Introduction to Symmetric Spectra

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Simplicial Sets

We will write $S = \{Fun(\Delta^{op}, \mathbf{Sets})\}$ for the category of simplicial sets and S_* for the category of pointed simplicial sets (in which we have a distinguished $* \in X_0$). For a discrete group G (considered as a 1-object category), we define $S_*^G = \{Fun(G, \mathbf{sSets})\}$ for the category of G-simplicial sets (where $*$ is a G-invariant basepoint). This category has a symmetric monoidal product $\wedge_G$: given $X, Y \in S_*$ we define $X \wedge Y = (X \times Y)/(X \vee Y)$, and given $X, Y \in S_*^G$ we define $X \wedge_G Y = X \wedge Y/$diagonal G-action. This is a closed symmetric monoidal product.

Proposition. *$Map_G(X, Y) = S_*^G(X \wedge \Delta[-]_*, Y)$ (with trivial G-action on Δ).*

Spectra

We now give the "Whitehead definition" of spectra.

Definition 1. A *spectrum* X is:

1. a sequence of pointed simplicial sets $(X_0, X_1, \ldots)$, and
2. pointed structure maps $\sigma : S^1 \wedge X_n \to X_{n+1}$.

(Note that smashing with the circle is just suspension: $S^1 = \Delta[1]/\partial\Delta[1]$.) A map of spectra $f : X \to Y$ is a sequence $f_n : X_n \to Y_n$ commuting with the structure maps. The category of spectra is denoted $\mathbf{Sp}^{\mathbb{N}}$.

Definition 2. An Ω-spectrum is a spectrum X such that each X_n is a Kan complex (i.e. a fibrant simplicial set) and the adjoint maps $\sigma^* : X_n \to Map(S^1, X_{n+1})$ are weak equivalences of simplicial sets.

Definition 3. A map $f : X \to Y$ of spectra is a *level equivalence* if each $f_n : X_n \to Y_n$ is a weak equivalence of simplicial sets.

We have the full subcategory $\Omega\mathbf{Sp}^{\mathbb{N}} \subseteq \mathbf{Sp}^{\mathbb{N}}$. From this we can define

$$Ho(\Omega\mathbf{Sp}^{\mathbb{N}}) = \Omega\mathbf{Sp}^{\mathbb{N}}[\text{level equiv.}^{-1}].$$

This is naturally equivalent to the stable homotopy category.

Symmetric spectra

We write Σ_p for the group of permutations of $\overline{p} = \{1, \ldots, p\}$. (Of course, $\overline{0} = \emptyset$.) We take $S^p = (S^1)^{\wedge p}$ for the p-sphere, the smash product of p copies of S^1. This comes with a left Σ_p-action.

Definition 4. A *symmetric spectrum* X consists of:

1. a sequence of pointed simplicial sets $(X_0, X_1, \ldots,)$;
2. a basepoint-preserving (left) action of Σ_n on X_n;
3. pointed structure maps $\sigma : S^1 \wedge X_n X_{n+1}$ such that the maps

$$\sigma^p = \sigma \circ (S^1 \wedge \sigma) \circ \cdots \circ (S^{p-1} \wedge \sigma) : S^p \wedge X_n \to X_{n+p}$$
$$S^i \wedge \sigma : S^i \wedge S^1 \wedge X_{n+p-i-1} \to S^i \wedge X_{n+p-i}$$

   are $\Sigma_p \times \Sigma_n$-equivariant.

A map $f : X \to Y$ of symmetric spectra consists of maps $f_n : X_n \to Y_n$ that are Σ_n-equivariant such that the diagram

$$\begin{array}{ccc} S^1 \wedge X_n & \longrightarrow & X_{n+1} \\ \downarrow {\scriptstyle S^1 \wedge f_n} & & \downarrow {\scriptstyle f_{n+1}} \\ S^1 \wedge Y_n & \longrightarrow & Y_{n+1} \end{array}$$

is Σ_{n+1}-equivariant. We denote by $\mathbf{Sp}^\Sigma$ the category of symmetric spectra.

We have the forgetful functor $U : \mathbf{Sp}^\Sigma \to \mathbf{Sp}^\mathbb{N}$.

Example (Suspension spectra)**.** Let $K \in S_*$. Then we define $\Sigma^\infty K = (K, S^1 \wedge K, S^2 \wedge K, \ldots)$. The Σ_n-action permutes the components of S^n and acts trivially on K. In particular, we have $\mathbb{S} = \Sigma^\infty S^0$, the *sphere spectrum.*

Smash products

The best thing about this category is that it has a really nice product which is actually fairly easy to write down.

