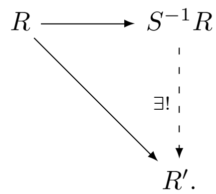
Model categories and Bousfield localization

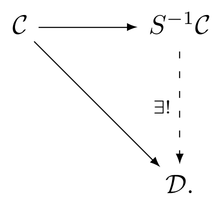
Enxin Wu

Localization

Recall that given a ring R and a $S \subseteq R$ is a multiplicatively closed subset, we can construct the localization $R \to S^{-1}R$, such that if $R \to R'$ send all elements of S to units then



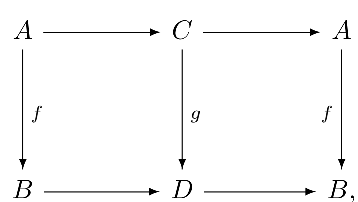
Analogously, if $\mathcal{C}$ is a category and S is a class of morphisms, we can construct a localization $\mathcal{C} \to S^{-1}\mathcal{C}$ such that if $\mathcal{C} \to \mathcal{D}$ is a functor sending all morphisms in S to isomorphisms then



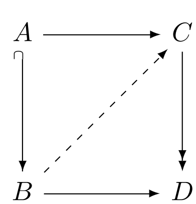
We'd like to construct the localization $S^{-1}\mathcal{C}$ by allowing morphisms to be (equivalence classes of) *zigzags*, i.e. $S^{-1}\mathcal{C}(a,b) = \{a \leftarrow \bullet \to \bullet \leftarrow \bullet \to \cdots \leftarrow \bullet \to b\}/\sim$. Unfortunately, $S^{-1}\mathcal{C}(a,b)$ is a proper class. How can we overcome this? One way is to use model categories.

Definition 1. A *model category* is a quadruple $(\mathcal{M}, W, C, F)$ of a category $\mathcal{M}$ and three classes of morphisms (called weak equivalences, cofibrations, and fibrations, resp.), satisfying:

1. $\mathcal{M}$ is complete and cocomplete.
2. W has the 2-out-of-3 property: if f and g are composable and two of $\{f, g, g \circ f\}$ are weak equivalences, then so is the third.
3. W, C, F are closed under retracts. (Recall that a *retract* is a diagram $B \to A \to B$ such that the composition is the identity.) This says that if we have a diagram of retracts

   then f is W/C/F iff g is the same.
4. If we have the solid arrows in

and $A \overset{\sim}{\hookrightarrow} B$ or $C \overset{\sim}{\twoheadrightarrow} D$ then the dotted arrow exists.

5. We can functorially factorize a morphism $A \to B$ as either $A \overset{\sim}{\hookrightarrow} C \twoheadrightarrow B$ or as $A \hookrightarrow D \overset{\sim}{\twoheadrightarrow} B$.

Generally, it is rather hard to check that a category is a model category. On the other hand, once we know that something is a model category, we get a huge amount of structure for free!

Theorem. *The homotopy category $Ho(\mathcal{M}) = W^{-1}\mathcal{M}$ is a category.*

Definition 2. We say that an object X is *cofibrant* if the unique map $\emptyset \to X$ is a cofibration. Similarly, we say that an object X is *fibrant* if the unique map $X \to *$ is a fibration.

Proof. The trick is to check that this gets around the set-theoretic issues. We obtain a "cylinder object" to do left homotopies and a "path object" to do right homotopies. Think about extending a map off $X \times \{0,1\}$ to a map off $X \times [0,1]$. Take $X \coprod X$. This has two maps to X, and we factorize this as $X \coprod X \hookrightarrow X \times I \overset{\sim}{\twoheadrightarrow} X$. (Here, $X \times I$ is our cylinder object.) Now we call two maps $f, g : X \to Y$ *homotopic* if we can factor $f \coprod g : X \coprod X \to Y$ through $X \times I$.

Now, given a model category $\mathcal{M}$, we can take the full subcategories of cofibrant objects $\mathcal{M}_c$, of fibrant objects $\mathcal{M}_f$, or of fibrant-cofibrant objects $\mathcal{M}_{cf}$, which gives us a diagram of functors

$$\begin{array}{ccc} \mathcal{M} & \longleftarrow & \mathcal{M}_c \\ \uparrow & & \uparrow \\ \mathcal{M}_f & \longleftarrow & \mathcal{M}_{cf}. \end{array}$$

When we apply Ho these all become equivalences, but the upshot is that we can actually write $Ho(\mathcal{M}_{cf}) \simeq \mathcal{M}_{cf}/\sim$! □

Example. The category **Top** of topological spaces is a model category. W is weak homotopy equivalences, F is Serre fibrations, and C consists of those maps with the "left lifting property" with respect to $W \cap F$.

