STAX

Contents

1 Introduction (Martin Olsson)

Motivation for stacks comes from:

- algebraic topology;

- moduli theory;
- not working with one arm tied behind your back.

In algebraic topology, we might like to talk about classifying spaces. But one quickly learns that algebraic varieties have such rigid structure that we can't make the same constructions. In topology, cohomology is a *representable functor*: for any abelian group A and nonnegative number n, there is a space $K(A,n)$ such that $H^n(X;A) \simeq [X, K(A,n)]$ (where the square brackets denote homotopy classes of maps). In algebraic geometry, $H^1(X;A)$ is defined in terms of *torsors* and $H^2(X;A)$ is defined in terms of *gerbes*, which are certain kinds of stacks. It is an extremely profitable business to think about cohomology classes in this way.

However, the story of stacks really begins with moduli theory: the classification of algebraic varieties. For example, curves are classified by their genus (so at $g=0$ we have only $\mathbb{P}^1$). But what about $\{X^2+Y^2+Z^2=0\} \subset \mathbb{P}^2$? If we're over the real numbers, we probably don't want to call this $\mathbb{P}^1$! To give the correct answer takes a lot of machinery involving Grothendieck topologies etc., but the final answer is that the moduli stack of $\mathbb{P}^1$ should be thought of as $BPGL_2$.

Without going into details, we have the following picture. If S is a space, we can make a *fiber bundle* $\mathbb{P}^1 \hookrightarrow X \twoheadrightarrow S$ – each fiber is a copy of $\mathbb{P}^1$, but $X \ncong S \times \mathbb{P}^1$. For example, if $S = S^1$ is the circle, we can cover S^1 with two open intervals U_1 and U_2 over which the bundle is trivial, and then to specify such a bundle we must specify how the bundles are supposed to glue on $U_1 \cap U_2$. These gluings can be nontrivial – indeed, PGL_2 gives the automorphisms of $\mathbb{P}^1$ – and so we can get nontrivial bundles over S. To do this properly in algebraic geometry requires *faithfully flat descent*, *étale topology*, and other things.

At $g=1$, there are two interesting things that happen. First of all, we can look at pointed genus-1 curves (E,e) (i.e. elliptic curves) or unpointed genus-1 curves. As it turns out, one can make families of $E \to S$ where all fibers are genus-1 curves and there exists an étale surjection $S' \to S$ so that $E \times_S S'$ is a scheme but E is an algebraic space which is not a scheme. An étale morphism should just be thought of as a local homeomorphism, so $S' \to S$ should be thought of as something like a covering space. Then in $S' \times_S S' \rightrightarrows S' \to S$, choices $p_1^*E' \xrightarrow{\sim} p_2^*E'$ plus cocycle conditions should glue to an element E' over S'. It is impossible to do this in the world of schemes.

Example 1. Consider the two (pointed) elliptic curves $E_1 : y^2 = x^3 - Dx$ and $E_2 : y^2 = x^3 + Dx$ over $\mathbb{Q}$. These are isomorphic once we have a cube-root of D, but they are non-isomorphic over $\mathbb{Q}$ itself. This illustrates the fact that the statement "elliptic curves are classified by their j-invariant" really only holds over algebraically closed fields. Thus, the functor $\mathcal{M}_{1,1}$ which takes a scheme S to an elliptic curve $E \to S$ over S (ie. each fiber is an elliptic curve, with basepoints given by a section $e : S \to E$) has a map $\mathcal{M}_{1,1} \to \mathbb{A}^1_j$, which we call a *coarse moduli space*.

Example 2. When $g \geq 2$, we have a map $\mathcal{M}_g \to M_g$; the first is a stack and the second is a singular space. If for example we'd like to study intersection theory, the correct place for defining cycles is $\mathcal{M}_g$.

2 Background: schemes as functors (Andrew Niles)

2.1 Category theory

Let $\mathcal{C}$ be a (locally small) category. For any $X \in \mathcal{C}$, we have a functor $h_X = \mathrm{Hom}_{\mathcal{C}}(-,X) : \mathcal{C}^{op} \to \mathbf{Sets}$.

Lemma 1 (Yoneda)**.** *For any functor $F : \mathcal{C}^{op} \to \mathbf{Sets}$, the function $\mathrm{Hom}_{Fun}(h_X, F) \to F(X)$ defined by $f \mapsto f_x(\mathrm{id}_X)$ is a bijection.*

Corollary 1. *The mapping $X \mapsto h_X$ defines a fully faithful functor $\mathcal{C} \to Fun(\mathcal{C}^{op}, catSets)$. Thus we will use X and h_X interchangeably.*

We say that a functor $F : \mathcal{C}^{op} \to \mathbf{Sets}$ is *representable* by an object $X \in \mathcal{C}$ if there is an isomorphism of functors $h_X \cong F$. We make the enormous abuse of notation and say that F "is" an object of $\mathcal{C}$ (since it's in the essential image of the Yoneda embedding).

Let $F, G : \mathcal{C}^{op} \to \mathbf{Sets}$ and let $f : F \to G$ be a morphism of functors. We say that f is *relatively representable* if for every $X \in \mathcal{C}$ and every morphism of functors $h_x \to G$, the fiber product $F \times_G h_X$ is representable. The fiber product is by definition computed pointwise, i.e. $(F \times_G h_X)(Y) = F(Y) \times_{G(Y)} h_X(Y)$.

2.2 Schemes

Let us fix $\mathcal{C} = Aff_S$, the category whose objects are affine schemes $\mathrm{Spec}(R)$ with a fixed morphism $\mathrm{Spec}(R) \to S$ and whose morphisms are just morphisms of S-schemes. (If $S = \mathrm{Spec}(A)$ is affine, then this is just $(\mathbf{Alg}_A)^{op}$.) Note that for any S-scheme X, we still have a functor $h_X : Aff_S^{op} \to \mathbf{Sets}$.

Let $F, G : Aff_S^{op} \to \mathbf{Sets}$ be functors and let $f : F \to G$ be a morphism of functors. We say that f is an *affine open (resp. closed) immersion* if

- f is relatively representable, and
- for all $h_X \to G$, the morphism $F \times_G h_X \to h_X$ in the diagram

$$\begin{array}{ccc} F \times_G h_X & \longrightarrow & h_X \\ \downarrow & & \downarrow \\ F & \longrightarrow & G \end{array}$$

  "is" an open (resp. closed) immersion.

That is, we choose an isomorphism $h_Y \cong F \times_G h_X$, and then the morphism $h_Y \to h_X$ corresponds to an open (resp. closed) immersion $Y \to X$ of schemes.

2.3 Sheaves

Recall the notion of a *sheaf* $\mathcal{F}$ (of sets) on a scheme X. This is a functor $\mathcal{F} : Op(X)^{op} \to \mathbf{Sets}$ with the property that for any open cover $\{U_i\}$ of an open subset $U \subset X$,

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is an equalizer diagram (in **Sets**).

Now, let $F : Aff_S^{op} \to \mathbf{Sets}$ be a functor. We say that $\mathcal{F}$ is a *big Zariski sheaf* if for all $U \in Aff_S^{op}$ and for all open affine covers $\{U_i\}$ of U,

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is an equalizer diagram (in **Sets**). (Note that we actually take the intersection to be a fiber product in the category Aff_S^{op}, so there's no worry that this won't be affine.) Of course this is very similar to the usual sheaf condition; the only difference is that U is no longer sitting inside a fixed scheme X. We will soon learn that this is nothing but a sheaf on a "Grothendieck topology".

A morphism $f : F \to G$ of big Zariski sheaves (which is by definition just a morphism of functors) is called *surjective* if for all $U \in Aff_S^{op}$ and $u \in G(U)$, there is a covering $\{U_i\}$ of U in Aff_S such that $u|_{U_i} \in \mathrm{Im}(f_{U_i} : F(U_i) \to G(U_i))$ for all i.

Proposition 1. *Let $F : Aff_S^{op} \to$ **Sets** be a functor. Then F is representable by a separated (not necessarily affine) S-scheme if and only if the following three conditions hold:*

1. *F is a big Zariski sheaf.*
2. *$\Delta : F \to F \times F$ is an affine closed immersion.*
3. *There is a surjection $\coprod_i h_{X_i} \to F$ of big Zariski sheaves, where each $h_{X_i} \to F$ is an affine open immersion.*

Also, the functor $Sch/S \to Fun(Aff_S^{op}, \mathbf{Sets})$ given by $X \to h_X$ is fully faithful, and the above precisely describes this functor.

This is nice because it can be a lot easier to describe a functor than to describe a scheme that represents it.

Proof of ($\Leftarrow$). Let us begin with the X_i. Since each $h_{X_i} \to F$ is an affine open immersion, then the fiber products $h_{X_i} \times_F h_{X_j}$ are affine, say $h_{X_i} \times_F h_{X_j} \cong h_{V_{ij}}$ with open immersions $X_i \hookleftarrow V_{ij} \hookrightarrow X_j$. Then we define X to be the X_i glued along the V_{ij}. $\square$

If we want to talk about more general, not necessarily separated schemes, we need to make the following modification. We say that $f : F \to G$ is *representable by separated schemes* if for any $h_U \to G$ with $U \in Aff_S$, the fiber product $F \times_G h_U$ is representable by a separated U-scheme. Then, we change condition 2 to be that "$\Delta : F \to F \times F$ is *representable by separated schemes*", and we change condition 3 to be that the same maps are just open immersions (not necessarily affine).

Example 3. Consider $\mathbb{P}^n_{\mathbb{Z}}$. We know from Hartshorne that this represents a particular functor P^n. If we pretend that we don't know that this functor is representable, then we can run through the conditions of the above result to recover the scheme $\mathbb{P}^n_{\mathbb{Z}}$.

Example 4. Let S be a Noetherian scheme, let $f : X \to S$ be a morphism of finite type, let $\mathcal{L}$ be a relatively ample invertible sheaf on X, and let $\mathcal{F}$ be a coherent sheaf on X. Then we have the functor

$$\underline{Quot}(\mathcal{F}/X/S) : (Sch/S)^{op} \to \mathbf{Sets}$$

given by

$$S' \mapsto \{\mathcal{F}_{S'} \twoheadrightarrow \mathcal{G} \text{ s.t. } \mathcal{G} \in Coh(X_{S'}) \text{ flat over } S' \text{ with proper support}\}.$$

Grothendieck proved that this functor is actually representable by a scheme. For example, if we take $\mathcal{F} = \mathcal{O}_X$ then we get a Hilbert scheme.

3 Sites and sheaves (Slater Stich)

Definition 1. Let $\mathcal{C}$ be a category. A *pretopology* on $\mathcal{C}$ is the data of for each $X \in \mathcal{C}$, a collection $Cov(X)$ of "covering families" of morphisms with codomain X, subject to the axioms:

1. If $V \to X$ is an isomorphism, then $\{V \to X\} \in Cov(X)$. ("Isomorphisms are covers.")
2. If $\{X_i \to X\}_{i \in I} \in Cov(X)$ and $Y \to X$ is an arbitrary morphism in $\mathcal{C}$, then $\{Y \times_X X_i \to Y\}_{i \in I} \in Cov(Y)$. ("The pullback of a cover is a cover." In particular, these pullbacks exist.)
3. If $\{X_i \to X\}_{i \in I_j} \in Cov(X)$ and for all i we have $\{V_{ij} \to X_i\}_{i \in I} \in Cov(X_i)$, then $\{V_{ij} \to X_i \to X\}_{i \in I, j \in I_j} \in Cov(X)$. ("Is something covers a cover, then it's a cover too.")

Example 5 (A small topological site)**.** The primordial example of a pretopology is when we take $\mathcal{C} = Open(X)$ for some topological space X (whose morphisms are inclusions). For $V \subseteq U$, let's identify V with the map $V \to U$. Define a pretopology by setting $Cov(U)$ to be the set of open covers of U. Let us check the axioms.

1. $\{U\}$ covers U.
2. If $\{U_i\}$ covers U and $V \subseteq U$, then $\{U_i \cap V\}$ covers U. (Pullback in this category is just given by intersection.)
3. If $\{U_i\}$ covers U and $\{V_{ij}\}$ covers U_i for each i, then $\{V_{ij}\}$ covers U.

Example 6 (The big topological site)**.** Let $\mathcal{C} = \mathbf{Top}$, the category of topological spaces. For a space X, we define $Cov(X)$ to be the set of collections of jointly surjective open immersions; that is, an element of $Cov(X)$ is a family $\{f_i : X_i \to X\}$ where f_i is an open immersion such that $X = \cup_i f_i(X_i)$. It is not hard to check that this satisfies the axioms for a pretopology. (We could instead demand that each f_i is a local homeomorphism, and then we'd recover the *étale site*.)

So, rather than talking about sheaves on a particular topological space, we will be able to talk about sheaves on all topological spaces at once!

Definition 2 (first definition of sieve)**.** Let $\mathcal{C}$ be a category. An *"internal" sieve* (though nobody uses the adjective) on an object $X \in \mathcal{C}$ is a set of morphisms with codomain X such that if $Y \xrightarrow{f} X$ is in the set, then so is $Z \xrightarrow{g} Y \xrightarrow{f} X$ for any morphism $Z \xrightarrow{g} Y$ in $\mathcal{C}$. (We might say that these sets are requires to be "downward closed".)

Example 7. In the first example (of a small topological site), this definition just requires that if the morphism $V \subseteq U$ in $Open(X)$ is in a sieve, then so is $W \subseteq V \subseteq U$ for any morphism $W \subseteq V$ in $Open(X)$.

