Circle actions on manifolds and complex elliptic genera

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This topic is rather classical, relating to the origins of the interactions between homotopy theory and geometry. However, there are a number of still-unexplored corners.

1 Genera for oriented and spin manifolds

For us, a *genus* is a ring homomorphism $\varphi : \Omega^{SO} \otimes \mathbb{Q} \to A$ where A is some $\mathbb{Q}$-algebra. This is computable by characteristic numbers. For spin manifolds, recall that we have the short exact sequence $1 \to \mathbb{Z}/2 \to \mathrm{Spin}(n) \xrightarrow{\lambda_2} SO(n) \to 1$; a *spin structure* on an oriented manifold X, considered as having a chosen $SO(n)$-principal bundle structure $SO(n) \to P \to X$, is a $\mathrm{Spin}(n)$-principal bundle $\mathrm{Spin}(n) \to Q \to X$ such that the diagram

$$\begin{array}{ccc} Q & \longleftarrow & \mathrm{Spin}(n) \\ \downarrow & & \downarrow \lambda_2 \\ P & \longleftarrow & SO(n) \\ \downarrow & & \\ X & & \end{array}$$

commutes. Of course, this exists iff $w_2(X) = 0$. Now, if we have an S^1-action on X, this lifts to an action on P; if the action lifts to Q, we call the action *even*; otherwise, we call it *odd*. We write $I_*^{\mathrm{Spin}}, I_*^{\mathrm{Spin},\ odd} \subset \Omega_*^{\mathrm{Spin}} \otimes \mathbb{Q} \simeq \Omega^{SO} \otimes \mathbb{Q}$ for the ideals generated by spin manifolds with effective S^1-action and odd S^1-action, respectively. Of course, it is natural to ask about the structure of these two ideals.

Theorem 1 (Atiyah-Hirzebruch, 1970)**.** $I_*^{\mathrm{Spin}} = \ker(\hat{A})$*, where* $\hat{A} : \Omega_*^{\mathrm{Spin}} \otimes \mathbb{Q} \to \mathbb{Q}$ *is the index of the Dirac operator.*

Proof sketch. For the inclusion $\subset$, one applies the equivariant Atiyah-Singer index theorem. For the inclusion $\supset$, one constructs enough manifolds (as it turns out, quaternionic projective spaces) to generate $\ker(\hat{A})$. $\square$

Theorem 2 (Landweber-Bosari-Ochanine-Witten)**.** $I_*^{\mathrm{Spin},\ odd} = \ker(\varphi_2) = \mathrm{sign}(q, \mathcal{L}X)$*, where* $\varphi_2 : \Omega_*^{\mathrm{Spin}} \otimes \mathbb{Q} \to \mathbb{Q}[\delta, \varepsilon] \subset \mathbb{Q}[[q]]$ *(via q-expansion), and* sign *denotes the signature.*

2 Genera for stably almost complex manifolds

For us, a *stably almost complex manifold* is a manifold with a complex structure on the stable tangent bundle. These have the bordism ring Ω_*^U, and $\Omega_*^U \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{C}P_1, \mathbb{C}P_2, \ldots]$. We now have the short exact sequence $1 \to \mathbb{Z}/N\mathbb{Z} \to U(1) \xrightarrow{\lambda_N} U(1) \to 1$, and we define a Spin-$N$ *manifold* as a lift (analogously to above) of the determinant bundle for the $U(n)$-bundle associated to the almost complex structure on a manifold X. Such a structure S exists iff $N | c_1(X)$. When it does, it defines a line bundle L over X with $L^{\otimes N} = \det TX$. Given an S^1-action on X, it has a *type* $t \in \mathbb{Z}/N\mathbb{Z}$, which is the obstruction to a lifting to S. As before, we have the ideals $I_*^{N,t} \subset \Omega_*^{U,N} \otimes \mathbb{Q} \simeq \Omega_*^U \otimes \mathbb{Q}$.

Theorem 3 (H)**.** $I_*^{N,0} = \bigcap_{i=1}^{N-1} \ker \chi(-, L^{\otimes i})$*, where* χ *is the Todd genus, and* $I_*^{N,1} = \bigcap_{n>1,\ n|N} \ker(\varphi_n)$*.*

Recall that χ can be considered as the index of the Dolbeault operator ∂_*. Moreover, we have $\varphi_{ell} : \Omega_*^U \otimes \mathbb{Q} \to \mathbb{Q}[A, B, C, D] \subset \mathbb{Q}[Y, Y^{-1}][[q]]$ (where $|A| = 1$, $|B| = 2$, $|C| = 3$, and $|D| = 4$), where

$$\varphi_{ell}(X) = \chi_Y(q, \mathcal{L}X) = \chi_Y \left(X, \bigotimes_{n=1}^{\infty} \Lambda_{Yq^n} T^* \otimes \bigotimes_{n=1}^{\infty} \Lambda_{Y^{-1}q^n} T \otimes \bigotimes_{n=1}^{\infty} S_{q^n}(T \oplus T^*) \right);$$

here, $\chi_Y = \sum \mathrm{Ind}(\partial_*^p) \cdot Y^p$ is the index of the Dolbeault operator acting on holomorphic p-forms, and $\Lambda_t E = \bigoplus_i \Lambda^i E \cdot t^i$, and $S_t E = \bigoplus_i S^i E \cdot t^i$, and $\varphi_n = \chi(q, \mathcal{L}X)|_{Y=e^{2\pi i/n}}$ takes values in $M_*(\Gamma_1(n))$.

Now, there is the following recent observation: $\chi_Y(q, \mathcal{L}X) = \chi_Y(X) + \chi_Y(X, T^* \oplus T \oplus T^*Y + TY^{-1})q + \ldots$. When X is K3, one coefficient here is $90 = 2 \cdot 45$; this factor 45 is the dimension of a particular irreducible representation of M_{24}. In fact, all coefficients are the dimensions of representations of M_{24}; this is suggested by physics. If we write $\chi_Y(q, \mathcal{L}\mathrm{K3})$ in terms of representations of the $N = 4$ supersymmetric Viraso algebra, as physics suggests. This all leads to "Mathieu moonshine", which was proved in 2010 by three Japanese mathematical physicists.

We would like a string-theoretic action of M_{24} on a K3 surface; an action of M_{23} was given by Mukai (and later explained further by Kondo). One can examine the conjugacy classes of geometrically realizable elements $g \in M_{23} \subset M_{24}$, and look at the g-equivariant elliptic genus $\chi_Y(g; q, \mathcal{L}\mathrm{K3})$. This agrees with the McKay-Thompson series.

Let us compare the situation to monstrous moonshine. In 1979, Conway conjectured that in the expansion $j - 744 = q^{-1} + 0 + 196884q + \cdots$, all the coefficients are dimensions of irreducible representations of the monster group. Hirzebruch explained this as an action on X_{24}, and Borcherds finally gave a full proof of the conjecture.

We end with a conjecture: $I_*^{N,t} = \bigcap_{n>1, t \nmid N} \ker \varphi_N$. The speaker's thesis student N. Ahmed has proved that this is true for $N < 12$ and $(N, t) \neq (6, 3)$ and $(N, t) \neq (10, 5)$.