Invertible topological field theories and differential cohomology

Peter Teichner

We will be using the Atiyah-Segal notion here, extended down to points. Recall that

$$d\text{-}\mathtt{TFT}_{A.S.} = \mathtt{Fun}^{\otimes}(d\text{-}\mathtt{Cob}, d\text{-}\mathtt{Vect}).$$

The source and target on the right side are symmetric monoidal d-categories. Specifically, $d - Cob$ has has it k-morphisms the k-dimensional compact smooth manifolds with corners, for $0 \leq k \leq d-1$, and *diffeomorphism classes* of such manifolds at $k = d$ for the d-morphisms. The business of demanding that one "goes down to points" is motivated by physics; specifically, Baez and Dolan formulated the *cobordism hypothesis* in 1995, which says that in the framed case, a d-dimensional TFT Z is determined by $Z(\mathrm{pt})$ along with certain dualizing data.

Example 1. Let's look at the case $d = 1$. The objects of $1\text{-}\mathtt{Cob}$ are compact 0-manifolds. By symmetric monoidality, we can just consider the single point, and we have $\mathrm{pt} \mapsto Z(\mathrm{pt}) \in \mathtt{Vect}_{\mathbb{C}}$. Then, the morphisms are compact 1-manifolds, where the interval must be sent to $\mathrm{id}_{Z(\mathrm{pt})}$. Then, S^1 is an endomorphism of the empty 0-manifold, and again by symmetric monoidality, it follows that $S^1 \mapsto \dim_{\mathbb{C}} Z(\mathrm{pt})$. But the interval is categorically more complicated than the circle, and it gives us a nondegenerate multiplication and comultiplication on $Z(\mathrm{pt})$.

Now, let's consider the framed case. Framed just means oriented here, and then the backwards-C morphism gives a pairing $Z(-\mathrm{pt}) \otimes Z(+\mathrm{pt}) \to \mathbb{C}$, and so $Z(+\mathrm{pt})^* \cong Z(-\mathrm{pt})$, and indeed Z is determined by $Z(+\mathrm{pt})$. This is the cobordism hypothesis for $d = 1$.

In fact, Hopkins and Lurie proved the cobordism hypothesis for $d = 2$, and Lurie later extended to $d > 2$. The methods of proof use (∞, d)-categories, which are essential for their inductive argument. Specifically, they keep the diffeomorphism groups of d-manifolds as $(d + 1)$-morphisms. But these are topological groups, and so this can be considered as giving k morphisms for all $k > d$. This is called an (∞, d)-category since all morphisms above dimension d are invertible. Of course, on the vector space side, we must remember that the space of linear maps of vector spaces form a space themselves, which gives an (∞, d)-category structure there too. So, we write

$$d\text{-}\mathtt{TFT}_{H.L.} = \mathtt{Fun}^{\otimes}_{\infty}(d\text{-}\mathtt{Bord}, d\text{-}\mathtt{Vect}_{\infty}).$$

Here, $d\text{-}\mathtt{Bord}$ is an (∞, d)-category where the d-morphisms are topological spaces, and so we take homotopy groups just by considering paths, paths of paths, etc., in those spaces. Thus, we say that $\pi_0(d\text{-}\mathtt{Bord}) = d\text{-}\mathtt{Cob}$.

Now, we have a map $d\text{-}\mathtt{TFT}_{H.L.} \to B(d\text{-}\mathtt{Vect}_{\infty})$ given by $Z \mapsto Z(\mathrm{pt})$, which the cobordism hypothesis predicts to be an equivalence. This is very difficult. However, we can make one enormous simplification, namely that our theory is *invertible*. That is,

$$d\text{-}\mathtt{TFT}^{\times}_{H.L.} = \mathtt{Fun}^{\otimes}(d\text{-}\mathtt{Bord}, d\text{-}\mathtt{Pic}),$$

where $d\text{-}\mathtt{Pic}$ is the full subcategory of $d\text{-}\mathtt{Vect}_{\infty}$ of tensor-invertible vector spaces, i.e. complex lines. This category $d\text{-}\mathtt{Pic}$ is now an $(\infty, 0)$-category, a/k/a an ∞-groupoid. But these "are" just topological spaces (or perhaps rather Kan simplicial sets). But the tensor structure makes $d\text{-}\mathtt{Pic}$ into a symmetric monoidal $(\infty, 0)$-category; since tensor product here is invertible, this is moreover grouplike. Thus, $d\text{-}\mathtt{Pic}$ is precisely a Picard ∞-groupoid; under the previous equivalence, this gives us a spectrum via (the appropriate notion of) geometric realization. This means that for any Picard ∞-groupoid $\mathcal{C}$,

$$d\text{-}\mathtt{TFT}(\mathcal{C})^{\times}_{H.L.} = \mathtt{Fun}^{\otimes}_{\infty}(d\text{-}\mathtt{Bord}, \mathcal{C})^{\times} \simeq \mathtt{Sp}(|d\text{-}\mathtt{Bord}|, |\mathcal{C}|).$$

In fact, the homotopy type of the right side was previously computed by Randal-Williams and by Galatius, Madsen, Tillmann, and Weiss. Let's consider the framed case $d\text{-}\mathtt{Bord}^f r$. Then by the Thom-Pontrjagin construction, $|d\text{-}\mathtt{Bord}^f r| = \mathbb{S}$, the sphere spectrum. So in this case, the above is just equivalent to $|\mathcal{C}|$ itself!