Definition 3 (nice definition of sieve)**.** Let $\mathcal{C}$ be a category. A *sieve* on $X \in \mathcal{C}$ is a subfunctor of $h_X = \mathrm{Hom}_{\mathcal{C}}(-, X)$ (i.e. a subobject of h_X in $\hat{C} = Fun(\mathcal{C}, \mathbf{Sets})$, which in this case just means a natural choice of subset for each $X \in \mathcal{C}$).

This specifies exactly the same data as the previous definition.

We can now finally define a *topology*.

Definition 4. Let $\mathcal{C}$ be a category. A *topology* on $\mathcal{C}$ is the data of for each $X \in \mathcal{C}$ a collection $\mathcal{S}(X)$ of sieves on X, called *covering sieves*, such that:

- $h_X \in \mathcal{S}(X)$.
- If $S \in \mathcal{S}(X)$ and $Y \to X$ is any morphism in $\mathcal{C}$, then $S \times_{h_X} h_Y \in \mathcal{S}(Y)$.
- Let $S \in \mathcal{S}(X)$ and let T be any sieve on X. Suppose that for every $(Y \to X) \in S$ we have that the pullback $T \times_{h_X} h_Y$ is in $\mathcal{S}(Y)$. Then $T \in \mathcal{S}(X)$.

We should think of this last axiom as follows. To say that $(Y \xrightarrow{f} X) \in S$ is to say that $f \in S(Y)$. These should roughly correspond to natural transformations $S \to h_X$.

A good way to think about this definition is to bring Yoneda into the game. Recall that the Yoneda embedding $y : \mathcal{C} \to \hat{\mathcal{C}} = Fun(\mathcal{C}^{op}, \mathbf{Sets})$ is given by $X \mapsto h_X$. Now, there may very well be subobjects in $\hat{\mathcal{C}}$ of h_X which aren't in the (essential) image of $\mathcal{C}$. (This is analogous to embedding a topological space into a new topological space with a finer topology. But we wouldn't want to do this with the discrete topology, because that's too much to be useful.) We'll call maps $S \to h_X$ in $\hat{\mathcal{C}}$ "nice" if they live in a sieve. Then the axioms say:

1. Every object is a nice subobject of itself.
2. The property of being a nice subobject is closed under pullbacks along maps in the image of the Yoneda embedding.
3. If $S \to h_X$ is nice, and $T \to h_X$ is any morphism, and the pullback of $T \to h_X$ is ??????

Definition 5. A *site* is a category with a topology.

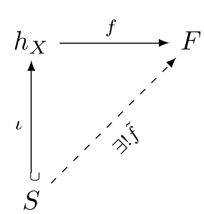
Remark 1. The topology associated to a pretopology is given by saying that $S \in \mathcal{S}(X)$ iff there is some $K \in Cov(X)$ with $K \subseteq S$. (Going in the other direction requires fiber products to exist.)

Definition 6. Let $\mathcal{C}$ be a site, and let $F : \mathcal{C}^{op} \to \mathbf{Sets}$ be a functor. We say that F is a *sheaf* (with respect to the topology on $\mathcal{C}$) if for every object $X \in \mathcal{C}$ and every covering sieve $S \in \mathcal{S}(X)$ (with $\iota : S \to h_X$) the map

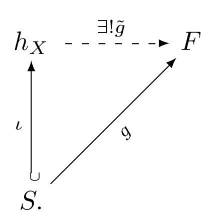
$$\mathrm{Hom}_{\hat{\mathcal{C}}}(S, F) \xleftarrow{-\circ\iota} \mathrm{Hom}_{\hat{\mathcal{C}}}(h_X, F) \cong F(X)$$

is a bijection.

This is encoded in the diagrams



and



In this framework, sheafification is really nice.

Definition 7. Let $F : \mathcal{C}^{op} \to \mathbf{Sets}$ be any functor. Define

$$\mathcal{L}(F)(X) = \varinjlim_{S \in \mathcal{S}(X)} \mathrm{Hom}_{\hat{\mathcal{C}}}(S, F).$$

This is a separated presheaf for all F, and if F is separated then $\mathcal{L}(F)$ is a sheaf. (A separated presheaf is when the map in the sheaf condition is injective (but not necessarily surjective).)

Note that this limit on the right side is actually just

$$\lim_{S} \mathrm{Hom}_{\hat{\mathcal{C}}}(h_X, F) = \lim_{S} F(X) = F(X).$$

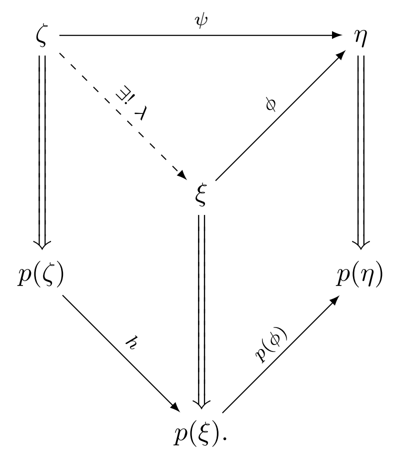
4 Fibered categories (Aaron Mazel-Gee)

In the same way that a sheaf is a special sort of functor, a stack will be a special sort of sheaf of groupoids (or a special special sort of groupoid-valued functor). It ends up being advantageous to think of the groupoid associated to an object X as living "above" X, in large part because this perspective makes it much easier to study the relationships between the groupoids associated to different objects. For this reason, we use the language of *fibered categories.*

We note here that throughout this exposition we will often say *equal* (as opposed to isomorphic), and we really will mean it.

4.1 Definitions and basic facts

Definition 8. Let $\mathcal{C}$ be a category. A *category over* $\mathcal{C}$ is a category $\mathcal{F}$ with a functor $p : \mathcal{F} \to \mathcal{C}$. A morphism $\xi \overset{\phi}{\to} \eta$ in $\mathcal{F}$ is called *cartesian* if for any other $\zeta \in \mathcal{F}$ with a morphism $\zeta \overset{\psi}{\to} \eta$ and a factorization $p(\zeta) \overset{h}{\to} p(\xi) \overset{p(\phi)}{\to} p(\eta)$ of $p(\psi)$ in $\mathcal{C}$, there is a unique morphism $\zeta \overset{\lambda}{\to} \eta$ giving a factorization $\zeta \overset{\lambda}{\to} \eta \overset{\phi}{\to} \eta$ of ψ such that $p(\lambda) = h$. Pictorially,



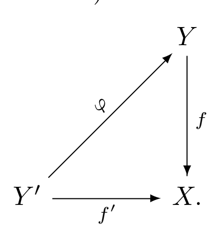
In this case, we call ξ a *pullback* of η along $p(\phi)$. The reason for this will be made clear(er) in a moment.

For each $U \in \mathcal{C}$, we have the category

$$\mathcal{F}(U) = \begin{cases} \mathrm{ob}(\mathcal{F}(U)) = \{\xi \in \mathcal{F} \colon p(\xi) = U\}, \\ \mathrm{Hom}_{\mathcal{F}(U)}(\xi', \xi) = \{f \in \mathrm{Hom}_{\mathcal{F}}(\xi', \xi) \colon p(f) = \mathrm{id}_U\}. \end{cases}$$

We call this the *fiber of* $\mathcal{F}$ *over* U.

Example 8. Let $\mathcal{C}$ be a category and choose $X \in \mathcal{C}$. Define the *over-category* $\mathcal{C}/X$, whose objects are objects-over-X (i.e. objects $Y \in \mathcal{C}$ with a specified map $Y \overset{f}{\to} X$) and whose morphisms are commutative diagrams



The functor $p : \mathcal{C}/X \to \mathcal{C}$ sending $Y \xrightarrow{f} X$ to Y and the above morphism to $Y' \xrightarrow{\varphi} Y$ makes $\mathcal{C}/X$ into a category over $\mathcal{C}$. The fiber $(\mathcal{C}/X)(Y)$ is just the discrete category $\mathrm{Hom}_{\mathcal{C}}(Y, X)$.

Definition 9. A *fibered category over* $\mathcal{C}$ is a category $p : \mathcal{F} \to \mathcal{C}$ over $\mathcal{C}$ such that for every morphism $U \xrightarrow{f} V$ in $\mathcal{C}$ and object $\eta \in \mathcal{F}(V)$, there exists a cartesian arrow $\phi : \xi \to \eta$ such that $p(\phi) = f$ (i.e. $\xi \in \mathcal{F}(U)$). A morphism of fibered categories is a functor of categories-over-$\mathcal{C}$ sending cartesian arrows to cartesian arrows. Given two morphisms $g, g' : \mathcal{F} \to \mathcal{G}$ of fibered categories over $\mathcal{C}$, a *base-preserving natural transformation* $\alpha : g \to g'$ is a natural transformation of functors such that for every $\xi \in \mathcal{F}$, the morphism $\alpha_\xi : g(\xi) \to g'(\xi)$ projects to an identity arrow (i.e. is a morphism in $\mathcal{G}(p_F(\xi))$). We write $HOM_{\mathcal{C}}(\mathcal{F}, \mathcal{G})$ the category whose objects are morphisms of fibered categories over $\mathcal{C}$ and whose morphisms are base-preserving natural transformations.

Exercise 1. Figure out how pullbacks work in the fibered category $p : \mathcal{C}/X \to \mathcal{C}$.

Lemma 2. *If* $p : \mathcal{F} \to \mathcal{C}$ *is a fibered category, then every morphism* $\psi : \zeta \to \eta$ *factors as*

$$\zeta \xrightarrow{\lambda} \xi \xrightarrow{\phi} \eta,$$

where ϕ *is cartesian and* λ *projects to an identity arrow (i.e. is a morphism in* $\mathcal{F}(p(\zeta))$*).*

This is just saying that we can factor an arrow by first moving around inside our fiber and then applying a cartesian arrow. The idea here is that any map of bundles covering a map of bases factors through the pullback bundle.

Lemma 3. *Let* $g : \mathcal{F} \to \mathcal{G}$ *be a morphism of fibered categories over* $\mathcal{C}$ *such that for every object* $U \in \mathcal{C}$ *the induced functor* $g_U : \mathcal{F}(U) \to \mathcal{G}(U)$ *is fully faithful. Then* g *is a fully faithful functor.*

So, the full-faithfullness of a morphism of fibered categories can be checked fiberwise.

Proposition 2. *A morphism of fibered categories* $g : \mathcal{F} \to \mathcal{G}$ *over* $\mathcal{C}$ *is an equivalence of fibered categories (with respect to* $HOM_{\mathcal{C}}$*, i.e. such that the isomorphisms between composite functors and identity functors are base-preserving) iff every* $g_U : \mathcal{F}(U) \to \mathcal{G}(U)$ *is an equivalence of categories.*

So, whether a morphism of fibered categories is an equivalence can also be checked fiberwise.

4.2 The 2-Yoneda lemma

Recall the "strong" Yoneda lemma, which says that given a category $\mathcal{C}$ and an object $X \in \mathcal{C}$, the natural function

$$\begin{array}{rcl} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{op}, \mathbf{Set})}(h_X, F) & \to & F(X) \\ \varphi & \mapsto & \varphi(\mathrm{id}_X) \end{array}$$

is a bijection. This implies the "weak" Yoneda lemma, which we get when we replace F with h_Y for some $Y \in \mathcal{C}$. Another way of saying this is that the Yoneda functor $h : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathbf{Set})$ given by $X \mapsto h^X$ is fully faithful, i.e. the Yoneda functor is an *embedding.*

In our fibered context, we have new, souped up versions of the Yoneda lemmas.

Theorem 1 (strong 2-Yoneda lemma)**.** *Let* $p : \mathcal{F} \to \mathcal{C}$ *be a fibered category. Then the natural functor*

$$\begin{array}{rcl} HOM_{\mathcal{C}}((\mathcal{C}/X), \mathcal{F}) & \to & \mathcal{F}(X) \\ (g : \mathcal{C}/X \to \mathcal{F}) & \mapsto & g(\mathrm{id}_X) \end{array}$$

is an equivalence of categories.

Thus, to understand the fiber of $p : \mathcal{F} \to \mathcal{C}$ over X, all we have to do is probe $\mathcal{F}$ for sections over the entire over-category of X!

Corollary 2 (weak 2-Yoneda lemma)**.** *Let* $X, Y \in \mathcal{C}$*. Then*

$$\begin{array}{rcl} HOM_{\mathcal{C}}((\mathcal{C}/X), (\mathcal{C}/Y)) & \to & (\mathcal{C}/Y)(X) = \mathrm{Hom}_{\mathcal{C}}(X, Y) \\ (g : \mathcal{C}/X \to \mathcal{C}/Y) & \mapsto & g(\mathrm{id}_X) \end{array}$$

is an equivalence of categories.

Similarly, this corollary says that we can embed the category $\mathcal{C}$ into the 2-category of fibered categories over $\mathcal{C}$. Henceforth, we will not make a distinction between objects of $\mathcal{C}$ and fibered categories over $\mathcal{C}$ (i.e. we will simply write X for $\mathcal{C}/X$ when undue no confusion will arise); in a sense, we have *inflated* our category to include "objects" whose Yoneda functors take values in categories instead of just in sets.

4.3 Categories fibered in groupoids

4.3.1 ... coming from co/groupoid objects

Let us change gears for a moment. We will make a natural construction which will end up giving us a groupoid-valued functor, which will lead us to a source of many more groupoid-valued functors. We will then reinterpret these as fibered categories.

