Transchromatic generalized character maps (and more!)

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Fix a prime p, and let E_n be the n^{th} Morava E-theory. Then $E_n^0 \cong W(k)[[u_1, \ldots, u_{n-1}]]$. Choose any integer $t \in [0, n)$; then $L_t = (L_{K(t)}E_n)^0 \cong W(k)[[u_1, \ldots, u_{n-1}]][u_t^{-1}]^{\wedge}_{(p,u_1,\ldots,u_{t-1})}$, i.e. $K(t)$-localizing affects the (0^{th}) homotopy by inverting u_t and then re-completing at the ideal of all the u_i for $i < t$.

Now, recall that to E_n is associated the p-divisible group $\mathbb{G}_{E_n}[p] \to \mathbb{G}_{E_n}[p^2] \to \cdots$ (which we will notate simply as $\mathbb{G}_{E_n}$). Let us write $\mathbb{G}_{L_t}$ for the formal or p-divisible group associated to $L_{K(t)}E_n$. Then $\mathcal{O}_{\mathbb{G}_{E_n}[p^k]} \cong E_n^0(B\mathbb{Z}/p^k)$, a free E_n^0-module of rank p^{kn}, and $\mathcal{O}_{\mathbb{G}_{L_t}[p^k]} \cong L_{K(t)}E_n^0(B\mathbb{Z}/p^k)$, a free L_t-module of rank p^{kt}.

Our goal is to approximate E_n by using a lower-height cohomology theory (i.e. $L_{K(t)}E_n$). The easy case is that when X is a finite CW-complex; then $L_t \otimes_{E_n^0} E_n^0 X \xrightarrow{\cong} L_{K(t)}E_n^0 X$. But let's consider the harder case $X = BG$ for G a finite group. For example, let's take $G = \mathbb{Z}/p^k$. Then we have the canonical map $L_t \otimes_{E_n^0} E_n^0(B\mathbb{Z}/p^k) \to L_{K(t)}E_n^0(B\mathbb{Z}/p^k)$. Simply by counting ranks, we can see that this cannot be an isomorphism. What are we missing?

Consider the pullback p-divisible group $L_t \otimes \mathbb{G}_{E_n}$; the constituents are $L_t \otimes \mathbb{G}_{E_n}[p^k] = \text{Spec } L_t \times_{\text{Spec } E_n^0} \mathbb{G}_{E_n}[p^k]$. It turns out that the formal-étaleshort exact sequence here takes the form

$$0 \to \mathbb{G}_{L_t} \to L_t \otimes \mathbb{G}_{E_n} \to \mathbb{G}_{\text{ét}} \to 0,$$

where the formal part has height t and the étalepart has height $n - t$. So, it's $\mathbb{G}_{\text{ét}}$ that we weren't taking into account. Over L_t this is tricky, so our solution is to construct what we will call C_t, the univesal L_t-algebra equipped with an isomorphism $C_t \otimes \mathbb{G}_{E_n} \cong (C_t \otimes \mathbb{G}_{L_t}) \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^{n-t}$: not only does the sequence split, but the étalepart is actually made constant. We will write $\mathbb{G}_{C_t} = C_t \otimes \mathbb{G}_{L_t}$.

Lemma 1. *C_t is a ff [faithfully flat???] L_t-algebra.*

For a group G, let us write

$$G_p^{n-t} = \text{Hom}(\mathbb{Z}_p^{n-t}, G) = \{(g_1, \ldots, g_{n-t}) : [g_i, g_j] = e,\ g_i^{p^k} = e \text{ for suff. large } k\}.$$

Theorem 1 (S, Hopkins-Kuhn-Ravenel)**.** *There is a character map*

$$\phi_G^t : E_n^0(BG) \to \prod_{[\alpha] \in G_p^{n-t}/\sim} C_t \otimes_{L_t} L_{K(t)}E_n^0(B(C(\text{im } \alpha))),$$

and this becomes an isomorphism when we tensor the source up to C_t.

