Introduction to Algebraic K-Theory

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Due to its historical development, algebraic K-theory has two main components: the study of lower K-groups (K_0, K_1, and K_2), which have explicit algebraic descriptions used in applications, and the study of higher K-theory, which was introduced by Quillen. Quillen gave two different constructions for the higher K-groups - the $+$ construction and the Q-construction - and proved the"$+ = Q$" theorem, which states that his two definitions agree. The first definition was particularly useful in computations, while the second one was more appropriate for proving theorems.

K_0

Let M be an abelian monoid. Then we can define the Grothendieck group $Gr(M)$ to be the free abelian group on M modulo the relation $[m] + [n] = [m + n]$. For example, $Gr(\mathbb{N}) = \mathbb{Z}$.

Recall that topological K-theory (defined by Atiyah and Hirzebruch in 1959) sets the complex K-theory group to be

$$KU^0(X) = Gr(\text{Vect}_{\mathbb{C}}(X)),$$

where $\text{Vect}_{\mathbb{C}}(X)$ is the monoid of (isomorphism classes of) complex vector bundles over X, with addition given by Whitney sum of bundles. Similarly, real K-theory was defined as

$$KO^0(X) = Gr(\text{Vect}_{\mathbb{R}}(X)).$$

Definition 1. The 0^{th} K-group of a ring is defined as

$$K_0(R) = Gr(\mathcal{P}(R)),$$

where $\mathcal{P}(R)$ is the monoid of isomorphism classes of finitely-generated projective R-modules under direct sum.

Exercise 1. Why do we require finite generation? (Hint: Eilenberg swindle.)

Example. Let F be a field. Then $\mathcal{P}(F) \cong \mathbb{N}$ since every module is free, so $K_0(F) = \mathbb{Z}$.

Example. Let R be a PID. By the structure theorem for modules over a PID, we know that projective modules are free. Thus again $K_0(R) = \mathbb{Z}$.

Example. Let K be a number field, and let $\mathcal{O}_K$ be its number ring (a/k/a ring of integers). It's a theorem that every number ring is a Dedekind domain, and the structure theorem for finitely-generated modules over Dedekind domains tells us that if P is projective of rank n, then $P \cong \mathcal{O}_K^{n-1} \oplus I$, where I is a uniquely defined class in the class group $Cl(K)$. Thus $K_0(\mathcal{O}_K) = \mathbb{Z} \oplus Cl(K)$.

Example. If X is a compact Hausdorff space, then $KU^0(X) \cong K_0(C^0(X, \mathbb{C}))$. (Here C^0 denotes continuous functions.) This is called *Swan's theorem*.

There are many other applications. For example, *Wall's finiteness obstruction* is an algebraic-K-theoretic obstruction to a CW complex being homotopy equivalent to a finite CW complex.

K_1

If R is a ring, we can define $GL(R) = \text{colim}_n GL_n(R)$ along the inclusions $GL_n(R) \hookrightarrow GL_{n+1}(R)$ given by inclusion into the upper-left corner (with a 1 in the bottom-right corner and zeros elsewhere). We then define

$$K_1(R) = GL(R)/[GL(R), GL(R)].$$

Lemma (Whitehead)**.** $[GL(R), GL(R)] = E(R)$, *the elementary matrices.*

If R is commutative, then $\det : GL(R) \to R^\times$ induces a map $\det : K_1(R) \to R^\times$. The kernel of this map is denoted $SK_1(R)$. On the other hand, we can include $R^\times \hookrightarrow K_1(R)$ via the identification $R^\times \cong GL_1(R)$. This splits the determinant map, and hence $K_1(R) \cong R^\times \oplus SK_1(R)$.

Example. Let F be a field. In this case, $E(F) = SL(F)$, and hence $K_1(F) = F^\times$.

Example. Let K be a number field and $\mathcal{O}_K$ be its number ring. By a theorem of Bass-Milnor-Serre, $SK_1(\mathcal{O}_K) = 0$. Thus $K_1(\mathcal{O}_K) \cong \mathcal{O}_K^\times \cong \mu(K) \oplus \mathbb{Z}^{r_1+r_2-1}$. The latter isomorphism is by the Dirichlet unit theorem, $\mu(K)$ denotes the unit group of K, r_1 is the number of real embeddings of K, and r_2 is the number of conjugate pairs of complex embeddings of K.

Higher K-groups

First we give some motivation for studying higher K-theory.

Quillen-Lichtenbaum Conjecture

Recall the (analytic version of the) class number formula:

$$\lim_{s\to 0} \frac{\zeta_K(s)}{s^{r_1+r_2-1}} = -\frac{h_K}{w_K} R_K,$$

where

- $\zeta_K(s)$ is the Dedekind ζ-function (if $K = \mathbb{Q}$, this is just the Riemann ζ-function);
- h_K is the class number of K;
- w_K is the number of roots of unity in K;
- R_K is the regulator of K.

Observe that $h_K = |K_0(\mathcal{O}_K)_{tor}|$, while $w_K = |K_1(\mathcal{O}_K)_{tor}|$.

Conjecture (Quillen-Lichtenbaum)**.**

$$\lim_{s\to -m} \frac{\zeta_K(s)}{(s+m)^{rk(K_{2m+1}(\mathcal{O}_K))}} = \pm \frac{|K_{2m}(\mathcal{O}_K)|}{K_{2m+1}(\mathcal{O}_K)_{tor}}$$

This gives an indication that higher K-groups can have extremely deep implications.

Vandiver's Conjecture

We give another example of the deep connections between number theory and algebraic K-theory. The proof of the following conjecture would be a landmark in algebraic number theory because it would imply other important conjectured results.