Given a ring R, we define the groupoid $Q(R)$ of (monic) quadratic expressions and changes of variable by

$$Q(R) = \begin{cases} \mathrm{ob}(Q(R)) = \{x^2 + bx + c : b, c \in R\} \cong R \times R, \\ \mathrm{Hom}_{Q(R)}((b', c'), (b, c)) = \{r \in R : (x + r)^2 + b'(x + r) + c' = x^2 + bx + c\}. \end{cases}$$

A ring homomorphism $R \to S$ determines a functor $Q(R) \to Q(S)$. So, these constructions assemble into a functor $Q : \mathbf{Rings} \to \mathbf{Groupoids}$.

Everyone loves a (co)representing object. Luckily for everyone, then, it is not hard to see that $\mathrm{ob}(Q(R)) = \mathrm{Hom}_{\mathbf{Rings}}(\mathbb{Z}[b, c], R)$: there is no choice about where to send the copy of $\mathbb{Z}$, and then the free generators b and c select, well, b and c (the linear and constant coefficients of our quadratic expressions). Let us write $A = \mathbb{Z}[b, c]$. Moreover, $\mathrm{mor}(Q(R)) = \mathrm{Hom}_{\mathbf{Rings}}(A[r], R)$: the copy of A picks out the source of our morphism, and the free generator r selects the change of coordinates. Let us write $\Gamma = A[r]$.

So, the pair of sets $(\mathrm{Hom}(A, R), \mathrm{Hom}(\Gamma, R))$ has the structure of a groupoid (the set of objects and the set of morphisms), and this is functorial in the ring R. By a general principle called "the method of the universal example" (which is really just an application of Yoneda's lemma), we can actually extract quite a bit of structure. For example, there are *source* and *target* functions $s, t : \mathrm{mor}(Q(R)) \to \mathrm{ob}(Q(R))$; by setting $R = \Gamma$, we obtain maps $s, t \in \mathrm{ob}(Q(\Gamma)) = \mathrm{Hom}(A, \Gamma)$ that are the images of $\mathrm{id}_\Gamma \in \mathrm{Hom}(\Gamma, \Gamma)$. What happens is that a morphism in $Q(R)$ is given by a map $\Gamma \to R$, and to pick out the source we precompose with s to get $A \xrightarrow{s} \Gamma \to R$ (and similarly for the target). Explicitly, $s : A \to \Gamma$ is the standard inclusion while $t : A \to \Gamma$ is given by $t(b) = b + 2r$ and $t(c) = c + br$.

In this way, all the axioms of a groupoid manifest themselves as maps within the pair (A, Γ):

groupoid axioms	structure maps
every arrow has a source and a target	$s, t : A \to \Gamma$
every object has an identity arrow	$\epsilon : \Gamma \to A$
every arrow has an inverse	$i : \Gamma \to \Gamma$
composable arrows compose	$m : \Gamma \to \Gamma \otimes_{s,A,t} \Gamma$

Exercise 2. Work out the rest of the structure maps.

These maps satisfy various identities dictated by what they're supposed to encode (e.g. $s \circ \epsilon = \mathrm{id}_A = t \circ \epsilon$, an associativity diagram, etc.), all of which makes the pair (A, Γ) into a *cogroupoid in* **Rings**.

Exercise 3. Determine the cogroupoids and their structure maps associated to the following constructions:

- objects are $\{x^2 + bx + c \colon b, c \in R\}$, morphisms are $\{x \mapsto ex + r \colon e, r \in R, e^2 = 1\}$;
- objects are $\{ax^2 + bx + c \colon a \in R^\times, b, c \in R\}$, morphisms are $\{x \mapsto ux + v \colon u \in R^\times, v \in R\}$;
- objects are $\{*\}$, morphisms are $\{f(t) \in R[[t]] \colon f(0) = 0, f'(t) \in R^\times\}$ (under composition of power series);
- objects are $\{F(x, y) \in R[[x, y]] \colon F(x, y) = F(y, x), F(x, 0) = x, F(F(x, y), z) = F(x, F(y, z))\}$, morphisms are $\{f(x) \text{ as above} \colon F_1(f(x), f(y)) = f(F_2(x, y)) \rightsquigarrow F_1 \xrightarrow{f} F_2\}$;
- objects are $\{y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \colon a_i \in R\}$, morphisms are $\{(x, y) \mapsto (\lambda^2 x + r, \lambda^3 y + s\lambda^2 x + t) \colon \lambda \in R^\times, r, s, t \in R\}$.

(The antepenultimate example is called *the group of infinitesimal automorphisms of the line fixing 0*, the penultimate example is called *the groupoid of (1-dimensional) (commutative) formal group laws and isomorphisms*, and the ultimate example is called *the groupoid of elliptic curves in Weierstraß form and transformations preserving the isomorphism type of the curve.*)

Aside 1. A cogroupoid (A, Γ) in **Rings** (which is actually called a *Hopf algebroid*) admits a notion of *comodule*: this is an A-module with a coaction of Γ satisfying counitality and coassociativity. These form an abelian category with enough injectives (assuming the source map $s : A \to \Gamma$ (or equivalently the target map) is flat). These should of course be thought of as something like sheaves on the Hopf algebroid (A,Γ). There is a cohomology functor given by $M \mapsto H^*((A,\Gamma);M) = \mathrm{Ext}^*_{\mathbf{Comod}_{(A,\Gamma)}}(A,M)$, and in particular one might care about $H^*((A,\Gamma);A)$ (since A is a nice and canonical (A,Γ)-comodule), which we simply call "the cohomology of (A,Γ)". Different Hopf algebroids can have the same cohomology, if the groupoids they represent are equivalent or even *locally* equivalent (in the flat topology). This gives some indication that these cogroupoid objects should not quite be our final object of study. Indeed, the category of comodules on a groupoid-valued functor is 2-categorically equivalent to the category of quasi-coherent sheaves on its stackification.

Incidentally, note that when applied to an algebraically closed field, the second construction in the above exercise may give an equivalent groupoid to the one (A,Γ) that we've been talking about this whole time: we might hope to retract the groupoid of quadratic expressions onto the full subgroupoid of monic quadratic expressions (although the choice of arrow may not be able to be made canonical, in which case this could very well fail). Of course, over an arbitrary field these groupoids will in general be inequivalent. This suggests that (depending on our goals) we may want to pass to algebraic closure before applying our groupoid-valued functor. The map $k \to \overline{k}$ is faithfully flat, and so Spec $\overline{k} \to$ Spec k is a cover in the flat topology. So this may be some indication that the flat topology is the right one for us, since in this case the stackifications of these two groupoid-valued functors will be equivalent.

Now, let's take what we've got and flip it and reverse it. Applying Spec everywhere, we get a *groupoid (pair)* (Spec A, Spec Γ) in **AffSch**. This takes us back to geometry, and is more like what we're used to anyways (representing instead of corepresenting). We get structure maps just as before, which we'll even call by the same names. More explicitly and more generally, any pair of objects (X_0, X_1) in a category $\mathcal{C}$ is called a *groupoid in* $\mathcal{C}$ if it has maps

$$s : X_1 \to X_0,\ t : X_1 \to X_0,\ \epsilon : X_0 \to X_1,\ i : X_1 \to X_1,\ m : X_1 \times_{s,X_0,t} X_1 \to X_1$$

that satisfy the obvious identities coming from the definition of a groupoid. (This is a generalization of the notion of a *group object* in a category, whose contravariant Yoneda functor lands in **Groups**.)

Now, given an object $U \in \mathcal{C}$, we define a category

$$\{X_0(U)/X_1(U)\} = \left\{ \begin{array}{l} \mathrm{ob}(\{X_0(U)/X_1(U)\}) = X_0(U) \\ \mathrm{Hom}_{\{X_0(U)/X_1(U)\}}(u,u') = \{\xi \in X_1(U) : s(\xi) = u,\ t(\xi) = u'\}. \end{array} \right.$$

A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(V,U)$ gives a functor $f^* : \{X_0(U)/X_1(U)\} \to \{X_0(V)/X_1(V)\}$ by pullback, and $g^*f^* = (fg)^*$ (on the nose).

We finally define a category $\{X_0/X_1\}$. Its objects are given by pairs (U,u), where $U \in \mathcal{C}$ and $u \in \{X_0(U)/X_1(U)\}$. A morphism $(V,v) \to (U,u)$ in $\{X_0/X_1\}$ is given by a pair

$$(f \in \mathrm{Hom}_{\mathcal{C}}(V,U), \alpha \in \mathrm{Hom}_{\{X_0(V)/X_1(V)\}}(v, f^*u)).$$

We have a functor $p : \{X_0/X_1\} \to \mathcal{C}$ which make $\{X_0/X_1\}$ into a *category fibered in groupoids over* $\mathcal{C}$ (i.e. a fibered category over $\mathcal{C}$ where all fibers are groupoids).

4.3.2 ... and 2-categorical fiber product thereof

We first examine the situation for groupoids before generalizing to categories fibered in groupoids.

Definition 10. Let $\mathcal{G}_1 \xrightarrow{f} \mathcal{G} \xleftarrow{g} \mathcal{G}_2$ be a diagram of groupoids. We define the *2-categorical fiber product* $\mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2$ as follows. Its objects are triples (x,y,σ), where $x \in \mathcal{G}_\infty$, $y \in \mathcal{G}_2$, and $\sigma \in \mathrm{Hom}_{\mathcal{G}}(f(x), g(y))$. A morphism $(x',y',\sigma') \to (x,y,\sigma)$ is a pair of isomorphisms $a : x' \to x$ and $b : y' \to y$ such that the diagram

$$\begin{array}{ccc} f(x') & \xrightarrow{\sigma'} & g(y') \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ f(x) & \xrightarrow[\sigma]{} & g(y) \end{array}$$

commutes in $\mathcal{G}$.

There are projection functors $p_j : \mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2 \to \mathcal{G}_j$ and a natural isomorphism $\Sigma : f \circ p_1 \to g \circ p_2$ (which is induced by the collection of maps σ). The above construction is universal with respect to this fact. More precisely, if $\mathcal{H}$ is a groupoid equipped with functors $\alpha : \mathcal{H} \to \mathcal{G}_1$, $\beta : \mathcal{H} \to \mathcal{G}_2$ and an isomorphism $\gamma : f \circ \alpha \to g \circ \beta$ of functors, then there is a triple

$$(h : \mathcal{H} \to \mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2, \lambda_1 : \alpha \to p_1 \circ h, \lambda_2 : \beta \to p_2 \circ h)$$

making the diagram

$$\begin{array}{ccc} f \circ \alpha & \xrightarrow{f(\lambda_1)} & f \circ p_1 \circ h \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Sigma \circ h} \\ g \circ \beta & \xrightarrow[g(\lambda_2)]{} & g \circ p_2 \circ h \end{array}$$

commute, and this triple is unique up to unique isomorphism. We may summarize by saying that there is a contractible 1-groupoid of 2-categorical fiber products.

We now turn to categories fibered in groupoids over $\mathcal{C}$. Let $\mathcal{F}_1 \xrightarrow{f} \mathcal{F} \xleftarrow{g} \mathcal{F}_2$ be a diagram of categories fibered in groupoids over $\mathcal{C}$. Any other category $\mathcal{G}$ fibered in groupoids over $\mathcal{C}$ gives us a groupoid $HOM_{\mathcal{C}}(\mathcal{G}, \mathcal{F}_1) \times_{HOM_{\mathcal{C}}(\mathcal{G},\mathcal{F})} HOM_{\mathcal{C}}(\mathcal{G}, \mathcal{F}_2)$ by our previous construction (note that these are all groupoids!). Then any morphism $\mathcal{H} \to \mathcal{G}$ of categories fibered in groupoids over $\mathcal{C}$ induces a morphism

$$HOM_{\mathcal{C}}(\mathcal{H}, \mathcal{G}) \longrightarrow HOM_{\mathcal{C}}(\mathcal{H}, \mathcal{F}_1) \times_{HOM_{\mathcal{C}}(\mathcal{H},\mathcal{F})} HOM_{\mathcal{C}}(\mathcal{H}, \mathcal{F}_2)$$

of groupoids.

Theorem 2. *There exists a category $\mathcal{G}$ fibered in groupoids over $\mathcal{C}$ making the above functor an equivalence of categories for all $\mathcal{H}$. Any two such $\mathcal{G}$ are related by an isomorphism of categories fibered in groupoids over $\mathcal{C}$, and any two such isomorphisms are related by a unique 2-isomorphism.*

In short, there is a contractible 2-groupoid of 2-categorical pullbacks of categories fibered in groupoids over $\mathcal{C}$.

Exercise 4. Let G be a discrete group, $\mathcal{B}G$: **Top** $\to$ **Groupoids** be the groupoid of principal G-bundles. Check that

$$\text{pt} \times_{\mathcal{B}G} \text{pt} = G$$

(considered as a fibered category by the over-category construction, so in particular the resulting groupoid is always discrete).

Exercise 5. Let $X \to \mathcal{B}G$ classify $G \hookrightarrow E \twoheadrightarrow X$, and let $H \leq G$. Check that

$$X \times_{\mathcal{B}G} \mathcal{B}H = E \times_G G/H = E/H.$$

5 Faithfully flat descent (Andrew Dudzik)

First, we'll talk about the word "descent".

Example 9 (Motivating example)**.** Suppose that we have a sheaf $\mathcal{F}$ on a topological space X which has an open cover $\{U_i \to X\}$. Then we have restrictions $\mathcal{F}|_{U_i}$ on each U_i. When can we reverse this process? In other words, given some sheaves $\mathcal{F}_i$ on U_i, can we glue them to a sheaf $\mathcal{F}$ with $\mathcal{F}|_{U_i} = \mathcal{F}_i$? Of course, we can't just do this, we need to specify isomorphisms on the overlaps, and moreover these must be suitably compatible.