Example. The category $\mathbf{sSet} = \mathbf{Set}^{\Delta^{op}}$ of simplicial sets is a model category. There is a *Quillen equivalence* $|-| : \mathbf{sSet} \leftrightarrows \mathbf{Top} : S_*$, which means that these model categories "have the same homotopy theory". In **sSet**, W is the weak homotopy equivalences after $|-|$, C is the levelwise injections, and F consists of those morphisms with the "right lifting property" with respect to $W \cap C$. It is a hard theorem that these are exactly the Kan fibrations.

Example. The category $Ch_{\geq 0}(R)$ of chain complexes of R-modules is a model category. W is the quasi-isomorphisms, F is the levelwise surjections at levels ≥ 1, and C is the morphisms with the left lifting property with respect to $W \cap F$. If we discard our functoriality assumption on replacements, we can replace $A \in \mathbf{Mod}_R \subseteq Ch_{\geq 0}(R)$ by a projective resolution, which will be cofibrant.

(Left) Bousfield Localization

Definition 3. Let $(\mathcal{M}, W, C, F)$ be a simplicial model category (i.e. a model category enriched in **sSet**). Let S be a class of morphisms. We say that X is *S-local* if $\underline{\mathcal{M}}(B, X) \overset{\sim}{\to} \underline{\mathcal{M}}(A, X)$ for all $(A \to B) \in S$. We say that a morphism $X \to Y$ is an *S-local equivalence* if $\underline{\mathcal{M}}(Y, Z) \overset{\sim}{\to} \underline{\mathcal{M}}(X, Z)$ for all S-local objects Z. We denote the class of S-local equivalences by W_S. Write F_S for the class of morphisms with the right lifting property with respect to $W_S \cap C$. Then $\mathcal{M}' = (\mathcal{M}, W_S, C, F_S)$ is called the *(left) Bousfield localization* of $\mathcal{M}$ with respect to S.

Theorem (uniqueness)**.** *If the Bousfield localization exists, then it has the universal property*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\text{“id”}} & \mathcal{M} \\ & \searrow^{F} & \Big\downarrow{\scriptstyle \exists!\ \text{Quillen pairs}} \\ & & \mathcal{N} \end{array}$$

for any functor F to a model category $\mathcal{N}$ such that W_S is taken to weak equivalences which is part of a Quillen pair $F : \mathcal{M} \leftrightarrows \mathcal{N} : G$.

Theorem (existence)**.** *For certain model categories $\mathcal{M}$ and for S a set of morphisms, the Bousfield localization exists.*

Example. The categories **Top**, $\mathbf{Top}_*$, **sSet**, and $\mathbf{sSet}_*$ are all good model categories in this sense. When $\mathcal{C}$ is a small category and $\mathcal{M}$ is such a good category, then so is $\mathcal{M}^{\mathcal{C}}$.

Application 1. There is a universal model structure on $sPre(\mathcal{C}) = \mathbf{sSet}^{\mathcal{C}^{op}}$ (the simplicial presheaves on $\mathcal{C}$). By Bousfield-Kan, weak equivalences and fibrations are defined "sectionwise" (and cofibrations are defined by their lifting property). Universality means that for any functor $\mathcal{C} \to \mathcal{D}$ to a model category, we have

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & sPre(\mathcal{C}) \\ & \searrow & \Big\downarrow{\scriptstyle \exists \text{ Quillen pair}} \\ & & \mathcal{D}, \end{array}$$

where the top map is the Yoneda embedding. For example:

1. If $\mathcal{C} = \Delta$ $S = \{\Delta^n \to *\}$, then $sPre(\mathcal{C})_S \leftrightarrows$ **Top** is a Quillen equivalence.
2. If S is all hypercovers, then $sPre(\mathcal{C})_S$ is equivalent to $sPre(\mathcal{C})$ with Jardine's model structure, which is denoted by $sPre(\mathcal{C})_{Jardine}$.
3. If $\mathcal{C} = Sm_k$ (the site of smooth k-schemes with the Nisnevich topology) and $S = \{X \times \mathbb{A}^1 \to X \text{ hypercovers}\}$, then $sPre(\mathcal{C})_S$ is equivalent to Morel-Voevodsky's $\mathbb{A}^1$ model structure on $sPre(\mathcal{C})$.

Suggestion (Marcy Robertson)**.** Anyone who hasn't seen this before should try out the category of small categories and functors between them:

- weak equivalences are equivalences of categories;
- cofibrations are the functors $F : C \to D$ such that $\text{ob}(F) : \text{ob}(C) \to \text{ob}(D)$ is a monomorphism;
- fibrations are functors $F : C \to D$ such that for all $c \in C$ and isomorphisms $g : F(c) \to d$ in D, there is some $c' \in C$ and an $f : c \to c'$ in C such that $F(f) = g$. (This encodes a sort of "path-lifting" idea from topology.)

References:

- Dwyer & Spalinski, *Homotopy Theories and Model Categories*
- Hirschhorn, *Model Categories & Their Localizations*
- Hovey, *Model Categories*
- Quillen
- Gelfand & Manin
- Dugger, *Universal Homotopy Theory*