Example 2. Consider $\mathcal{C} = d\text{-}\mathtt{Pic}$. Then $|d\text{-}\mathtt{Pic}| = \Sigma^d H\mathbb{C}^{\times}$, considered as the spectrum $(\mathrm{pt}, \ldots, \mathrm{pt}, \mathbb{C}^{\times}, B\mathbb{C}^{\times}, B^2\mathbb{C}^{\times}, \ldots)$, where $\mathbb{C}^{\times}$ sits as the d-morphisms (where $\mathcal{C}^{\times}$ retains its topology). Thus

$$d\text{-}\mathtt{TFT}^{fr}_{H.L,}(X)^{\times}/\text{iso.} \cong H^d(X; \mathbb{C}^{\times}) \cong H^{d+1}(X; \mathbb{Z}).$$

(Here, we're just considering manifolds over a topological space X; then $|d\text{-}\mathtt{Bord}^{fr}(X)| \simeq X \wedge \mathbb{S}$.)

Now, returning to the Atiyah-Segal version, since $\pi_0(d\text{-}\mathtt{Bord}) = d\text{-}\mathtt{Cob}$, we just recover that

$$d\text{-}\mathtt{TFT}^{fr}_{A.S.}(X)^\times/\text{iso.} \cong H^d(X;\mathbb{C}^\times_\delta),$$

where δ denotes discretization. The natural map in sheaf cohomology $H^d(X;\mathbb{C}^\times_\delta) \to H^d(X;\mathbb{C}^\times)$ corresponds to the Bockstein $H^d(X;\mathbb{C}^\times_\delta) \to H^{d+1}(X;\mathbb{Z})$ (associated to the exponential short exact sequence of discrete groups).

Let's specialize further to $d = 1$. Then we get that $1\text{-}\mathtt{TFT}^{fr}_{H.L.}(X)^\times$ is the groupoid of line bundles on X, whereas $1\text{-}\mathtt{TFT}^{fr}_{A.S.}(X)^\times$ is the groupoid of *flat* line bundles (that is, functors $\pi_{\leq 1}(X) \to \mathtt{Pic}$).

Now, there is something that sits between flat line bundles and line bundles, namely line bundles with connection. Correspondingly, there is something that sits between $\mathbb{C}^\times_\delta$-cohomology and $\mathbb{C}^\times$-cohomology, namely *differential cohomology* $\hat{H}^d(X;\mathbb{C}^\times)$.

Theorem 1 (Pavlov,Stolz,T.)**.** *There is a notion of* smooth *on the TFT side, i.e. the sequence*

$$H^d(X;\mathbb{C}^\times_\delta) \to \hat{H}^d(X;\mathbb{C}^\times) \to H^d(X;\mathbb{C}^\times)$$

is isomorphic to a sequence

$$d\text{-}\mathtt{TFT}^{fr}_{A.S.}(X)^\times/\text{iso.} \to d\text{-}\mathtt{TFT}^{fr}_{sm}(X)^\times/\text{iso.} \to d\text{-}\mathtt{TFT}^{fr}_{H.L.}(X)^\times/\text{iso.}.$$

Proof outline. We need a notion of a smooth ∞-groupoid (or later, a smooth Picard ∞-groupoid, or even later, a smooth (∞, d)-groupoid). For these, we will use the category $\mathtt{PSt}$ of *prestacks*, namey functors in $\mathtt{Fun}(\mathtt{Man}^{op}, \mathtt{sSet})$. This is a model category with the very nice "local projective" model structure, whose fibrant objects are precisely the stacks, i.e. prestacks that satisfy descent. For smooth Picard ∞-groupoids we use $\mathtt{PSt}^\otimes = \mathtt{Fun}(\mathtt{Man}^{op} \times \Gamma^{op}, \mathtt{sSet})$.[1]

Now, we define

$$d\text{-}\mathtt{TFT}^{fr}_{sm}(X)^\times = \mathtt{Fun}^\otimes_{\mathtt{Man}}(d\text{-}\mathtt{Bord}^{fr}_{sm}(X), B^d_{sm}\mathbb{C}^\times),$$

where here we're doing everything in families over some manifold S. That is, the k-morphisms are given by maps $P \to X$ where we have a fiber bundle $\Sigma^k \to P \to S$, where Σ^k is a k-manifold. And for A any abelian Lie group, we apply $S \mapsto DK_\bullet(C^\infty(S, A)) \in \mathtt{sSet}$ and call this $B^d_{sm}A$.

There are two main steps in the proof. We must again examine the exponential sequences $\mathbb{Z}_\delta \to \mathbb{R}_\delta \to U(1)_\delta$ and $\mathbb{Z} \to \mathbb{R} \to U(1)$ and the natural map between these. This induces a map of associated long exact sequences. The resulting diagram is where differential forms enter the game, in $\int_\Sigma : \Omega_d(X)/\text{exact} \to [d\text{-}S_\bullet X, B^d_{sm}\mathbb{R}]$. The main lemma is that this is an isomorphism. □

The case of Chern-Simons is the case $X = \text{pt}/G$.

[1]For a nice explanation of this, see the survey paper of Dan Dugger.