Note that if we set $U = \coprod U_i$, then the collection $\{\mathcal{F}_i\}$ can be thought of as a single sheaf on U, and we have a "covering morphism" $U \to X$ along which the sheaf is supposed to *descend*. Since $U_i \cap U_j = U_i \times_X U_j$, we can think of the "compatibility" as living somehow in $U \times_X U$. We'll return to this.

Example 10 (Motivating example)**.** Suppose $k \hookrightarrow K$ is an inclusion of fields. For any k-vector space W, we have a natural K-vector space $W \otimes_k K$ of the same dimension. But suppose we start with a K-vector space V. Is there a "natural" k-vector space W with $W \otimes_k K$ being "naturally" isomorphic to V? Naively, certainly the answer is yes: we can choose a basis $\{e_i\} \subset W$ and then from $\bigoplus k \cdot e_i$. But can we do this without a basis? What data do we need exactly?

We now move into the technical part of the talk, that seeks to answer the questions laid out above.

Let $p : F \to C$ be a fibered category (i.e. we have a notion of *canonical pullback*). Let $X \xrightarrow{f} Y$ be a morphism in C. We define a category $F(X \xrightarrow{f} Y)$ as follows:

1. Objects are pairs (E, σ), where $E \in F(X)$ and σ is an isomorphism (in F) between the two pullbacks of E to $X \times_Y X$ along the projections $p_1, p_2 : X \times_Y X \rightrightarrows X$ such that the following diagram commutes:

$$\begin{array}{ccc} p_{23}^* p_2^* E \text{``}={}\text{''} p_{13}^* p_2^* E & \xleftarrow{p_{13}^*\sigma} & p_{13}^* p_1^* E \\ \uparrow {\scriptstyle p_{23}^*\sigma} & & \Vert \text{``}\ \text{''} \\ p_{23}^* p_1^* E \text{``}={}\text{''} p_{12}^* p_2^* E & \xleftarrow[p_{12}^*\sigma]{} & p_{12}^* p_1^* E. \end{array}$$

   Here, $p_{ij} : X \times_Y X \times_Y X \to X \times_Y X$ for $1 \le i < j \le 3$ denote the various projections, and the equalities are in scare quotes because these are actually only canonical isomorphisms.

2. Morphisms are just morphisms in F that are compatible with σ.

Note that an object $E' \in F(Y)$ induces *descent data*, as we set $E = f^* E'$ and take σ to be canonical – the two pullbacks along $X \times_Y X \rightrightarrows X \to Y$ are identified! (This is by definition of product.)

So, we have a functor $F(Y) \to F(X \xrightarrow{f} Y)$, the source we think of as our "gadgets" and the target we think of as our "descent data". When this is an equivalence of categories, we say that f is an *effective descent morphism.* For example, it is tautological that morphisms $\coprod U_i \to Y$ induced by open covers are effective descent morphisms for sheaves.

Example 11 (Revisiting the vector space example)**.** We showed that a choice of basis gives us a "descended" object. We'd like to phrase this in terms of descent data. We want an isomorphism $\sigma : V \otimes_k K \xrightarrow{\sim} K \otimes_k V$ of $K \otimes_k K$-modules. This is actually harder than it looks – we can't just reverse the factors, as we might first guess. But given a basis $\{e_i\}$ we can set $\sigma(ae_i \otimes b) = a \otimes be_i$. This is sneaky! It is routine to check the cocycle condition. This works because Spec $K \to$ Spec k is an effective descent morphism. This is an example of *faithfully flat descent.*

In a moment, we'll define the fppf topology, which will be finer than the Zariski or étale topologies, and yet we will still have the following theorem:

Theorem 3. *Any fppf covering $X \to Y$ is an effective descent morphism for quasi-coherent sheaves.*

In other words, we can assemble quasi-coherent sheaves on schemes by working fppf-locally instead of Zariski-locally. This is great! Much more is possible with the fppf topology.

Definition 11. An *fppf covering* is a morphism which is locally finitely presented, surjective, and flat.

Flatness is a strange condition. It seems that nobody can explain it intuitively. But it's not too hard to see why it's important. In general, we'd like to demand that pullbacks maintain geometry, and flatness says exactly that some "local vector" which is not pulled back to zero must be nonzero in any "local vector space". (We might demand that $x \neq 0 \Rightarrow f^*x \neq 0$, but surjectivity – or "faithfulness" – gives us this for free.)

The connection to descent is as follows. Take a ring map $A \to B$, and let M be a B-module. Take a free resolution of M and "descend it". The cocycle condition may tell you when you can do this, but we need something to prove that the new sequence is exact... and that something is faithful flatness.

The fppf topology is very useful. The étale topology (basically "local homeomorphisms") has trouble with inseparability (e.g. $\mathbb{F}_p(X^p) \hookrightarrow \mathbb{F}_p(X)$) and degeneration (e.g. $k[Z] \to k[X,Y,Z]/(XY - Z)$). The problem is that we get a singular fiber, so the map is not smooth but it is still relatively nice (for example, all fibers have the same dimension and genus). We have the following facts about the fppf topology:

- Zariski (pre?)sheaves are sheaves if they satisfy the sheaf condition for flat morphisms of affine schemes.
- h_X is a sheaf in the fppf topology. (We say that the fppf topology is *subcanonical.*)
- Descent works for quasi-coherent sheaves.

We end with a question: Is there a finer topology in which we can still perform descent?

6 Algebraic spaces (Yuhao Huang)

We begin with the definition.

Definition 12. Let S be a scheme. An *algebraic space over* S is a functor $X : (\mathbf{Sch}/S)^{op} \to \mathbf{Set}$ such that:

1. X is a sheaf on the big étale site on S.

2. The diagonal $\Delta : X \to X \times_S X$ is representable.

3. There exists an S-scheme $U \to S$ together with an étale and surjective morphism $U \to X$.

We will first explain this, and then compare it with the definition of a separated scheme.

Remark 2. Recall that the *big étale site* is just the category $\mathbf{Sch}/S$ with covers given by collections $\{U_i \to U\}$ of jointly-surjective étale morphisms.

Remark 3. Recall also that a morphism of functors $F \to G$ is called *representable* if given any scheme Y with a map $Y \to G$, the fiber product $Y \times_G F$ is representable. In particular, the diagonal $\Delta : X \to X \times_S X$ being representable is equivalent to saying that for any $U, V \to X$ (where U, V are schemes), $U \times_X V$ is again a scheme. This is because we have the cartesian diagram

$$\begin{array}{ccc} U \times_X V & \longrightarrow & U \times_S V \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta} & X \times_S X. \end{array}$$

Remark 4. Recall that a morphism $U \to X$ is étale and surjective iff for all $V \to X$ (where V is a scheme), $U \times_X V \to V$ is étale surjective.

Now, recall that we defined separated schemes as functors $F : (\mathbf{Aff}/S)^{op} \to \mathbf{Set}$ such that:

1. F is a sheaf on the big Zariski site;

2. F is covered by representable open subfunctors;

3. the diagonal $\Delta : F \to F \times F$ is closed.

We understand this definition because a scheme is after all just defined by gluing affine schemes along Zariski-opens. Thus an algebraic space is obtained by gluing affine schemes along étale neighborhoods.

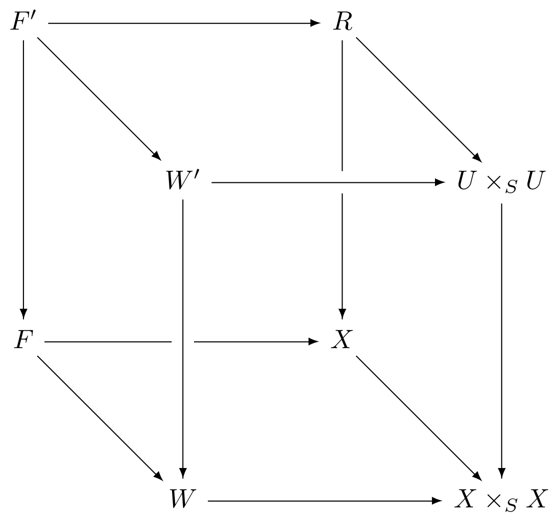
Since we know the diagonal is representable, we can develop the following alternate definition. First, if X is an algebraic space then we are guaranteed an étale surjective map $U \to X$ from an S-scheme $U \to S$. We can form $R = U \times_X U \to X$, and we have the two projections $R \rightrightarrows U$ which are both étale surjective. This is an *étale equivalence relation* (i.e. for any S-scheme T, the relation $p_1 \circ f \sim p_2 \circ f$ for $f : T \to R$ forms an equivalence relation on $U(T) = \mathrm{Hom}(T, U)$). Thus we have the following theorem.

Theorem 4. *If we have an étale equivalence relation $R \rightrightarrows U$ (i.e. both morphisms are étale and they define an equivalence relation on the set of T-points of U). Form the functor $T \mapsto U(T)/R(T)$, sheafify it with respect to the étale site Et_S, and call it X. Then X is an algebraic space.*

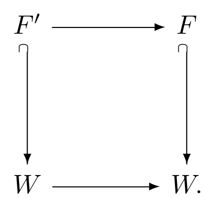
Proof. 1 and 3 are both clear. We must check 2, that $\Delta : X \to X \times_S X$ is representable. So suppose that $W \to X \times_S X$. We want $F = W \times_{X \times_S X} X$ to be a scheme. We will show this by quasi-affine descent. We know that $U \times_S U \to X \times_S X$ is a surjective morphism of étale sheaves. So there exists an étale surjective cover $W' \to W$ such that we get a factorization

$$\begin{array}{ccc} W' & \longrightarrow & U \times_S U \\ \downarrow & & \downarrow \\ W & \longrightarrow & X \times_S X. \end{array}$$

Now, we have



and 4 of the 6 faces are cartesian: namely, the bottom and right are cartesian by the definitions of R and F, so we can define F' by declaring the top to be cartesian, and hence the right side is automatically cartesian. (However, the front and back faces need not be cartesian.) The idea is to look at



We know that all of these but F are schemes and that both vertical arrows are monomorphisms (and hence separated), and we want to conclude that F is a scheme. We make the following reductions. First, we can assume W is affine since we can glue together the results at the end. Then we can also assume W' is affine. By Zariski's Main Theorem (the one by Deligne in EGA 4, not the one referenced in Hartshorne), we may reduce to the case that $F' \to W'$ is quasi-affine. Then F is a scheme by quasi-affine descent. (In summary, we know that F pulls back to a scheme under étale base change, and we have seen that this must descend to a scheme since $F' \to W'$ is quasi-affine. Recall that a morphism $X \to Y$ is *quasi-affine* if we can factor it as $X \hookrightarrow X' \to Y$, where $X \hookrightarrow X'$ is a quasi-compact open immersion and $X' \to Y$ is affine.) □

We now give some examples.

Example 12. Suppose a discrete group G acts on a scheme X. This action is free if $G \times X \to X \times X$ is a monomorphism (on the functors of points). Then, $G \times X \rightrightarrows X$ (where one map is the projection and the other is the action) is an étale equivalence relation, and the quotient X/G is an algebraic space.

Example 13. Here is an example of an algebraic space which is smooth but is not a scheme. (In 1 and 2 dimensions, smooth algebraic spaces are necessarily schemes.) Take two curves in $\mathbb{P}^3$ that intersect in 2 points. If we blow up the intersections in different orders and then glue them back together into an object Z [cf. Figure 24 in Appendix B or C of Hartshorne for the picture]. We can tell the result is not a scheme, because we have the algebraic equivalence $l_0 + m_0' \sim 0$: indeed, $m \sim l_0' + m_0'$ and $l \sim l_0 + m_0$, but also $m \sim m_0$ and $l \sim l_0'$. Now on Z we have an automorphism of $\mathbb{P}^3$ inducing an involution σ of Z given by $x_0 \leftrightarrow x_1$, $x_2 \leftrightarrow x_3$. Let $Z' = Z \backslash \mathrm{Fix}(\sigma)$. Then we can form the quotient Z'/σ, and then l_0 and m_0' will be identified. From basic intersection theory arguments, this cannot be a scheme.

7 Coarse moduli spaces, part 1 (Shelly Manber)

The classical way to think about invariant theory is just like Galois theory. Consider a field extension $K/\mathbb{Q}$ with Galois group G. Then $\mathbb{Q} = K^G$. The rings of integers are $\mathcal{O}_K$ (by definition/notation) and $\mathbb{Z}$, and over a prime ideal $p \subset \mathbb{Z}$ we have a finite number of prime ideals $p_1, \ldots, p_n$.

We distill the following nice properties.

1. G acts transitively on the p_i.
2. The map taking primes of $\mathcal{O}_K$ to primes of $\mathbb{Z}$ given by $p \mapsto p \cap \mathbb{Z}$ is surjective.
3. $\mathcal{O}_K$ is *integral* over $\mathbb{Z}$, i.e. every element is the root if a monic polynomial over $\mathbb{Z}$.
4. $\mathcal{O}_K$ is a finitely generated $\mathbb{Z}$-module.
5. Whenever we have a ring R and a map $\varphi : R \to \mathcal{O}_K$ such that $g \cdot \varphi(r) = \varphi(r)$ for all $r \in R$ and all $g \in G$, then φ factors through $\mathbb{Z}$.

We keep this in mind when we state the following result.

