E-theory* E_n, which has $(E_n)_* \cong \mathbb{W}_{\mathbb{F}_{p^n}}[[u_1,\ldots,u_{n-1}]][u^{\pm1}]$ (where $|u_i| = 0$ and $|u| = -2$).

This is totally intractable except for the case $n = 1$, where we have $\mathbb{G}_1 = \mathbb{Z}_p^\times$. This acts tautologically on $(E_1)_{-2} \cong \mathbb{Z}_pu$, and the action extends multiplicatively. In fact, $E_1 = KU\mathbb{Z}_p$. Now, we would like to relate the above fibration to the given spectral sequence. To compute the group cohomology, we need a resolution of the trivial $\mathbb{S}_1$-module $\mathbb{Z}_p$. (Recall that $\mathbb{S}_1 = \mathbb{Z}_p^\times$.) Now, $\mathbb{G}_1 = \mathbb{Z}_p^\times \cong U_1 \times \mu$ where $U_1 \subset \mathbb{Z}_p^\times$ are the units that are congruent to 1 mod p. So, we can begina resolution by mapping down to $\mathbb{Z}_p$ from $\mathbb{Z}_p[[\mathbb{G}_1/\mu]] \cong \mathbb{Z}_p[[\mathbb{Z}_p]]$. As it turns out, this gives rise to a two-stage resolution $C_\bullet$. In fact, the above fibration realizes this complex as its homotopy; Shapiro's lemma yields an isomorphism $\mathrm{Hom}_{\mathbb{Z}_p[[\mathbb{G}_1]]}(C_0,(E_1)_*) \cong \mathrm{Hom}_{\mathbb{Z}_p[\mu]}(\mathbb{Z}_p,(E_1)_*)$, which at $n = 1$ makes understandable the isomorphism $(E_n)_*(X_\bullet) \cong \mathrm{Hom}_{\mathbb{Z}_p[[\mathbb{G}_1]]}(C_\bullet,(E_n)_*$ (for a "resolution" $X_\bullet$ of $L_{K(2)}S^0$ that we will get to in a moment). The fact that $C_\bullet$ is a two-stage complex means there are only two cohomology groups, and these give rise to the kernel and cokernel of the map in homotopy induced from $\psi^{p+1} - \mathrm{id} : (KU\mathbb{Z}_p)^{h\mu} \to (KU\mathbb{Z}_p)^{h\mu}$.

We would like to do this at $n = 2$. To get at $H^*(\mathbb{G}_2,(E_2)_*X)$, we begin with $H^*(\mathbb{G}_2,\mathbb{F}_p[u^{\pm1}])$ (since $\mathbb{F}_p[u^{\pm1}] = (E_2)_*/(p,u_1)$). Very luckily, $\mathbb{G}_2$ is a virtual Poincaré duality group of dimension 4. For $p > 2$, $\mathbb{G}_2 \cong (\mathbb{G}_2)^\wedge \times \mathbb{Z}_p$, and the first factor is a Poincaré duality group of dimension 3 for $p > 3$ (virtual for $p = 3$). So, we really only need to understand the first cohomology. It turns out that the 0^{th} and 3^{rd} cohomology groups are $\mathbb{F}_p$, and the 1^{st} and 2^{nd} are $\mathbb{F}_p^2$.

Now, we can actually reduce even further to computing $H^*(S_2, \mathbb{F}_{p^2})$; here, $S_2 < \mathbb{S}_2$ a p-Sylow subgroup, and $\mathbb{S}_2 \cong S_2 \rtimes \mu\mathbb{W}$ (where $\mu\mathbb{W}$ is the roots of unity in the Witt vectors). So in fact, we want a minimal resolution of the trivial S_2-module $\mathbb{Z}_p$. This admits a four-stage resolution, where $C_0 \cong C_3 \cong \mathbb{Z}_p[S_2]$ and $C_1 \cong C_2 \cong \mathbb{Z}_p[S_2]^{\oplus 2}$. (Of course, this doesn't tell us about the differentials in the spectral sequence, but that's a whole other story.) We can actually consider this as a complex of modules for $\mathbb{S}_2$ (or even for $\mathbb{G}_2$, although this isn't so helpful), and this lifts to a complex of spectra which finally gives us our "resolution" $X_\bullet = L_{K(2)}S^0 \to X_0 \to X_1 \to X_2 \to X_3 \to X_4$. (To relate this all to the title of the conference, it turns out that $X_0 \simeq X_3 \simeq L_{K(2)}tmf$, and X_1 is a variant with level structure, and the map $X_0 \to X_1$ is the difference of maps induced by the functors $(C, H \subset C) \mapsto C$ and $(C, H \subset C) \mapsto C/H$, just like Nora talked about.) This is a a generalization of the fibration description of $L_{K(1)}S^0$ in terms of homotopy fixedpoint spectra. Although these spaces are still difficult, this resolution gives us a handle on where the various pieces of $L_{K(2)}S^0$ are coming from. The thesis of Keramanov makes this complex "explicit".

As for applications, Henn-Mahowald-Rezk at $p = 3$ calculated $\pi_* L_{K(2)}V(1)$, completing some work of Shimomura. Lader (a student of Henn) is currently trying to generalize this to $p \geq 3$. Moreover, Henn-Kalmanazov-Mahowald have computed $\pi_* L_{K(2)}V(0)$, correcting work of Shimomura. (This calculation can be fed into a computation of $\pi_* L_{K(2)}S^0 \otimes \mathbb{Q}_p$, which indicates that Shimomura's work must not be entirely correct.)