Definition 5. The category S_*^Σ of *symmetric sequences* has objects that are just symmetric spectra without the structure maps. Explicitly, they take the form $X = (\Sigma_0 \curvearrowright X_0, \Sigma_1 \curvearrowright X_1, \ldots, \Sigma_n \curvearrowright X_n, \ldots)$. Morphisms are sequences of equivariant maps. The *tensor product* of two symmetric sequences is given by

$$(X \otimes Y)_n = \bigvee_{p+q=n} (\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q).$$

(Here we're considering the group Σ_n as a category and then taking the nerve to obtain a simplicial set.)

Proposition. *The tensor product $\otimes$ makes S_*^Σ into a closed symmetric monoidal category. The unit is the sequence $(S^0, *, *, *, \ldots)$.*

The symmetric sequence $S = (S^0, S^1, S^2, \ldots)$, which underlies the sphere spectrum, is a commutative monoid in S_*^Σ (i.e. a ring object). Multiplication is given by $\mu : S \otimes S \to S$, and this is commutative.

Proposition (Hovey-Shipley-Smith)**.** *The category $\mathbf{Sp}^\Sigma$ is equal to the category of left modules over the commutative monoid S.*

Proof sketch. To say that X is a left module means that we have a map $m : S \otimes X \to X$. This datum is equivalent to a collection of $\Sigma_p \times \Sigma_q$-equivariant maps $M_{p,q} : S^p \wedge X_q \to X_{p+q}$. So given such a left S-module X, there exists a symmetric spectrum which has X as its underlying symmetric sequence with exactly these structure maps $\sigma_n = M_{1,n}$. Conversely, given a symmetric spectrum Y we can obtain structure maps $M_{p,q} = \sigma^p : S^p \wedge X_q \to X_{p+q}$. □

We continue to mimic algebra.

Definition 6. Given $X, Y \in \mathbf{Sp}^\Sigma$, the smash product $X \wedge Y$ is the left module

$$X \otimes_S Y = \operatorname{colim}(X \otimes S \otimes Y \underset{1\otimes m}{\overset{m\otimes 1}{\rightrightarrows}} X \otimes Y).$$

(This makes sense because S is commutative, so we don't need to be thinking of X as a right S-module.)

Definition 7. We call a symmetric spectrum $X \in \mathbf{Sp}^\Sigma$ an Ω-*spectrum* if $UX \in \mathbf{Sp}^\mathbb{N}$ is an Ω-spectrum. We call a morphism $f : X \to Y$ in $\mathbf{Sp}^\Sigma$ a *level equivalence* if Uf is a level equivalence in $\mathbf{Sp}^\mathbb{N}$.

Now we have $\Omega\mathbf{Sp}^\Sigma \subseteq \mathbf{Sp}^\Sigma$, and

$$Ho(\Omega\mathbf{Sp}^\mathbb{N}) = Ho(\Omega\mathbf{Sp}^\Sigma) = \Omega\mathbf{Sp}^\Sigma[\text{level equiv.}^{-1}].$$

Stable equivalences $f : X \to Y$ are equivalent to weak equivalences on $\mathbf{Sp}^\Sigma$, and now

$$\mathbf{Sp}^\Sigma[\text{stable equiv.}^{-1}] = Ho(\mathbf{Sp}^\Sigma) = Ho(\Omega\mathbf{Sp}^\Sigma) = Ho(\mathbf{Sp}^\mathbb{N}) = \mathcal{SHC}.$$

Ring spectra

Definition 8. We say that $R \in \mathbf{Sp}^{\Sigma}$ is a *ring spectrum* if R is a monoid (ring) in $(\mathbf{Sp}^{\Sigma}, \wedge)$. That is, we have maps $R \wedge R \to R$ and $\mathbb{S} \to R$.

Proposition. *Ring spectra are the same thing as $\mathbb{S}$-algebras.*

We say that a category $\mathcal{C}$ is a *stable model category* if $\mathcal{C}$ has a notion of homotopy and $Ho(\mathcal{C})$ is triangulated.

Theorem (Schwede-Shipley)**.** *All stable model categories are always (spectrally) Quillen equivalent to some* $\mathbf{Mod}-\mathcal{E}(G)$*. ($G$ is "generators" and $\mathcal{E}$ is "endomorphisms".) (This is just a version of Morita theory.)*