Theorem 5 (Main theorem)**.** *Let X_0 and X_1 be affine schemes forming a groupoid object in* **Schemes** *such that $s, t : X_1 \to X_0$ are finite and flat. Then there is a scheme Y and a morphism $\pi : X_0 \to Y$ such that $\pi \circ s = \pi \circ t$ which is universal among such pairs (Z, φ). (Another way of saying this is that*

$$X_1 \rightrightarrows X_0 \to Y$$

is a coequalizer diagram.) Furthermore, if $\alpha, \beta \in X_0$ such that $\pi(\alpha) = \pi(\beta)$, then there is some $z \in X_1$ such that $s(z) = x$ and $t(z) = y$.

(Note that coequalizers don't exist in general in **Schemes**.)

Now, let R be a ring and let G be a finite group which acts on R. Then we have two natural maps $R \rightrightarrows \prod_{g \in G} R$ (just the product of copies of R indexed by the elements of G): the diagonal map $r \mapsto (r, \ldots, r)$ and the action map $r \mapsto (g_1 \cdot r, \ldots, g_n \cdot r)$. Our running example will be $X_0 = \text{Spec } R$, $X_1 = \text{Spec Spec}(\prod_{g \in G} R)$. Then

$$R^G \hookrightarrow R \rightrightarrows \prod_{g \in G} R$$

is an equalizer diagram, so applying Spec gives us immediately that Spec R^G is the coequalizer for affine schemes; in fact, it is the coequalizer for all locally ringed spaces.

This setup enjoys the following nice properties:

1. π is finite (i.e. R is finitely generated as an R^G-module).
2. R is integral over R^G.
3. π is surjective.
4. For all $y \in Y = \text{Spec} R^G$, G acts transitively on $\pi^{-1}(Y)$.

Example 14. Consider the setup

$$\begin{array}{ccc} A & \longrightarrow & R \circlearrowleft G \\ \downarrow & & \downarrow \\ A' & \longrightarrow & A' \otimes_A R. \end{array}$$

We might ask: is $(R \otimes_A A')^G$ isomorphic to $R^G \otimes_A A'$? The answer in general is *no*. For example, take $A = K[\varepsilon]/\varepsilon^2$ for a field K with char $K = p$, take $A' = K$, take $R = A[t]$, and take $G = \mathbb{Z}/p\mathbb{Z}$. For a generator $g \in G$, we set $g \cdot t = t + \varepsilon$. Then, we'd have

$$\begin{array}{ccc} R & \longrightarrow & R \otimes_A A' \cong K[t] \\ \uparrow & & \uparrow \\ A & \longrightarrow & A' \end{array}$$

where all maps $A \to A'$ are $\varepsilon \mapsto 0$. Note that the G-action on $R \otimes_A A'$ is trivial, so the fixed ring is the entire thing.

Example 15. Take $R = K[s,t]$, with $G = \mathbb{Z}/2\mathbb{Z}$ acting as $s \mapsto -s$ and $t \mapsto -t$. Outiside of characteristic 2, this is not flat. So $R^G = K[s^2, t^2, st] \cong K[x,y,z]/(xy - z^2)$.

We now give a sketch of the proof of one of the more interesting features of our main theorem.

Lemma 4. *Suppose we have the hypotheses of the theorem. If $x, y \in X_0$ such that $\pi(x) = \pi(y)$, then there is some $z \in X_1$ such that $s(z) = x$ and $t(z) = y$.*

Proof sketch. The diagram $X_1 \rightrightarrows X_0 \to Y$ will be opposite to the diagram $A \hookrightarrow A_0 \rightrightarrows A_1$. The prime ideals $p_x, p_y \subset A_0$ will pull back to A by intersection, so then $p_x \cap A = p_y \cap A$.

We make the following claims. The first two are specific to our situation, the last three are general facts from commutative algebra.

1. There are finitely many primes over p_y.
2. A_0 is integral over A.
3. If $R \subseteq A$ such that A is integral over R and $p, q \subset A$ are distinct prime ideals with $p \cap R = q \cap R$, then $p \not\subset q$ and $q \not\subset p$.
4. If $I_1, \dots, I_n \subset R$ are prime and $J \subset \bigcup I_i$ then $J \subset I_i$ for some i.
5. If $A \to R$ is a finite flat map of rings and $\alpha \in R$ such that $N(\alpha) \in p \subset A$ (considering R as an A-module, so we can define norm as the determinant of the multiplication action) then $\alpha \in q \subset R$ where q is prime and moreover $p = q \cap A$.

Now, from Claim 1 we can write $q_1, \dots, q_n \subset A$ such that $(t^\#)^{-1}(q_i) = p_y$. If our lemma is false, then $(s^\#)^{-1}(q_i) \neq p_x$ for all i. By Claim 3, then $p_x \not\subset (s^\#)^{-1}(q_i)$ for any i. (If $p_x \subset (s^\#)^{-1}(q_i)$, then $s^\#(q_i) \cap A = p_y \cap A = p_x \cap A$.) Next, Claim 4 tells us that in fact $p_x \not\subset \bigcup s^\#(q_i)$. In particular, there is some $a \in p_x$ such that $a \notin s^\#(q_i)$ for all i. Our job, then, would be to construct an element $\sigma = N_{t^\#: A_0 \to A_1}(s^\#(a))$. It will turn out that $\sigma = A \cap p_x$. So finally to deduce that $\sigma \notin p_y$, we take $\alpha = s^\#(a)$. Now, if our lemma were to fail then we would have $s^\#(a) \subset q_i$, a contradiction. ☐

8 Quasicoherent sheaves of modules on algebraic spaces (Jason Ferguson)

Our goal is to define all the words in the title and to give at least one nontrivial example. We begin by reviewing the notion of sheaves of modules on schemes.

Definition 13. If $X = (|X|, \mathcal{O}_X)$ is a ringed space, then a *sheaf of modules* $\mathcal{F}$ on X is a sheaf of abelian groups on X, together with a morphism $\mathcal{O} \times \mathcal{F} \to \mathcal{F}$ of sheaves of sets, called the *action of $\mathcal{O}$ on $\mathcal{F}$, so that for all open $U \subset X$, the map $\mathcal{O}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ makes $\mathcal{F}(U)$ into an $\mathcal{O}(U)$-module.*

Example 16. Let A be a ring and M be an A-module. Then there is a sheaf of modules $\tilde{M}$ on Spec A which has that $\tilde{M}(D(f)) = M_f$, the localization of the module M, coming with the natural action of $\mathcal{O}(D(f)) = A_f$.

We would like generalize the notion of a sheaf of modules to any topos.

Definition 14. Let C be a site (i.e. a category with a Grothendieck topology). Let T be its topos (i.e. the category of sheaves of sets over C). A *sheaf of rings* $\mathcal{O}$ over C is a ring object in T. (Recall that saying $C \in T$ is a *ring object* is equivalent to saying that $\mathrm{Hom}_T(S, \mathcal{O})$ has the structure of a commutative ring which is functorial in S. In other words, we have a factorization $h_{\mathcal{O}} : T^{op} \to \textbf{Rings} \to \textbf{Sets}$ of the Yoneda functor of $\mathcal{O}$.) An *$\mathcal{O}$-module* $\mathcal{F}$ over C is an abelian group object in T together with an action morphism $\alpha : \mathcal{O} \times \mathcal{F} \to \mathcal{F}$ satisfying the usual diagrammatic axioms. (For example, the diagram

$$\begin{array}{ccc} \mathcal{O} \times \mathcal{O} \times \mathcal{F} & \xrightarrow{(\alpha,\alpha)} & \mathcal{F} \times \mathcal{F} \\ \downarrow {\scriptstyle +_{\mathcal{O}}, \mathrm{id}_{\mathcal{F}}} & & \downarrow {\scriptstyle +_{\mathcal{F}}} \\ \mathcal{O} \times \mathcal{F} & \xrightarrow[\alpha]{} & \mathcal{F} \end{array}$$

should commute.)

Notice that this definition reduces to the previous one when our site is just the category of opens in a scheme.

To talk about sheaves of modules on an algebraic space, we will need to choose the correct site. We first recall the following fact from algebraic geometry.

Proposition 3. *Let $f : X \to Y$ be a morphism of schemes. Then the following are equivalent:*

1. *f is étale .*
2. *There exist étale surjective morphisms $V \to Y$ and $U \to X \times_Y V$ as in the diagram*

$$\begin{array}{ccccc} U & \xrightarrow{\text{étale surj.}} & X \times_Y V & \longrightarrow & V \\ & & \downarrow & & \downarrow \text{étale surj.} \\ & & X & \longrightarrow & Y \end{array}$$

   such that the composition $U \to X \times_Y V \to V$ is étale .

3. *For all such setups, the composition is étale .*

(For a proof, see Ravi Vakil's notes.)

Definition 15. Let X be a scheme and Y be an algebraic space. We say that a morphism $f : X \to Y$ is *étale* if for all schemes T and morphisms $T \to Y$, the fibered product diagram

$$\begin{array}{ccc} X \times_Y T & \longrightarrow & T \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

(where the pullback is actually a scheme since Y is an algebraic space) has that $X \times_Y T \to T$ is an étale morphism of schemes.

(By the above proposition, it suffices to use our favorite cover.)

Definition 16. Pick an algebraic space Y. We define the *small étale site et_Y*. The objects are schemes X with an étale morphism $X \to Y$. Morphisms are commutative triangles (where it turns out that the new morphism will have to be étale , though this is not necessary for the definition). A collection $\{(X_i \to Y) \to (X \to Y)\}$ is a cover if $\coprod X_i \to X$ is surjective.

This is the same as the étale site when Y is just a scheme. However, note that there is no final object here.

Definition 17. If X and Y are both spaces, then we say that a morphism $X \to Y$ is *étale* if there exists a scheme V with an étale surjection $V \to Y$ and a scheme U with an étale surjection $U \to X \times_Y V$ for which $U \to X \times_Y V \to V$ is étale . (This is equivalent to this composition being étale for all such setups.)

We now let $\mathcal{O} : Et_Y \to \mathbf{Rings}$ be the sheaf of rings which for any étale morphism $X \to Y$ returns $\Gamma(X, \mathcal{O}_X)$. Then a sheaf of module over Y is a sheaf of $\mathcal{O}$-modules in Et_Y. This does indeed give sheaves of modules over every open. Given a scheme X, if $\mathcal{F}$ is a sheaf on Et_Y, then for any open $U \subset X$, the inclusion $U \hookrightarrow X$ is étale , so we get $\mathcal{F}(X) \to \mathcal{F}(U)$.

Theorem 6 (Miracle theorem)**.** *Now, let X be a scheme and $\mathcal{F}$ be a sheaf of modules on X. Then the following are equivalent:*

1. *There is an open affine cover $\{\text{Spec } A_i\}$ of X so that $\mathcal{F}|_{\text{Spec } A_i} \cong \tilde{M}_i$ for some A_i-module M_i.*
2. *For all open affines $\text{Spec } A$ in X, $\mathcal{F}_{\text{Spec } A} \cong \tilde{M}$ for some A-module M.*
3. *There exists an open cover $\{U_i\}$ of X so that $\mathcal{F}|_{U_i}$ is a quotient of two free sheaves. That is, for each i we have an exact sequence $(\mathcal{O}|_{U_i})^{\oplus I_i} \to (\mathcal{O}|_{U_i})^{\oplus J_i} \to \mathcal{F}|_{U_i} \to 0$. (The I_i and J_i may be hugely infinite.)*

Sheaves on X with these properties are called quasicoherent.

Definition 18. Let $\mathcal{F} : Et_Y \to \mathbf{Sets}$ be a sheaf of modules on Y. Then $\mathcal{F}$ is called *quasicoherent* if for each $(X \to Y) \in Et_Y$, the Zariski sheaf $\mathcal{F}_X$ on X determined by $\mathcal{F}$ is quasicoherent (in the sense above) and for every morphism $g : X \to X'$ in Et_Y, the resulting morphism $g^*\mathcal{F}_{X'} \to \mathcal{F}_X$ of sheaves on X is an isomorphism.

Example 17. One of the first examples of a quasicoherent sheaf on a scheme is an *ideal sheaf.* If $i : Z \hookrightarrow X$ is a closed immersion of schemes, we can look at $\mathrm{Ker}(\mathcal{O}_X \to i_*\mathcal{O}_Z)$, which is an $\mathcal{O}_X$-module. This is called the *ideal sheaf* of Z. These are in bijection with closed immersions of schemes. Now, if i is actually a closed immersion of algebraic spaces, we can also get an *ideal sheaf* of modules on X via, for schemes T, the fibered product

$$\begin{array}{ccc} Z \times_X T & \longrightarrow & T \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X. \end{array}$$

Since the lower horizontal map is a closed immersion, then the upper horizontal map is closed too. Moreover, since X is an algebraic space, then $Z \times_X T$ is actually a scheme. Now, we send $(T \to X)$ to $\Gamma(T, \mathrm{Ker}(\mathcal{O}_T \to i_*\mathcal{O}_{Z\times_X T}))$. In fact, these classify closed immersions into X up to isomorphism.

Example 18. The sheaf of differentials Ω^1 on an algebraic space X is a quasicoherent sheaf.

9 Algebraic stacks (Doosung Park)

If S is a scheme, then $\mathbf{Sch}/S$ is a big étale site. To motivate algebraic stacks, recall that finite projective limits are representable in $\mathbf{Sch}/S$, but for arbitrary limits this is not true. Consider étale -surjective maps $X \rightrightarrows Y$: we have a cokernel in the category of sheaves, $X \rightrightarrows Y \to Z$, but in general $Y \times_Z Y \neq X$. In fact, $Y \times_Z Y = X$ iff Z is an algebraic space. For example, if $G \times X \rightrightarrows X$ is an action of an étale group scheme G, in general the quotient $[X/G]$ is not an algebraic space. (The object $[X/G]$ will be explained later.)

