Motivation and Basics of Localization

Viraj Navkal

Localization

Let $\mathcal{C}$ be a category and $w \subset Mor(\mathcal{C})$ be a class of morphisms. A *localization* of $\mathcal{C}$ at w is a functor $l : \mathcal{C} \to w^{-1}\mathcal{C}$ such that $l(f)$ is an isomorphism for all $f \in w$ and that l is universal with respect to this property.

Bousfield Localization

Now, let $\mathcal{T} = Ho(Sp)$ denote the stable homotopy category and let E be a spectrum representing the homology theory $E_* = \pi_*(E \wedge -)$. We say that a map $f : A \to B$ of spectra is an *E-equivalence* if $E_*f : E_*A \to E_*B$ is an isomorphism. A space X is called *E-acyclic* if $E_*X = 0$ (we will be loose with reducedness of our theories), and a space Y is called *E-local* if for every E-acyclic space X, $[X, Y] = 0$. A map $\phi_E : A \to A_E$ is called an E-localization if it is an E-equivalence and A_E is E-local.

We make the following observations.

1. If $X \to A \xrightarrow{f} B \to \Sigma X$ is an exact triangle in $\mathcal{T}$, then we see that f is an E-equivalence iff X is E-acyclic. Thus we can reformulate the condition that Y is E-local to say that $f^* = [f, Y] : [B, Y] \to [A, Y]$ is an isomorphism whenever $f : A \to B$ is an E-equivalence.

2. E-localization is "natural". More precisely,

$$\begin{array}{ccc} A & \xrightarrow{E\simeq} & A_E \\ \downarrow & & \downarrow{\scriptstyle \exists!} \\ B & \xrightarrow[E\simeq]{} & B_E. \end{array}$$

   That is, $[A_E, B_E] \xrightarrow{\cong} [A, B_E]$.

Proposition. *The following are equivalent:*

1. *The localization $l : \mathcal{T} \to w^{-1}\mathcal{T}$ at the E-equivalences has a right adjoint r.*

2. *E-localizations exist; that is, for each $A \in \mathcal{T}$, there is an E-localization $\phi_E : A \to A_E$.*

Proof. First, suppose 1 is true. Then we can let ϕ_E be the unit $1 \to rl$ of the adjunction. Conversely, we can define $l : \mathcal{T} \to w^{-1}\mathcal{T}$ by $A \mapsto A_E$. □

Theorem (Bousfield, 1979)**.** *E-localizations exist for any spectrum E.*

For $X \in \mathcal{T}$, we denote by $\langle X \rangle$ the smallest full subcategory of $\mathcal{T}$ generated by X (taking arbitrary coproducts, cones, suspensions, and desuspensions).

We need the following rather tricky lemma, which we will not prove.

Lemma. *The subcategory of E-acyclic objects is of the form $\langle aE \rangle$ for some $aE \in \mathcal{T}$.*

Hence we can detect whether a spectrum Y is E-local just by looking at $[aE, Y]$: Y is E-local iff $[aE, Y] = 0$.

Proof of theorem. Let γ be the first infinite cardinal greater than the number of stable cells in aE. (In triangulated language, we may also say that aE is γ-small.) Now, observe that an E-localization $\phi_E : A \to A_E$ is a final E-equivalence: for any E-equivalence $A \to B$, we have

$$\begin{array}{ccc} A & \xrightarrow{E\simeq} & B \\ & \searrow^{\phi_E} & \downarrow{\scriptstyle \exists!} \\ & & A_E. \end{array}$$

We'd like to therefore set ϕ_E to be the homotopy colimit over all E-equivalences $A \to B$, but of course this is nonsense since these don't form a set, so we must somehow cut down this class so that it's definitely a set. (The following sometimes goes by the name "small object argument".) We define a tower

$$A = A_0 \overset{E\simeq}{\to} A_1 \overset{E\simeq}{\to} \cdots \overset{E\simeq}{\to} A_\alpha \overset{E\simeq}{\to} A_{\alpha+1} \overset{E\simeq}{\to} \cdots A_\gamma = A_E$$

as follows. For any ordinal α, we define

$$A_{\alpha+1} = cone\left(\coprod_{[aE, A_\alpha]} aE \to A_\alpha\right).$$

Now any map $aE \to A_\alpha$ becomes trivial when we postcompose into $A_{\alpha+1}$. The fibers at each stage will be built up out of copies of A_E, which guarantees that each of these maps are E-equivalences. For a limit ordinal β, we let $A_\beta = \mathrm{hocolim}_{\alpha<\beta} A_\alpha$. To see that A_γ is E-local, note that any map $aE \to A_\gamma$ must factor through some A_α for some $\alpha < \gamma$, and so the map dies at $A_{\alpha+1}$. □

Smashing Localizations

A spectrum E is called *smashing* if the E-localization functor L_E agrees with the functor $A \mapsto L_E S \wedge A$.

Proposition. *If E is a ring spectrum and $E = E \wedge S \to E \wedge E$ is an equivalence (or in other words $\eta : S \to E$ is an E-equivalence), then E is smashing.*

Proof. We proceed in the following steps.

1. $E \wedge A$ is always E-local.
2. $L_E S = E$.
3. $A \to E \wedge A$ is an E-equivalence.

We begin with step 1. We need to show that if X is E-acyclic then $[X, E \wedge A] = 0$. We have the diagram

$$\begin{array}{ccccc} X & \longrightarrow & E \wedge A & & \\ \downarrow & & \downarrow & \searrow^{=} & \\ E \wedge X & \longrightarrow & E \wedge E \wedge A & \longrightarrow & E \wedge A. \end{array}$$

But $E \wedge X = 0$, so $X \to E \wedge A$ must be trivial.

For step 2, we simply observe that $\eta : S \to E$ is an E-localization.

For step 3, we have that $E \wedge A \overset{(E\wedge\eta)\wedge A}{\to} E \wedge E \wedge A$ is an equivalence. □

So for example, if E is the Moore spectrum on $\mathbb{Z}[p^{-1}]$ (i.e. $\pi_n(E) = 0$ for $n < 0$, $\pi_0(E) = \mathbb{Z}[p^{-1}]$, and $H_n E = 0$ for $n > 0$) then E is a ring spectrum. As it turns out, this follows from the fact that $\mathbb{Z}[p^{-1}] \overset{\cong}{\to} \mathbb{Z}[p^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}]$.

In general, it is an open question which spectra are smashing. However, if B is a finite CW-spectrum, then there is a map $\psi : S \to S^B$ which is a $[B, -]_*$-trivialization (i.e. $\Sigma^n B, S^B = 0$ for all n and ψ is initial with respect to this property), then S^B is a ring spectrum which satisfies the conditions of the proposition: $S^B = S^B \wedge S \to S^B \wedge S^B$ is an equivalence. Hence S^B is smashing.

Conjecture (Bousfield)**.** *If E is smashing, then $L_E = L_{S^B}$ for some finite CW-spectrum B.*