Note that when $t = 0$, $C_0 \otimes \mathbb{G}_{E_n} = (\mathbb{Q}_p/\mathbb{Z}_p)^n$, so that we recover the Hopkins-Kuhn-Ravenel character map. C_0 is a $p^{-1}E_n^0$-algebra, so $C_0^0(BG) = C_0^0$ and the codomain takes the form $Cl(G_p^n, C_0^0)$ (the class functions). In the particular case that $n = 1$, then $E_1 = K_p$, and then this specializes to the Chern character.

We also need to talk about transfers. Let $H \leq G$ be a subgroup. Then we have a transfer map $\text{Tr}_{E_n} : E_n^0(BH) \to E_n^0(BG)$, and this sits in a commutative diagram

$$\begin{array}{ccc}
E_n^0(BH) & \xrightarrow{\phi_H^t} & \prod_{[\beta] \in H_p^{n-t}/\sim} C_t^0(B(C(\text{im } \beta))) \\
\Big\downarrow {\scriptstyle \text{Tr}_{E_n}} & & \Big\downarrow \\
E_n^0(BG) & \xrightarrow{\phi_H^t} & \prod_{[\alpha] \in H_p^{n-t}/\sim} C_t^0(B(C(\text{im } \alpha)))
\end{array}$$

where we are writing C_t for the cohomology theory $C_t \otimes_{L_t} L_K(t)E_n$. As a further notational simplification, we write $\phi^t_G[\alpha] : E^0_n(BG) \to C^0_t(B(C(\text{im }\alpha)))$ for the projection.

Observe that $(G/H)^{\text{im }\alpha}$ carries an action of $C(\text{im }\alpha)$.

Theorem 2 (S, Hopkins-Kuhn-Ravenel)**.** *Let $x \in E^0_n(BH)$. Then*

$$\phi^t_G[\alpha](\text{Tr}_{E_n}(x)) = \sum_{[gH] \in (G/H)^{\text{im }\alpha}/C(\text{im }\alpha)} \text{Tr}_{C_t}(\phi^t_H[g\alpha g^{-1}](x)).$$

Next, we need to discuss a little algebraic geometry. We've already noticed that Spec $E^0_n(B\mathbb{Z}/p^k) \cong \mathbb{G}_{E_n}[p^k]$; we want to extend this dictionary to (the classifying spaces of) more groups. Write $\text{sub}_{p^k}(\mathbb{G}_{E_n}) : E^0_n\texttt{-alg} \to \texttt{Set}$ take R to the set of subgroups of $R \otimes \mathbb{G}_{E_n}$ of order p^k. Let $I_{\text{tr}} \subset E^0_n(B\Sigma_{p^k})$ be the ideal generated by the image of the transfer from $\Sigma^{\times p}_{p^{k-1}} \subset \Sigma_{p^k}$.

Theorem 3 (Strickland)**.** $\text{sub}_{p^k}(\mathbb{G}_{E_n}) \cong \text{Spec } E^0_n(B\Sigma_{p^k})/I_{\text{tr}}$*, and moreover* $\text{sub}_{p^k}(\mathbb{G}_{E_n})$ *is a finite flat scheme over* Spec E^0_n*, and its rank is equal to the number of subgroups of order p^k of $(\mathbb{Q}_p/\mathbb{Z}_p)^n$.*

Let's work out a simple example of what this means for us.

Example 1. Let $G = \Sigma_p$, $H = e$, and $t = n-1$. Note that if $[\alpha] \in G^{n-t}_p = (\Sigma_p)^1_p$, then just $\alpha : \mathbb{Z}_p \to \Sigma_p$ up to conjugacy. So there are two conjugacy classes, that of e and that of everything else. The respective centralizers $C(\text{im }\alpha)$ are Σ_p and $\mathbb{Z}/p$, respectively. So our transfer square is just

$$\begin{array}{ccc} E^0_n & \longrightarrow & C^0_{n-1} \\ {\scriptstyle \text{Tr}_{E_n}}\big\downarrow & & \big\downarrow \\ E^0_n(B\Sigma_p) & \longrightarrow & C^0_{n-1}(B\mathbb{Z}/p) \times C^0_{n-1}(B\Sigma_p). \end{array}$$