Recall that a presheaf F on $\mathbf{Sch}/S$ is an *S-groupoid* (an S-category fibered in groupoids) has $F(U)$ is a discrete category with objects $F(U)$. If $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Z} \to \mathcal{Y}$ are morphisms in S-groupoids, we get $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$, where the objects of $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})(U)$ are diagrams $x \to f(x) \to g(y) \leftarrow y$ and the morphisms are diagrams of such zigzags.

Definition 19. $\mathcal{X}$ is called a *prestack* if for U/S and $x, y \in ob(\mathcal{X}(U))$, $\underline{Isom}(x, y) : \mathbf{Sch}/U \to \mathbf{Set}$ is a sheaf. This takes $(V \to U)$ to $\mathrm{Hom}_{\mathcal{X}(V)}(x_V, y_V)$. A prestack $\mathcal{X}$ is called a *stack* if for any covering $(f_i : V_i \to U)$, every *descent data* (i.e. objects $x_i \in ob(\mathcal{X}(V_i))$ with isomorphisms $f_{ji} : x_i|_{V_{ij}} \to x_j|_{V_{ij}}$ satisfying the cocycle condition) is effective; that is, there is some $x \in ob(\mathcal{X}(U))$ such that $f_i : x|_{v_i} \to x_i$ are isomorphisms and $f_j|_{V_{ji}} = f_{ji} \circ f_i|_{V_{ji}}$.

Proposition 4. *A presheaf X is a prestack iff X is separated, and from there, it is a stack iff X is a sheaf. Moreover, a stack $\mathcal{X}$ is a sheaf iff $|\mathrm{Hom}_{\mathcal{X}(U)}(x, y)| \in \{0, 1\}$ in some universe. (This is basically the same as saying that there are no nontrivial automorphisms.)*

Definition 20. Assume $\mathcal{X}$ is a prestack. We can define its *stackification* $\tilde{\mathcal{X}}$: an object of $\tilde{\mathcal{X}}(U)$ consists of a covering of U along with descent data, and morphisms are morphisms of descent data over the intersections of the coverings that are compatible with the gluing data.

Recall that an S-sheaf $\mathcal{X}$ is an algebraic space if there is a schematic étale cover $X \to \mathcal{X}$. (This is equivalent to the definition given in the book.) In some universe, one can equivalently say that there is a representable smooth cover $X \to \mathcal{X}$.

Definition 21. $\mathcal{X}$ is called a *Deligne-Mumford stack* if there is a representable étale cover $X \to \mathcal{X}$.

Example 19. Let X_1, X_0 be algebraic spaces, and let $s, t : X_1 \rightrightarrows X_0$ be smooth covers. Suppose we also have the structure morphisms m, i, ε satisfying the diagrammatic axioms for a groupoid. We then have the quotient $X_0(U)/X_1(U)$, whose objects are $X_0(U)$ and whose morphisms are $X_1(U)$. This is an actual groupoid. This is called $[x_0/X_1]'$. We have a morphism $X_1 \rightrightarrows X_0 \to [X_0/X_1]'$, a cokernel in the category if S-groupoids. This is a prestack, but it is not a stack in general. Its stackification $[X_1/X_0]$ will be the cokernel in the category of stacks. Then we do indeed recover that $X_1 = X_0 \times_{[X_0/X_1]} X_0$.

Proposition 5. *In the above example, $[X_1/X_0]$ is an algebraic stack.*

Proof. We want $X_0 \to \mathcal{X}$ to be a representable smooth cover. If U is a scheme with a map to $\mathcal{X}$, we need $X_0 \times_{\mathcal{X}} U$ to be an algebraic space. Now, given this setup, we have $U' \to U \to \mathcal{X}$ an étale cover such that the composition factors through X_0. So we obtain the diagram

$$\begin{array}{ccc} X' & \longrightarrow & U' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & U \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & \mathcal{X} \end{array}$$

(in which Y and X' are algebraic spaces).

Now, write $x = (x_0, u_0'), x' = (x_1, u_1') \in X'(V)$. Then $\mathrm{Hom}(x, x') = \mathrm{Hom}(x_0, x_1)$, which is either empty or a singleton. Since Y is a sheaf, then $U' \to U$ is a schematic étale cover and $X' \to Y$ is a schematic étale cover. Thus by definition Y is an algebraic space. □

Proposition 6. *The diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable.*

Proof. Let $X_0 \to \mathcal{X}$ be a smooth representable cover. Then we have $X_1 = X_0 \times_{\mathcal{X}} X_0 \rightrightarrows X_0 \to \mathcal{X}$, so $\mathcal{X} = [X_0/X_1]$. So, Δ is representable iff for all $U, V \to \mathcal{X}$, $U \times_{\mathcal{X}} V$ is an algebraic space.

We recall the following properties:

- If P is a property of morphisms which is local on the source, and we have $X \xrightarrow{f} Y \xrightarrow{g} Z$ where f is smooth, then:
  - If g has P, then gf has P.
  - If f is surjective and gf has P, then g has P.
- If P is a property of morphisms which is local on the target, and we have a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

  where g is a smooth cover, then:
  - If f has P, then f' has P.
  - If f' has P, then f has P.

Definition 22. We say that a morphism $\mathcal{X} \to \mathcal{Y}$ has P if for any cartesian diagram

$$\begin{array}{ccccc} Y'' & \xrightarrow{g} & Y' & \longrightarrow & Y \\ & & \downarrow & & \downarrow{\scriptstyle p} \\ & & \mathcal{X} & \longrightarrow & \mathcal{Y} \end{array}$$

such that p and g are smooth, then h has P. (Here, Y', $\mathcal{X}$, and $\mathcal{Y}$ are algebraic stacks and Y'' and Y are algebraic spaces.)

Now, if $\mathcal{X} \to \mathcal{Y}$ is representable and P is local on the target, then this has P if for any one diagram

$$\begin{array}{ccc} Y' & \xrightarrow{h} & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ \mathcal{X} & \longrightarrow & \mathcal{Y} \end{array}$$

such that g is a smooth cover, then h has P. We claim that for any such diagrams, h has P. (This is proved in the book.) $\square$

Exercise 6. Take k to be a field of characteristic p, and take μ_p to be the group scheme of p^{th} roots of unity. This is not smooth. Then $B\mu_p$ represents μ_p-torsors. Prove that this is a smooth algebraic stack.

10 Quasicoherent sheaves on algebraic stacks (Katrina Honigs)

Throughout, we'll be thinking of quotient stacks $[X/G]$.

10.1 Stacks and algebraic stacks

Despite our love of checking details, we will eschew this pathway for today's talk to give an intuitive picture.

Definition 23. A *stack* is a fibered category $p : F \to C$ over a site so that for any covering morphism $f : X \to Y$, we have selected pullbacks $f^* : F(Y) \to F(X)$, and $\mathcal{E} : F(Y) \overset{sim}{\to} F(X \to Y)$ is an equivalence of categories. Here, $X \to Y$ induces a cartesian diagram

$$\begin{array}{ccc} X \times_Y & \xrightarrow{pr_2} & X \\ {\scriptstyle pr_1}\downarrow & & \downarrow \\ X & \longrightarrow & Y, \end{array}$$

and $F(X \to Y)$ is the set of pairs $E \in F(X)$ and isomorphisms $\sigma : pr_1^* E \overset{\sim}{\to} pr_2^* E$ satisfying the cocycle condition. $F(X \to Y)$ is called the *category of descent* for the morphism $f : X \to Y$, and in fact the functor $\mathcal{E} : F(Y) \to F(X \to Y)$ is given by $E_0 \mapsto (f^* E, pr_1^* f^* E_0 \cong pr_2^* f^* E_0)$.

We like to think of equivalences of categories in the following helpful way. Basically, the categories encode the same objects and their interrelationships, up to renaming. For example, finite-dimensional k-vector spaces are equivalent to the natural numbers, where a morphism in the latter is just an $m \times n$ matrix.

Definition 24. An *algebraic stack* is a stack $\mathcal{X}$ satisfying:

1. The diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable (or equivalently, given $T, T' \to \mathcal{X}$ where T, T' are objects, the $T \times_{\mathcal{X}} T'$ is representable (by an algebraic space)).

2. We have a smooth surjective morphism $X \to \mathcal{X}$, where X is a scheme. (This is often called an *atlas*.)

(In order to specify a sheaf on an algebraic stack, it will suffice to specify it on X along with some descent data. This ends up being incredibly convenient.)

We now turn to our example, the quotient stack. Suppose we have an S-scheme X admitting an action of a smooth group S-scheme G, given by $G \times X \to X$. We can then define the quotient stack $[X/G]$. This is defined as follows. The objects of $[X/G](T)$ are corners $T \overset{\pi}{\leftarrow} P \overset{h}{\to} X$, where the fibers of π are torsors and h is a G-equivariant morphism. On the ground, this means we have an action G_T on P, where G_T is G pulled back to T such that

$G_T \times P \xrightarrow{\sim} P \times_T P$ given by $(g, p) \mapsto (gp, p)$. Thus we have a diagram

$$\begin{array}{ccc} G_T \times_T P & \xrightarrow{action} & P \\ \downarrow{\scriptstyle \mathrm{id}\times h} & & \downarrow{\scriptstyle h} \\ G_T \times_T X_T & \xrightarrow[action]{} & X_T. \end{array}$$

Example 20. For example, we have $BG = [*/G]$, the classifying space for a group. (We generally take $X = \text{Spec } k$.) In any of these cases, we have the atlas $X \to [X/G]$. This ends up being an algebraic stack.

Example 21. Another example is $\mathbb{P}^n = [\mathbb{A}^{n+1}\backslash\{0\}/\mathbb{G}_m]$, where the action $\mathbb{G}_m \times \mathbb{A}^{n+1}\backslash\{0\} \to \mathbb{A}^{n+1}\backslash\{0\}$ is given by $\lambda \times (a_0, \ldots, a_n) \mapsto (\lambda a_0, \ldots, \lambda a_n)$. Thus, a morphism $T \to \mathbb{P}^n$ is the same as a corner $T \leftarrow P \to \mathbb{A}^{n+1}\backslash\{0\}$, which we know is the same as a line bundle L on T along with generating sections $s_0, \ldots, s_n \in \Gamma(T, L)$. This in turn is equivalent to $\mathcal{O}^{n+1} \twoheadrightarrow L$.

Example 22 (first approximation to $\mathcal{M}_{1,1}$)**.** The stack $\mathcal{M}_{1,1}$ parametrizes "marked elliptic curves". That is, an object is $(S, (E, e))$, where S is a scheme and (E, e) is a marked elliptic curve. Namely, in good cases we can get to Legendre normal form $y^2 = x(x-1)(x-\lambda)$, where $\lambda \in U_\lambda = \mathbb{A}^1\backslash\{0, 1\}$. However, not all λ give different elliptic curves: $X_\lambda \cong X_{\lambda'}$ iff $\lambda' \in \{\lambda, 1-\lambda, 1/\lambda, 1/(1-\lambda), (\lambda-1)/\lambda, \lambda/(\lambda-1)\}$. These transformations are an action of the symmetric group S_3 on U_λ. Thus we have the quotient stack $[U_\lambda/S_3]$, and in fact there is a nice embedding $\mathcal{M}_{1,1} \subset [U_\lambda/S_3]$.

10.2 Quasicoherent sheaves on stacks

Let $\mathcal{X}$ be a stack on a site. We can define the *lisse-étale site*, whose objects are smooth maps $T \to \mathcal{X}$ from a scheme T, and morphisms are commutative triangles. Coverings are just étale coverings. Then, a sheaf on $\mathcal{X}$ will just be a sheaf on this lisse-étale site. We then get an embedding $QCoh(\mathcal{X}_{l-e}) \hookrightarrow \mathbf{Mod}_{\mathcal{O}_X}$, which is an exact functor in X.

To be precise, if $\mathcal{X}$ is an algebraic stack and we have an atlas $X_0 \to \mathcal{X}$, then we get an equivalence of categories $QCoh(\mathcal{X}) \to QCoh(X_0 \to \mathcal{X})$. For a sheaf $\mathcal{F}_0$ on our atlas X_0 with isomorphism $\sigma : \alpha^*\mathcal{F}_0 \xrightarrow{\sim} \beta^*\mathcal{F}_0$ (along with descent information), we can build $X_0 \times X_0 \rightrightarrows_\beta^\alpha X_0\mathcal{X}$, and then a map $T \to \mathcal{X}$ pulls all the way back to $T_0 \times T_0 \rightrightarrows_q^p T_0 \to T$, and we get the desired isomorphism $p^*\rho_0^*\mathcal{F}_0 \cong q^*\rho_0^*\mathcal{F}_0$, where $\rho_0 : T_0 \to X_0$ and $\rho_1 : T_0 \times t_0 \to X_0 \times X_0$.

Example 23. We can take an atlas Spec $k \to BG$. Then, giving a quasicoherent sheaf on BG amounts to giving a vector space over k along with some descent data. In fact, $QCoh(BG) \cong Rep_k(G)$. We show that these are equivalent by relating each of them to morphisms $BG^{pre} \to Vect_k$, where BG^{pre} is the prestack of BG and $Vect_k$ is the stack which takes T to k-vector spaces over $\Gamma(T, \mathcal{O}_T)$. (The prestack BG^{pre} takes a scheme T to a "copy of G" (i.e. G as a one-object category) for each connected component of T.) Any morphism from the prestack BG^{pre} to a stack must factor uniquely through the stackification BG. Now, such a morphism of prestacks is easily seen to be exactly G-representations over k, i.e. a map $G \to GL_n$, where $GL_n(T)$ gives $n \times n$ matrices with coefficients in $\Gamma(T, \mathcal{O}_T)$. On the other hand, a quasicoherent sheaf on BG can be given on the atlas Spec k, along with descent data.

