# On certain nilmanifolds and their $f$-invariants

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## 1 Introduction

Recall the *stable stems* $\pi_*^s S^0$, which form a cornerstone of algebraic topology. These are generally approached using spectral sequences, most successfully the Adams-Novikov spectral sequence. We will write this as $E_2[MU] = \mathrm{Ext}^{s,t}_{MU_*MU}(MU_*, MU_*) \Rightarrow \pi_{t-s}^s S^0$. This yields a filtration on $\pi_*^s S^0$, and we can try to construct invariants that are adapted to this filtration. The first is simply the *degree*, $d : \pi_*^s S^0 \to \mathbb{Z}$, but this one is not so interesting since it vanishes on all positive-dimensional classes. The next is the Adams $e$-invariant, $e : \pi_{2r+1}^s S^0 \to \mathbb{Q}/\mathbb{Z}$. This invariant is $KU$-based, whereas the degree is $H\mathbb{Z}$-based. By analogy, Laures defined the $f$-invariant, $f : \pi_{2r+2}^s S^0 \to \underline{\underline{D}}^{\Gamma_1(N)}_{r+2} \otimes \mathbb{Q}/\mathbb{Z}$, where the first tensor factor in the target is essentially the ring of divided congruences for modular forms on $\Gamma_1(N)$. (Recall that the ring of divided congruences detects congruences between $q$-expansions of modular forms.) The beauty of this is that it factors through the 2-line of the ANSS.

What does this all mean for the framed bordism ring of manifolds? First, $d$ counts points (with signs). Next, $e$ is related to index theory on manifolds with smooth boundary (specifically the Atiyah-Patodi-Singer index theorem); one can compute the $e$-invariant of a parallelized manifold $M$ via the reduced $\eta$-invariant of the Dirac operator,

$$\eta(D) = \left.\left(\sum_{\sigma\in\mathrm{Spectrum}(D)} \frac{\sigma}{|\sigma|}(1+s)\right)\right|_{s=0}.$$

(Formally, this just sums normalizations of the various eigenvalues.) This is difficult in general, but Deninger and Singhof were able to compute this for *Heisenberg nilmanifolds*, showing that these represent generators for im $J_{2r+1}$ (up to a factor of 2).

What about $f$? By work of Bunke, Naumann, and vB, this can still be related to index theory, but unfortunately it must be carried out on manifolds with corners of codimension 2. The problem is that you can't work on the manifold itself. However, there are some exceptions. Here is a recent one: Given a hermitian line bundle $\lambda \to M^{2r+1}$ where $M$ is a parallelized smooth closed manifold, one can reconstruct $f$ on the circle bundle $S(\lambda)$ as

$$f[S(\lambda)] \equiv \sum_{n\geq 1}\left(\sum_{d|n} \zeta_N^{-n/d}\cdot \eta^{red}(D_M\otimes\lambda^d) - \zeta_N^{n/d}\eta^{red}(D_M\otimes\lambda^{-d})\right) q^n.$$

In principle this might not be much worse than the above formula, but this equivalence is taken mod $\underline{\underline{D}}_{r+2} + \mathbb{Z}[\zeta_N, 1/N][[q]] + \mathbb{R}\cdot\varphi$ for a particular modular form $\varphi$ of weight $r+2$ for $\Gamma_1(N)$ with the constant term removed. So, we have a lifting problem.

## 2 Some nilmanifolds

Let $\Lambda$ be a positive-definite even lattice, i.e. a free $\mathbb{Z}$-module of rank $n$, equipped with a quadratic form $Q : \Lambda \to \mathbb{N}_0$. We also set $B(x,y) = Q(x+y) - Q(x) - Q(y)$. Then we can make $\mathbb{Z}\times\Lambda\times\mathbb{Z}\times\Lambda$ into a group $\Gamma_\Lambda$ by setting

$$(z,c,b,a)\cdot(z',c',b',a') = (z+z'+B(a,c')+Q(a)b', c+c'+ab', b+b', a+a').$$

If we extend this over $\mathbb{R}$, we get a contractible nilpotent Lie group $G_\Lambda$, which is 3-step nilpotent and which has center $Z(G_\Lambda)\cong\mathbb{R}$. This implies that the quotient manifold $\Gamma_\Lambda\backslash G_\Lambda$ is a right principal circle bundle over $M_\Lambda = \Gamma_\Lambda\backslash G_\Lambda/Z(G_\Lambda)$.

We parallelize $M_\Lambda$ as follows. Choose an identification $\Lambda \cong \mathbb{Z}^r$, and then this extends to an identification $G_\Lambda \cong \mathbb{R}^{2r+2}$, and the standard basis here gives us left-invariant vector fields that descent to the quotient. So we have a tautological line bundle $\lambda$ with canonical connection. One can compute that

$$\eta^{red}(D \otimes \lambda^d) = \sum_{\rho \in dis(\Lambda_d)} \left(\frac{1}{2} - \overline{Q_d}(\rho)\right) \cdot sign(d^r) \equiv \eta^{red} \bmod \mathbb{Z}.$$

# 3 ... and their $f$-invariants

We have some good news.

**Corollary 1.** *Let $\Lambda$ be a positive-definite even lattice of rank 2. Then $[\Gamma_\Lambda \backslash G_\Lambda] \neq 0 \in \pi_6^s S^0 \cong \mathbb{Z}/2$ iff $|dis(\Lambda)| = \det(B_{ij}) \not\equiv 0 \bmod 2$.*

But then we have some bad news.

**Corollary 2.** *For any positive-definite even lattice $\Lambda$ of rank greater than 2, then $[\Gamma_\Lambda \backslash G_\Lambda]$ is in Adams-Novikov filtration greater than 2.*

**Remark 1.** If we take into account that there are no elements in Adams-Novikov filtration above 2 in degrees less than 18, we obtain that for $3 \leq r \leq 7$, the associated element in $\pi_*^s S^0$ is trivial. We can revisit the case $r \geq 8$ once someone defines higher invariants!

# 4 ... put into perspective

There are far too many nilmanifolds, so we want to cut down this class of examples somewhat. Take $\hat{G}$ to be a connected simple Lie group (over $\mathbb{R}$). This has an Iwasawa decomposition $\hat{G} = N \cdot A \cdot K$ (where $K$ is a maximal compact subgroup, $A$ is flat, and $N$ is nilpotent). But even better, nilmanifolds that are covered by such $N$ occur naturally when one attempts to compactify certain suitable locally symmetric quotients $\hat{\Gamma} \backslash \hat{G}/K$. (This uses the theory of Borel-Serre compactifications.) The important point here is that $\Gamma | \backslash N$ is a "model corner", and we don't want to necessarily consider it directly, but rather up to its $d$-, $e$-, and $f$-invariants.

Do these give rise to interesting representations of elements of $\pi_*^s S^0$? Let's take stock. First of all, Heisenberg nilmanifolds do. Next, parts 2 and 3 of this talk give an answer for $\hat{G} \cong SO_0(2, 2+r) \cong SO_0(B(-,-))$ (the group that stabilizes the given bilinear form). Current work suggests that we also get interesting results in the remaining rank-2 case, namely, the irreducible hermitian symmetric domains. Unfortunately, the computations become quite nasty.