Conjecture (Vandiver's conjecture)**.** *Let $K = Q(\zeta_p)^+$ be the maximal real subfield of the p-th cyclotomic field $\mathbb{Q}(\zeta_p)$. Then p does not divide the class number of K.*

It has been shown that Vandiver's conjecture is equivalent to the statement that $K_{4i}(\mathbb{Z}) = 0$ for all i. This should seem quite amazing.

Definition of higher K-groups

Recall that complex topological K-theory is a generalized cohomology theory given by the 2-periodic spectrum $KU = \{BU, \Omega BU, BU, \Omega BU, \ldots\}$, while real topological K-theory is given by the 8-periodic spectrum $KO = \{BO, \ldots, \Omega^8 BO, \ldots\}$. Note that as topological groups, $O \simeq GL(\mathbb{R})$ and $U \simeq GL(\mathbb{C})$. By analogy with topological K-theory, one might try to define higher algebraic K-groups for a ring R using the space $BGL(R)$, where again, $BGL(R)$ stands for the classifying space of the group $GL(R)$. However, now $GL(R)$ is a discrete group, so $BGL(R)$ is a $K(GL(R), 1)$. This suggests one needs to make a modification to the construction, and Quillen defined the higher K-groups by using the Quillen $+$-construction of $BGL(R)$:

$$K_i(R) = \pi_i(BGL(R)^+) \text{ for } i \geq 1,$$

where the $+$-construction is taken with respect to the commutator subgroup of $GL(R)$. Recall that the plus-construction kills a perfect subgroup of the fundamental group and is an isomorphism on homology.

Definition 2. We define the *K-theory space* to be $K(R) = K_0(R) \times BGL(R)^+$, and then for all $i \geq 0$ we can set $K_i(R) = \pi_i(K(R))$.

Let's check:

$$K_1(R) = \pi_1(BGL(R)^+) = \pi_1(BGL(R))/[GL(R), GL(R)] = GL(R)/[GL(R), GL(R)],$$

so the new definition is consistent witht he previous definition of K_1.

Relation of the $+$-construction to group completions

There are many ways to get the $+$-construction, and they all yield homotopy equivalent spaces. One way (e.g. the way Hatcher does it) is by attaching 2- and 3-cells. A more conceptual way to obtain the space $BGL(R)^+$ is as the basepoint component of the "group completion" of the H-space $\coprod_{n=0}^{\infty} BGL_n(R)$, and this is a key point in the proof of the $+ = Q$ theorem.

We recall the following definition of a group completion of a space. Recall that for a homotpy-associative and -commutative H-space X $\pi_0(X)$ is an abelian monoid, $H_0(X; R)$ is the monoid ring $R[\pi_0(X)]$, and the integral homology $H_*(X; R)$ is an associative graded-commutative ring with unit.

Definition 3. Let X be a homotopy-commutative and -associative H-space. A map $f : X \to Y$, where Y is a homotopy-commutative and -associative H-space, is called a *group completion* if:

- $\pi_0(Y) = Gr(\pi_0(X))$, and
- $H_*(Y; R) = \pi_0(X)^{-1} H_*(X, R)$ for any coefficients R.

The idea in equivariant algebraic K-theory is to replace the classifying bundles for $GL_n(R)$ in $\coprod BGL_n(R)$ with equivariant bundles.

Quillen's motivation behind this definition of higher K-groups was to compute the K-groups of finite fields, which he was able to do by describing $BGL(R)^+$ as the homotopy fiber of a computable map.

The Q-construction

There was a need to prove "fundamental theorems of K-theory" that had already been defined for K_0 and K_1, such as the additivity, the resolution, the devissage, and the localization theorems. However, Quillen wasn't able to do this using the definition we've already seen. So instead he gave a new definition using the Q-construction, generalized the theorems, and then proved the $+ = Q$ theorem, i.e., that the two definitions agree.

We will use the notion of *exact categories* (e.g. abelian categories, additive categories with split-exact sequences (e.g. $\mathcal{P}(R)$)). We will denote monomorphisms by $\hookrightarrow$ and epimorphisms by $\twoheadrightarrow$.

Definition 4. Let $\mathcal{C}$ be an exact category. We construct the category $Q\mathcal{C}$ with $\text{ob}(Q\mathcal{C}) = \text{ob}(\mathcal{C})$ and with $Q\mathcal{C}(A, B)$ consisting of isomorphism classes of diagrams $A \twoheadleftarrow C \hookrightarrow B$.

Definition 5. The *classifying space* of a category $\mathcal{C}$ is given by the composition

$$\mathbf{Cat} \stackrel{\text{Nerve}}{\to} \mathbf{sSet} \stackrel{|-|}{\to} \mathbf{Top}.$$

Functors of categories turn into maps of classifying spaces, and natural transformations turn into homotopies of maps.

Exercise 2. Why do natural transformations correspond to homotopies?

Exercise 3. Compute the classifying space of the category with one object and two morphisms (where the non-identity morphism squares to the identity). (Hint: This is the group $\mathbb{Z}/2$, thought of as a category.)

Definition 6. Given an exact category $\mathcal{C}$, we define $K(\mathcal{C}) = \Omega B(Q\mathcal{C})$, and then we set

$$K_i(\mathcal{C}) = \pi_i(K(\mathcal{C})) = \pi_{i+1}(BQ\mathcal{C}).$$

$+ = Q$

Theorem (+=Q)**.** *For a ring R,*

$$\Omega BQ\mathcal{P}(R) \simeq K_0(R) \times BGL(R)^+.$$

The proof is incredibly involved, and would probably take two 45-minute talks to expose. The strategy of the proof is be to define an intermediate object, the category $S^{-1}S$ associated to a certain symmetric monoidal category S, and to show that $B(S^{-1}S)$ is homotopy equivalent to each of the terms in the equivalence we are trying to prove.