Exercise 7. Take an algebraic group G (e.g. GL_n). This has a Lie algebra $\mathfrak{g}$, which admits the adjoint action of G. For every torsor over a scheme T, we can specify a representation. What is an explicit recipe?

11 $\mathcal{M}_g$ is a Deligne-Mumford stack (Martin Olsson)

We begin with a theorem.

Theorem 7. *If $\mathcal{X}/S$ is an algebraic stack, then $\mathcal{X}$ is Deligne-Mumford iff the diagonal $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is formally unramified.*

This is a key result which motivates many constructions. We will define "formally unramified" in a moment.

The idea is as follows. We begin with some moduli problem $\mathcal{M}$ that we want to prove has a "moduli space". We might try to sharpen from algebraic stack, to DM stack, to coarse moduli space. So we have the following rough recipe.

1. Show that $\mathcal{M}$ is an algebraic stack, usually by writing it as $[X/G]$ where X is a scheme (coming from a Hilbert scheme) and G is an algebraic group acting on X (e.g. GL_n, PGL_n).

2. Check that the diagonal is "formally unramified" (along with possibly some finiteness condition) to get to the DM setting.

3. Use Keel-Mori to get a (coarse) moduli space in the classical geometric sense.

Remark 5. The end result isn't necessarily an algebraic space. Different techniques are needed to get the moduli space as a quasiprojective scheme.

Today, we will focus on $\mathcal{M}_g$, the moduli stack of genus-g curves, for $g \geq 2$.

Definition 25. If k is an algebraically closed field, then a *genus-g curve* over k is by definition a smooth proper connected scheme C/k of dimension 1 (so that every localization gives a field or a dvr) with $h^0(C, \Omega^1_C) = g$. (If $k = \mathbb{C}$, this agrees with the usual definition.) More generally, if S is any scheme, a *genus-g curve over S* is a proper, flat (or equivalently smooth) morphism of schemes $f : C \to S$ such that for every algebraically closed field k and point $x : \operatorname{Spec} k \to S$, the fiber $C_x = C \times_S \operatorname{Spec} k$ is a genus-g curve over k.

Definition 26. Define $\mathcal{M}_g \to$ **Schemes** to be the fibered category with fiber over a scheme S to be the groupoid of genus-g curves over S. (This is the pseudofunctor approach; we haven't said what the morphisms are. The right way to do this is to say that $\mathcal{M}_g$ has objects $(S, C \to S)$ with C/S a genus-g curve and morphisms are cartesian squares.)

Theorem 8. *$\mathcal{M}_g$ is a DM stack.*

Proof. The key here is *canonical embedding*. We first begin with descent, which is not obvious. Suppose $S' \to S$ is an étale surjection, and suppose $f : C' \to S'$ is a genus-g curve. This gives us $S' \times_S S' \rightrightarrows S' \to S$, and descent data is an isomorphism $\sigma : p_1^* C' \to p_2^* C'$ satisfying the cocycle condition. We need to be able to produce a curve C/S. Certainly we could make the quotient as an algebraic space, which is sometimes all we can do. But here we can do better.

What do we know how to descend, anyways? Mainly we can just descend quasicoherent sheaves. So somehow we need to turn the whole question into quasicoherent sheaves. From the section on descent, we know that this gives us descent in particular for closed subschemes (as they are given by quasicoherent sheaves of ideals). So, let $E' = f_*(\Omega^1_{C'/S'})^{\otimes 3}$, which comes with a map $f^* E' \to (\Omega^1_{S'/C'})^{\otimes 3}$. This is a locally free sheaf of finite rank $5g - 5$. If we have any morphism of schemes $T \to S'$, then we get $f_T : C_T \to T$ (for C_T the pullback), and we get an isomorphism $g^* E' \to f_*(\Omega^1_{C_T/T})^{\otimes 3}$ as well as a commutative diagram

$$\begin{array}{ccc} C' & \hookrightarrow & \mathbb{P}E' \\ & \searrow & \downarrow \\ & & S'. \end{array}$$

So now, by this property of base change, we get an isomorphism $\overline{\sigma} : p_1^* E' \to p_2^* E'$ over $S' \times_S S'$ satisfying the cocycle condition, and we can descend down to E over S. Going back to our curves, we get $C' \hookrightarrow \mathbb{P}E' \to \mathbb{P}E$, and this gives us

$$\begin{array}{ccc} C & \hookrightarrow & \mathbb{P}E \\ & \searrow & \downarrow \\ & & S. \end{array}$$

Thus $\mathcal{M}_g$ is a stack.

Now, let's consider the variant $\tilde{\mathcal{M}}_g$, which associates to a scheme S a curve $Cf :\to S$ along with a fixed isomorphism $\lambda : \mathcal{O}_S^{5g-5} \xrightarrow{\sim} f_*(\Omega^1_{C/S})^{\otimes 3}$. Now we have the transformation $\tilde{\mathcal{M}}_g \to \mathrm{Hilb}_{\mathbb{P}^{5g-6}} = \{\text{closed subschemes of } \mathbb{P}^{5g-6}\}$, given by $(C \to S, \lambda) \mapsto (C \hookrightarrow \mathbb{P}(f_* \Omega^1_{C/s})^{\otimes 3} \xrightarrow{\sim} \mathbb{P}_S^{5g-6})$. One can actually show that there is no stackiness here: this is actually (equivalent to) a functor. Since $\mathrm{Hilb}_{\mathbb{P}^{5g-6}}$ is representable, this ends up implying that $\tilde{\mathcal{M}}_g$ is representable. But now, we just need to quotient out by the choice of λ. When we sort this out, we get that $\mathcal{M}_g \simeq [\tilde{\mathcal{M}}_g/GL_{5g-5}]$, where the action of $g \in GL_{5g-5}(\mathcal{O}_S)$ on $(C \to S, \lambda)$ is that $g * (C \to S, \lambda) = (C \to S, \lambda \circ g)$. One we know that $\mathcal{M}_g$ is a stack, it follows that it really is this indicated stack quotient. Thus $\mathcal{M}_g$ is an algebraic stack.

The next step is to figure out why this is DM. Recall that a stack is *Deligne-Mumford* is it admits an étale surjection from a scheme (instead of just a flat surjection). By the theorem, we have to check that $\Delta : \mathcal{M}_g \to \mathcal{M}_g \to \mathcal{M}_g$ is *formally unramified*: if $S_0 \hookrightarrow S$ is a closed immersion (of affine schemes, we can assume) defined by $J \subset \mathcal{O}_S$ with $J^2 = 0$ and we have the solid diagram

$$\begin{array}{ccc} S_0 & \hookrightarrow & S \\ \downarrow & \swarrow & \downarrow {\scriptstyle (c_1,c_2)} \\ \mathcal{M}_g & \xrightarrow[\Delta]{} & \mathcal{M}_g \times \mathcal{M}_g, \end{array}$$

then there exists at most one way to fill in the dotted arrow.

What does this *really* mean? First of all, $(c_1, c_1) : S \to \mathcal{M}_g \times \mathcal{M}_g$ really gives us two curves $c_i : S \to \mathcal{M}_g$; to say that this (2-categorically) factors through the diagonal is to give us an *isomorphism* between them. We are assuming that we have specified an isomorphism between their reductions to S_0. So the dotted arrow actually needs that this isomorphism is the identity on the reduction. So, Δ is formally unramified iff for all $S_0 \overset{J}{\hookrightarrow} S$ as before and C/S a genus-g curve, there are no nontrivial automorphisms of C/S reducing to the identity over S_0. (This is why people say that *being Deligne-Mumford is the same as having no infinitesimal automorphisms over a thickened neighborhood*.) Assuming S and S_0 are affine, deformation theory tells us that this is the same as saying that $H^0(C_0, T_{C_0/S_0}) \otimes J = 0$. Since this is supposed to be true for all J, we actually need that $H^0(C_0, T_{C_0/S_0}) = 0$, and in fact we can even take $S_0 = \mathrm{Spec}\ k$ for a field k. From Riemann-Roch, this turns out to be exactly equivalent to the statement that $g \geq 2$. This implies that $\mathcal{M}_g$ is a Deligne-Mumford stack. □

For the Keel-Mori theorem, what we really want is that $\Delta_{\mathcal{M}_g}$ is finite. Given $(C_1, C_2) : S \to \mathcal{M}_g \times \mathcal{M}_g$, we get the cartesian diagram

$$\begin{array}{ccc} \mathrm{Isom}(C_1, C_2) & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathcal{M}_g & \longrightarrow & \mathcal{M}_g \times \mathcal{M}_g, \end{array}$$

and we get a subscheme structure $\mathrm{Isom}(C_1, C_2) \to \mathrm{Hilb}_{C_1 \times C_2}$ given by $(f : C_1 \overset{\sim}{\to} C_2) \mapsto \Gamma_f \subset C_1 \times C_2$, where Γ_f denotes the graph of f. We need to show that $\mathrm{Isom}(C_1, C_2)$ is proper over S. For this we use the *valuative criterion*: given a dvr V, $K = \mathrm{Frac}(V)$, $C_1, C_2/V$, $\sigma_K : C_{1,K} \to C_{2,K}$, there should exist a unique $\sigma : C_1 \to C_2$ extending σ_K in the diagram

$$\begin{array}{ccc} C_{i,K} & \hookrightarrow & C_i \\ \downarrow & & \downarrow \\ \mathrm{Spec}\ K & \hookrightarrow & \mathrm{Spec}\ V. \end{array}$$

For curves, this is a classical result in the classification of varieties.

12 Coarse moduli spaces and the Keel-Mori theorem (Piotr Achinger)

12.1 Moduli problems

Often we want to organize some family of objects into a scheme, stack, or algebraic space. (For instance, we might want to give the Picard group the structure of a scheme.) For this to make sense, we need some notion of coherence of our the objects that we're trying to classify. That is, we want to be able to say when two such objects are "close", so that we can speak of families. As families – whatever they should be – can always be pulled back, we get a moduli functor $F : \mathbf{Sch}^{op} \to \mathbf{Sets}$ which sends a scheme T to the set of families of objects over T. In fact, we obtain a category fibered in groupoids, and we obtain (something like) a lax 2-functor $\mathcal{F} : \mathbf{Sch}^{op} \to \mathbf{Gpds}$.

If F is representable (by a scheme or an algebraic space X), we call X a *fine moduli space*.

Example 24. $\mathbb{P}^n$ is a fine moduli space. More generally, the Grassmannians $\mathcal{G}(k, n)$, the Picard scheme Pic (in nice cases), and the Quot scheme are all fine moduli problems. (Here we take the convention that $\underline{\mathrm{Pic}(X)}(T) = \mathrm{Pic}(X \times T)/\pi^*\mathrm{Pic}(T)$.)

The nice thing about representability is of course the Yoneda lemma: we have a universal family over X classified by id_X. Unfortunately, usually F is not even a sheaf! If our objects have automorphisms, then we get "nontrivial isotrivial families" (i.e. families in which all the fibers are isomorphic).

Example 25. If we were topologists, we would take the circle covered by two open intervals, an object x with a nontrivial automorphism, and then glue by id_x on one overlap and by σ on the other overlap.

We could do this algebraically as follows. If k is a function field or $\mathbb{F}_p$ (or something with nontrivial Galois group) and we have X/k and an extension l/k of degree 2, then the base change $X \times \mathrm{Spec}\ l$ is a cover of X admits different descent data from the usual one via the Galois action on Spec l.

The first possible solution is to talk about fibered categories instead, and we can hope that it is a "nice stack". The second possible solution is to consider a *coarse moduli space*.

Example 26 (elliptic curves)**.** The classical theory over an algebraically closed field K tells us that elliptic curves are classified by the j-invariant, $j(\mathcal{E}) \in K$. So we have a transformation $F \to \mathbb{A}^1$ (implicitly using the Yoneda embedding). We'd like to think that $\mathbb{A}^1$ is a "nice" moduli space. It is not a fine moduli space for this problem, but nevertheless we have that:

1. this is a bijection for $T = \mathrm{Spec}\ k$ when $k = \overline{k}$;

2. this is initial map from F to a scheme.

Incidentally, we now have another example of a nontrivial isotrivial family. We use the elliptic curves over $\mathbb{Q}$ defined by $\mathcal{E}_\pm : y^2 = x^3 \pm Dx$ where $\sqrt{D} \notin \mathbb{Q}$.

Definition 27. An algebraic space X and a map $\tau : F \to h_X$ is a *coarse moduli space* if:

1. τ is a bijection for geometric points from $T = \mathrm{Spec}\ k$ when $k = \overline{k}$;

2. τ is a universal map to an algebraic space.

We lose the fact that every family is pulled back from a universal family; a family gives a map to the coarse moduli space, but not all such maps give families.

12.2 The Keel-Mori theorem

We would like to compare these two potential solutions (fibered categories and coarse moduli spaces).

Definition 28. Given a stack $\mathcal{X}$ and a map $\pi : \mathcal{X} \to X$ for an algebraic space X, we call this pair a *coarse moduli space* if:

1. τ induces $|\mathcal{X}(T)| \to X(T)$ is a bijection for $T = \mathrm{Spec}\ \overline{k}$ (where absolute value denotes isomorphism classes);

2. τ is a universal map to an algebraic space.