One can work out that $C^0_{n-1} \to C^0_{n-1}(B\mathbb{Z}/p)$ is the zero map, and $C^0_{n-1} \to C^0_{n-1}(B\Sigma_p)$ is the usual inclusion. So, we learn that $C_{n-1} \otimes \text{sub}_p(\mathbb{G}_{E_n}) \cong \text{sub}_p(\mathbb{G}_{C_{n-1}}) \amalg \mathbb{G}_{C_{n-1}}[p]$. What's going on here is the underlying fact that $R \otimes \text{sub}_{p^k}(\mathbb{G}_{E_n}) \cong \text{sub}_{p^k}(R \otimes \mathbb{G}_{E_n})$, so in our case, we get that $C_{n-1} \otimes \text{sub}_p(\mathbb{G}_{E_n}) \cong \text{sub}_p(\mathbb{G}_{C_t} \oplus \mathbb{Q}_p/\mathbb{Z}_p)$, and under this isomorphism, $\mathbb{G}_{C_t} \oplus \mathbb{Q}_p/\mathbb{Z}_p \mapsto \mathbb{Q}_p/\mathbb{Z}_p$ and $H \mapsto$ either e or $\mathbb{Z}/p$.

Now, let's try a harder example. We will be very specific, so that we can count things up and learn something new.

Example 2. Take $n = 2$, $t = 1$, $G = \Sigma_4$, and $H = \Sigma_2 \times \Sigma_2$. Now we've got $C_1 \otimes \text{sub}_4(\mathbb{G}_{E_2}) = \text{sub}_H(\mathbb{G}_{C_1}) \amalg \mathbb{G}_{C_1}[4] \amalg Y$ for some mysterious Y. (The left side has rank 7, while on the right side the first guy has rank 1 and the second guy has rank 4, so Y has rank 2; this is how we know there's the Y there in the first place.) Now we've got the table

$[\alpha]$	$C(\text{im }\alpha)$
e	Σ_4
$(12)(34)$	D_8
(12)	$\mathbb{Z}/2 \times \mathbb{Z}/2$
(1234)	$\mathbb{Z}/4$

and from this we can conclude that $Y = \text{Spec } C^0_1(BD_8)/I_{\text{tr}}$. So, Y somehow represents the cohomology of more subgroups.

Let's take another example and see if we can write down the subgroups explicitly.

Example 3. Take $t = n-1$, $G = \Sigma_{p^2}$, and $H = (\Sigma_p)^{\times p}$. Then we have

$[\alpha]$	$C(\text{im }\alpha)$
e	Σ_{p^2}
$(1 \cdots p)((p+1) \cdots (2p)) \cdots$	$\mathbb{Z}/p \wr \Sigma_p$
$(1 \cdots p^2)$	$\mathbb{Z}/p^2$

Now we have

$$C_{n-1} \otimes \mathrm{sub}_{p^2}(\mathbb{G}_{E_n}) = \mathrm{sub}_{p^2}(\mathbb{G}_{C_{n-1}}) \amalg \mathbb{G}_{C_{n-1}}[p^2] \amalg \mathrm{sub}_p(\mathbb{G}_{C_{n-1}}) \times \mathrm{sub}_p(\mathbb{Q}_p/\mathbb{Z}_p) \amalg ???.$$

The question marks are conjecturally filled by $X_0 \times (\mathbb{Z}/p \times \mathbb{Z}/p \backslash (0,0)) \amalg X_1 \times (\mathbb{Z}/p \backslash 0)$, where the X_i are defined by the pullback diagrams

$$\begin{array}{ccc} X_i & \longrightarrow & \mathrm{sub}_{p^2}(\mathbb{G}_{C_{n-1}}) \\ \downarrow & & \downarrow \\ \mathrm{sub}_{p^i}(\mathbb{G}_{C_{n-1}}) & \longrightarrow & \mathrm{sub}_p(\mathbb{G}_{C_{n-1}}) \amalg \mathrm{sub}_p(\mathbb{G}_{C_{n-1}}). \end{array}$$