Structure within Bousfield Lattices

Luke Wolcott

This talk will attempt to incorporate things from all the previous talks. The following categories have come up:

- the stable homotopy category $\mathcal{S}$;
- $D(R)$ for a commutative ring R;
- $D(R)$ for a commutative Noetherian ring R;
- $\mathcal{C}((kG)^*)$ (BCR);
- the stable module category $stmod(kG)$;
- $Stable(A_*)$ for A_* the dual Steenrod algebra;
- general tensored triangulated categories.

In fact these are all tensored triangulated categories, and all but the last are monogenic (i.e. generated by the unit object) *axiomatic* stable homotopy categories (i.e. they are tensored-triangulated, have arbitrary set-indexed coproducts, and the unit S is a small (i.e. $[S, \coprod X_\alpha] = \coprod [S, X_\alpha]$) weak generator (i.e. $\pi_*(X) = 0$ iff $X = 0$)). (We could add the equivariant stable homotopy category, but then we'd have to remove the word "monogenic".)

One can define a Bousfield lattice for any axiomatic stable homotopy category.

Definition 1. Recall that a *thick* subcategory of a triangulated category of finite (=compact) objects is a subcategory that is full, triangulated, and closed under summands. A *localizing* subcategory is one that is full, triangulated, and closed under coproducts.

There are two equivalence relations we can use to define Bousfield classes: we can use homology or cohomology. Cohomology is a bit scary and icky, so we'll stick to homology. Note that in all the above categories we have a version of Brown representability, so all cohomology theories take the form $[X, E]_*$; for homology theories to get something covariant *the most natural thing we can do* is have $[S, E \wedge X]_*$.

Definition 2. Let $\mathcal{C}$ be a (monogenic) axiomatic stable homotopy category, X an object. The *(homological) Bousfield class* is given by $\langle X \rangle = \{W : X \wedge W = 0\} = \{W : X_* W = 0\}$ (by the weak generator assumption). These are called X_**-acyclic*. We say that X and Y are *Bousfield equivalent* if $\langle X \rangle = \langle Y \rangle$. The collection of Bousfield classes is called the *Bousfield lattice*. We define a partial order by reverse inclusion, $\langle X \rangle \leq \langle Y \rangle$ whenever $W \in \langle Y \rangle \Rightarrow W \in \langle X \rangle$. Under this partial order, the maximum element is $\langle S \rangle$ and the minimum element is $\langle 0 \rangle$.

Example. Easy examples of spectra that are not homotopy equivalent but are nevertheless Bousfield equivalent include the facts that $\langle H\mathbb{Z} \rangle = \langle S \rangle$ and that $\langle E \rangle = \langle E \vee E \rangle$. The unit object S always generates the thick subcategory of *finite objects*, and the localizing category it generates is the entire category.

Proposition. *Every $\langle X \rangle$ is a localizing subcategory.*

Proof. Suppose $A \to B \to C$ is a triangle. Then $A \wedge X \to B \wedge X \to C \wedge X$ is a triangle. So if two of these are trivial, then so is the third. Moreover, if $W_\alpha \in \langle X \rangle$ then $X \wedge (\coprod W_\alpha) = \coprod (X \wedge W_\alpha) = 0$, so this is closed under coproducts. □

We have a *join* (a/k/a least upper bound) operation coming from the fact that we're looking at a lattice; this is given by $\langle X \rangle \vee \langle Y \rangle = \langle X \vee Y \rangle$. We have a well-defined operation $\langle X \rangle \wedge \langle Y \rangle = \langle X \wedge Y \rangle$; this is less than both $\langle X \rangle$ and $\langle Y \rangle$, but in general it is *not* the greatest lower bound. However, we'd like to define the *meet* (a/k/a greatest lower bound) by $\langle X \rangle \bigwedge \langle Y \rangle = \vee_{\langle W \rangle \leq \langle X \rangle,\ \langle W \rangle \leq \langle Y \rangle} \langle W \rangle$, but this may not be defined! However, there *is* in fact a *set* of homological Bousfield classes, so this is legal. This is what gives us meets and joins, which makes us a lattice.

Within this lattice we can look at sublattices, in particular there's one where $\bigwedge$ *is* the meet operation. This makes us into a distributive lattice, and inside of that we have a Boolean algebra. In spectra, every smashing localization corresponds to an object in this Boolean algebra, so studying it may be a way at getting at the telescope conjecture.

As we heard before, given an object E we get a Bousfield localization functor L_E that kills exactly $\langle E \rangle$, the E-acyclics. We might summarize this by saying that "homological localization exists". In fact, every localization functor is determined either by the things that it kills (a/k/a its acyclics) or its image (a/k/a its locals). Thus if $\langle E \rangle = \langle F \rangle$ then $L_E = L_F$. So we can think of the Bousfield lattice as exactly the set of ways in which we can homologically localize.

In the stable homotopy category, we have $\mathcal{F}$, the finite objects. We then have the *type* stratification

$$\cdots \subsetneq C_2 \subsetneq C_1 \subsetneq C_0 = \mathcal{F}.$$

Here $C_n = \langle K(n-1) \rangle \cap \mathcal{F}$. The *class invariance theorem* tells us that if $X, Y \in \mathcal{F}$, then $\langle X \rangle \leq \langle Y \rangle$ iff $type(Y) \leq type(X)$. So we have

$$\langle 0 \rangle \leq \langle F(\infty) \rangle = \langle H\mathbb{F}_p \rangle \leq \cdots \leq \langle F(1) \rangle \leq \langle S \rangle = \langle F(0) \rangle.$$

The reason we use homology instead of cohomology is that it's unknown in ZFC whether cohomological localization exists! (However,, it's known given an additional "large cardinal" set-theoretic axiom.) Moreover, nobody knows whether there's a *set* of cohomological Bousfield classes.

It can be shown that

$$\{\text{homological Bousfield classes}\} \subseteq \{\text{cohomological Bousfield classes}\} \subseteq \{\text{localizing subcategories}\},$$

but nobody knows whether any of these are strict containments or actual equalities or what.