We have the following general existence theorem. (Supposedly it was known to Deligne and Artin long before Keel and Mori.)

Theorem 9 (Keel-Mori,Conrad)**.** *Let $\mathcal{X}$ be an Artin stack over a scheme S which is locally of finite presentation. Suppose the diagonal of $\mathcal{X}$ is finite. Then there exists a coarse moduli space $\pi : \mathcal{X} \to X$. Moreover:*

1. *X is separated and locally of finite type (at least when S is noetherian);*

2. *π is proper;*

3. *this is preserved by flat base change, i.e. given a flat morphism $X' \to X$ then $\mathcal{X}' = X' \times_X \mathcal{X}$ with the map $\mathcal{X}' \to X'$ is a coarse moduli space.*

Remark 6. The assumption of finite diagonal is crucial. This is easily translated to saying that given $t_1, t_2 \in \mathcal{X}(T)$, then functor $\underline{\mathrm{Isom}}(t_1, t_2) : \mathbf{Sch}/T \to \mathbf{Sets}$ (which must be an a sheaf since $\mathcal{X}$ is a stack) is actually a scheme. This is not quite the same as demanding finite automorphism groups, but it's not that far off.

Remark 7. The condition that $\mathcal{X} \to \mathcal{Y}$ is *proper* is similar to the definition for schemes (universally closed, separated, and of finite type).

Remark 8. If $\pi : \mathcal{X} \to X$ is a proper coarse moduli space, then this is something like a universal homeomorphism. That is, we have that $\pi^{-1} : \mathrm{Open}(X) \to \mathrm{Open}(\mathcal{X})$ is a bijection.

Remark 9. The existence of a coarse moduli space is actually local on the stack $\mathcal{X}$. That is, if we have a covering $\mathcal{X} = \bigcup_i \mathcal{X}_i$ and we have $\pi_i : \mathcal{X}_i \to X_i$, then $\mathcal{X}_{ij}$ is open in $\mathcal{X}_i$ and $\mathcal{X}_j$, and so by the previous remark we have that X_{ij} is open in X_i and X_j. So we can glue the X_i along the X_{ij} to get X and $\pi : \mathcal{X} \to X$.

Remark 10. The idea of the proof is a "slice argument": If the stack is $X_1 \rightrightarrows X_0$, you start by looking for a subspace of X_0 so that when you restrict the groupoid you get the same stack. This actually ultimately requires some very delicate localization, however.

12.3 Tameness and quasicoherent sheaves

Definition 29. Suppose $\mathcal{X}$ is a Deligne-Mumford stack which is separated and of finite type over S. We say that $\mathcal{X}$ is *tame* if for all geometric points $x \in \mathcal{X}(\mathrm{Spec}\ olk)$, char $\overline{k}$ does not divide $|\mathrm{Aut}(x)|$ (which is finite by the Deligne-Mumford condition). (That is, $\mathcal{X}$ doesn't "mess with the characteristic".)

Proposition 7. *Suppose we have a Deligne-Mumford stack $\mathcal{X}$ which is separated and of finite type over S with finite diagonal. By the Keel-Mori theorem, we have some coarse moduli space $\pi : \mathcal{X} \to X$. Then X is tame iff the pushforward functor $\pi_* : QCoh\ \mathcal{X} \to QCoh\ X$ is exact.*

Thus for instance, in the tame case we can compute stack cohomology by pushing forward to the coarse moduli space.

Proof sketch. If we work étale -locally, then this functor is just taking invariants of the group action. □

Example 27. Recall that $\mathcal{M}_{1,1}$ is the moduli stack of elliptic curves. Over a base scheme S, we get smooth and proper morphisms $f : \mathcal{E} \to S$ such that geometric fibers are elliptic curves along with a section $e : S \to \mathcal{E}$. Then, we have the following facts:

1. $\mathcal{M}_{1,1}$ is a smooth separated Deligne-Mumford stack of finite type over Spec $\mathbb{Z}$;
2. the j-invariant functor $j : \mathcal{M}_{1,1} \to \mathbb{A}^1$ is a coarse moduli space.

Remark 11. Given any moduli problem (say a nice Deligne-Mumford stack with irreducible coarse moduli space): generically, how far off are we for having a family over the coarse moduli space? This translates into restricting the stack to the generic point of the coarse moduli space. This gives us something called a *gerbe*, which gives us a cohomology class. There is a whole industry of people who look at various examples and try to to determine whether or not this is trivial.

13 Gerbes (Peter Mannisto)

13.1 Torsors

We first give a definition for sets. Given a group G and a nonempty set S, we say that S is a *G-torsor if* the map

$$\begin{array}{ccc} G \times S & \longrightarrow & S \times S \\ (g, s) & \mapsto & (g \cdot s, s) \end{array}$$

is an isomorphism of sets. Alternatively, we can simply say that the G-action on S is free and transitive. Alternatively, we can just say that $S \cong G$ as G-sets.

Really, we want to have a topology around, so that we can begin "twisting" our G-torsors. For this talk, we will take $\mathcal{C}$ to be a site with all finite limits and with final object X (e.g. $(\mathbf{Sch}/X)_{\mathrm{fppf}}$, $(\mathbf{Sch}/X)_{\text{étale}}$, $(\mathbf{Top}/X)$).

Definition 30. Suppose μ is a sheaf of groups on $\mathcal{C}$, and take P to be a sheaf of μ-sets (i.e., we have a map of sheaves $\rho : \mu \times P \to P$ such that for any $u \in \mathcal{C}$, $\mu(U) \times P(U) \to P(U)$ is a $\mu(U)$-set action). We say that P is a *μ-torsor* if:

1. there is a cover $\{X_i \to X\}$ such that $P(X_i) \neq \emptyset$ for all i, and
2. for any $U \in \mathcal{C}$ with $P(U) \neq \emptyset$, the map

$$\begin{array}{ccc} \mu(U) \times P(U) & \longrightarrow & P(U) \times P(U) \\ (g, p) & \mapsto & (g \cdot p, p) \end{array}$$

   is an isomorphism of sets.

This really is exactly what people mean when they say *principal G-bundle*.

Remark 12. If $P(U) \neq 0$, then $P|_U \cong \mu|_U$. Indeed, by changing sites to $\mathcal{C}/U$, we can assume that U is the final object. To obtain a map $P \to \mu$, given $s \in P(U)$, by Yoneda's lemma we think of this as a map $s : U \to P$, and then we get

$$\mu \xrightarrow{(\mathrm{id},s)} \mu \times P \xrightarrow{\rho} P.$$

From here, it is not hard to check that this map must be an isomorphism.

Remark 13. Any morphism $f : P \to P'$ of μ-torsors (i.e. a morphism respecting the μ-action) is an isomorphism. Indeed, given such a morphism of sheaves over X (i.e. sheaves on $\mathcal{C}$, which happens to have terminal object X), to check that f is an isomorphism, we can check locally on X. Assuming $U \to X$ is a (one-element) cover such that $P(U) \neq \emptyset$ and $P'(U) \neq \emptyset$, by the previous remark we see that $P|_U \cong \mu|_U$ and $P'|_U \cong \mu|_U$ (considering μ as a μ-torsor via its left μ-action), and so

$$\mu|_I \simeq P|_U \xrightarrow{f|_U} P'|_U \simeq \mu|_U,$$

Remark 14. Let $\underline{\mathrm{Aut}}(P)$ denote the sheaf of μ-torsor automorphisms, and suppose μ is abelian. Then we have an isomorphism $\mu \xrightarrow{\sim} \underline{\mathrm{Aut}}(P)$ given by $1 \mapsto \mathrm{id}$. (If we didn't require μ to be abelian, then we'd have a left action on one side and a right action on the other.)

We denote by $B\mu$ the fibered category of μ-torsors over $\mathcal{C}$. The objects are $(U \to X, P)$ where P is a torsor on $\mathcal{C}/U$, and the morphisms are cartesian diagrams

$$\begin{array}{ccc} P' & \longrightarrow & P \\ \downarrow & & \downarrow \\ U' & \longrightarrow & U. \end{array}$$

This has the following properties.

1. $B\mu(U) \neq 0$ for any $U \in \mathcal{C}/X$.
2. Given two μ-torsors P and P' over U, there is a cover $U' \to U$ such that $P|_{U'} \cong P'|_{U'}$. (All torsors are "locally isomorphic".)

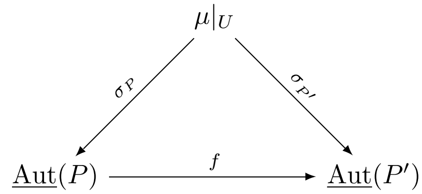
13.2 Gerbes

We begin with the definition of a gerbe. This is kind of like a torsor where you haven't specified the group. One should think of it as a *twisted classifying stack*.

Definition 31. A *gerbe* $\mathcal{G}$ on $\mathcal{C}/X$ is a stack on $\mathcal{C}/X$ such that:

1. There is a cover $\{X_i \to X\}$ such that $\mathcal{G}(X_i) \neq \emptyset$ for each X_i. (This implies local triviality for *any* object U, by pulling back the cover to $\{U \times_X X_i \to U\}$.)
2. Given $P, P' \in \mathcal{G}(U)$, there is a cover $\{U_i \to U\}$ such that $P|_{U_i} \cong P'|_{U_i}$ for each i.

Definition 32. Now, let μ be a sheaf of *abelian* groups on $\mathcal{C}/X$. Then a *μ-gerbe* is a gerbe $\mathcal{G}$ together with, for each $(U \to X, P) \in \mathcal{G}$, an isomorphism $\sigma_P : \mu|_U \to \underline{\mathrm{Aut}}(P)$, such that for any morphism $f : P \to P'$ in $\mathcal{G}(U)$, the diagram



commutes.

Proposition 8. *Given a gerbe $\mathcal{G}$ such that $\underline{\mathrm{Aut}}(P)$ is abelian for every $(P, U) \in \mathcal{G}$, then $\mathcal{G}$ has a structure of μ-gerbe for some μ.*

We need abelianity here because we make an arbitrary choice of isomorphism $P \xrightarrow{\sim} P'$ to give a *canonical* isomorphism $\underline{\mathrm{Aut}}(P) \to \underline{\mathrm{Aut}}(P')$.

We have the following analogous facts for μ-gerbes.

1. Any morphism of μ-gerbes $\mathcal{G} \to \mathcal{G}'$ is an isomorphism. (A morphism is of course just a morphism of stacks that respects the isomorphisms $\mu|_U \xrightarrow{\sim} \underline{\mathrm{Aut}}(P)$.) (To see that this is essentially surjective, we only need to show it locally, just as we can check surjectivity of a morphism of sheaves all the way down on stalks. If $P' \in \mathcal{G}'(U)$, then we can take a the cover $U' \to U$ with $\mathcal{G}(U') \neq \emptyset$. Let $P \in \mathcal{G}(U')$. then $f(P) \cong P'|_{U'}$ after taking some cover $U'' \to U'$. We won't check full-faithfulness, but it requires the μ-gerbe condition.)

2. $\mathcal{G} \cong B\mu$ as μ-gerbes iff $\mathcal{G}(X) \neq \emptyset$ (for X the final object).

What can we do with gerbes? Well, there is a bijection (which is difficult to even construct) between isomorphism classes of μ-gerbes on $\mathcal{C}/X$ and $H^2(X, \mu)$. Similarly, there is a bijection between isomorphism classes of μ-torsors on $\mathcal{C}/X$ and $H^1(X, \mu)$.

Example 28. Let us consider $\mu = \mathbb{G}_m$ on $(\mathbf{Sch}/X)_{et}$. For a scheme X, an *Azumaya algebra* $\mathcal{A}$ on X is a quasicoherent sheaf of $\mathcal{O}_X$-algebras (in the étale (or fppf) topology) such that there exists a cover $U \to X$ with $\mathcal{A}|_U \cong \mathcal{E}nd(E)$ for some locally free sheaf E on U of finite rank. So, as an example, if $X = \mathrm{Spec}\ k$, then $\mathcal{A}$ is a *central simple algebra* (e.g. $X = \mathrm{Spec}\ \mathbb{R}$, $\mathcal{A} = \mathbb{H}$). There's a gerbe $\mathcal{G}$ associated to any Azumaya algebra $\mathcal{A}$. This has $\mathcal{G}(U)$ with objects the locally free sheaves E on U, and with morphisms the isomorphisms $\sigma : \mathcal{E}nd(E) \xrightarrow{\sim} \mathcal{A}|_U$. This is in fact a $\mathbb{G}_m$-gerbe, and the corresponding map from the *Brouwer group* $Br(X)$ (i.e. the similarity classes of Azumaya algebras) to gerbes on X (i.e. $H^2(X, \mathbb{G}_m)$) is an injection. ($\mathcal{A} \sim \mathcal{B}$ means that $\mathcal{A} \otimes \mathcal{E}nd(E) \cong \mathcal{B} \otimes \mathcal{E}nd(F)$ for some locally free sheaves E and F.) When X is quasiprojective, then this is an isomorphism.

Exercise 8. Let k be a field.

1. There is a map $\mathcal{M}_{1,1,k} \to \mathbb{A}^1_k$. If we delete $\{0, 1728\}$ in the target and in the fiber, show that we get a $\mathbb{Z}/2$-gerbe.

2. Deduce that there is a curve $E \to \mathbb{A}^1 - \{0, 1728\}$ with for every $j \in \mathbb{A}^1$, $E_a \to a$ has j-invariant $j(E_a) = a$.