Introduction to S-Modules

Rolf Hoyer

Definition 1. A *prespectrum* is a collection of spaces $D_n \in \mathbf{Top}$ with structure maps $\Sigma D_n \to D_{n+1}$. If we have a prespectrum E such that the adjoint maps $E_n \xrightarrow{\cong} \Omega E_{n+1}$ are homeomorphisms, then we call E a *spectrum.*

Example. Let $X \in \mathbf{Top}$. Then we have a suspension prespectrum $\Sigma^\infty X = \{\Sigma^n X, \Sigma\Sigma^n X \xrightarrow{\cong} \Sigma^{n+1} X\}$. We can "spectrify" this by taking our n^{th} space to be $\mathrm{colim}_k \Omega^k \Sigma^{n+k} X$. By the compactness of the sphere, the map from each space to the next will be a homeomorphism.

The inclusion $\mathcal{S} \to \mathcal{P}$ of spectra intro prespectra has a left adjoint $L : \mathcal{P} \to \mathcal{S}$, given by $LD_n = \mathrm{colim}_k \Omega^k D_{n+k}$ (assuming $D_n \to \Omega D_{n+1}$ are closed inclusions). Then, if $D \in \mathcal{P}$ and $X \in \mathbf{Top}$, we can define the smash product prespectrum $D \wedge X = \{D_n \wedge X\}$ and the function prespectrum $F(X, D) = \{F(X, D_n)\}$. If E is a spectrum, then $F(X, E)$ will be too. But $E \wedge X$ is not, so we use $L(E \wedge X)$ instead. (In general, limits-style constructions preserve spectra and colimits don't.)

But really, we want a smash product on the level of *spectra.* We might guess that we should set $(D \wedge D')_n = D_i \wedge D'_{n-i}$ for some (essentially arbitrary) i. This is a gross construction, but it does yield a symmetric monoidal smash product on the homotopy category $ho(\mathcal{S})$ assuming that $i \to \infty$ and $n - i \to \infty$. But this involves choices, which gives us the willies.

Definition 2. A *universe* $\mathcal{U}$ is a countably-infinite-dimensional real vector space with an inner product. An *indexing* A of $\mathcal{U}$ is an increasing chain of finite-dimensional subspaces which exhausts $\mathcal{U}$. (For example, if $\mathcal{U} = \mathbb{R}^\infty$ we can take A to be $\{0\} \subseteq \mathbb{R}^1 \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3 \subseteq \cdots$.) Given a universe $\mathcal{U}$ with an indexing A, we define categories of prespectra $\mathcal{PU}$ and spectra $\mathcal{SU}$ to be choices of spaces DV for each $V \in A$ with structure maps $\Sigma^{W-V} DV \to DW$ whenever $V \subseteq W$ (such that $DV \xrightarrow{\cong} \Omega^{W-V} DW$ if D is to be a spectrum); here $W - V$ denotes the orthogonal complement of V in W.

Exercise 1. If $A \subseteq B$ are indexings on a universe $\mathcal{U}$, then the forgetful functor from spectra indexed on B to spectra indexed on A is an equivalence of categories. Hence we can ignore indexing.

Moreover, this construction is natural in our choice of universe. Given a linear isometry $f : \mathcal{U} \to \mathcal{U}'$ of universes, we can form a functor $f^* : \mathcal{SU}' \to \mathcal{SU}$ via $(f^*E)V = E(fV)$. In fact, this has a left adjoint $f_* : \mathcal{SU} \to \mathcal{SU}'$. We can construct a pairing $\mathcal{SU} \times \mathcal{SU}' \to \mathcal{S}(\mathcal{U} \oplus \mathcal{U}')$ by taking $(E \wedge E')(V \oplus V') = EV \wedge E'V'$, and in the case $\mathcal{U} = \mathcal{U}'$ we can choose $f : \mathcal{U} \xrightarrow{\cong} \mathcal{U} \oplus \mathcal{U}$ and use our pullback f^* to make this external smash product into an internal smash product. This can be taken as $f^*(E \wedge E')$ or as $(f^{-1})_*(E \wedge E')$. But of course, this has one huge gaping flaw: the choice of f is just as arbitrary as the choices of i and $n - i$ before.

So, we write $\mathcal{L}(i) = \mathrm{Isom}(\mathcal{U}^i, \mathcal{U})$ for the (contractible) space of linear isometries. Instead of choosing some $f \in \mathcal{L}(2)$, we'll choose all of them at once! Given $E \in \mathcal{SU}$, a space $X \in \mathbf{Top}$, and a map $A \to \mathrm{Isom}(\mathcal{U}, \mathcal{U}')$, there is a construction called the *twisted half-smash product*, denoted $A \ltimes E \in \mathcal{SU}'$, with some very nice properties.

Theorem. *Given a homotopy equivalence $A \to A'$ of spaces, then the associated map $A \ltimes E \to A' \ltimes E$ is as well (as long as E is "tame").*

So in particular we can take $\mathcal{L}(2) \ltimes (E \wedge E')$, and this will be equivalent to $f_*(E \wedge E')$ for all points $\{f\} \xhookrightarrow{\simeq} \mathcal{L}(2)$.

Now we've got something canonical, but unfortunately it's not associative. Nevertheless, we get structure maps

$$\begin{aligned} \mathcal{L}(n) \times \mathcal{L}(i_1) \times \cdots \times \mathcal{L}(i_n) &\to \mathcal{L}(i_1 + \cdots i_n) \\ f, g_1, \ldots, g_n &\mapsto f \circ (g_1 \oplus \cdots \oplus g_n). \end{aligned}$$

These form what is called an *operad.* We get composition from $\mathcal{L}(1) \times \mathcal{L}(1) \to \mathcal{L}(1)$, and the unit is given by $\{\mathrm{id}\} \cong \mathcal{L}(0) \to \mathcal{L}(1)$. The functor $\mathbb{L} = \mathcal{L}(1) \ltimes -$ gives us a *monad* on $\mathcal{SU}$. (This is an endofunctor with a "composition" natural transformation $\mu : \mathbb{L} \circ \mathbb{L} \to \mathbb{L}$ (given by the composition map above) and a "unit" natural transformation $\eta : 1 \to \mathbb{L}$ (given by the unit map above (note that $E \cong \mathcal{L}(0) \ltimes E$)).)

Definition 3. We define the category $\mathcal{S}[\mathbb{L}]$ of $\mathbb{L}$-*spectra* to be the algebras over $\mathbb{L}$, i.e. spectra M with action map $\mathbb{L}M \to M$.

Note that for any spectrum M, $\mathbb{L}M$ is an $\mathbb{L}$-spectrum.

As an aside, we now have the structure in place to be able to define A_∞- and E_∞-ring spectra through this operad.

We look at $\mathcal{L}(2) \ltimes E \wedge F$ for E and F $\mathbb{L}$-spectra. We have that $\mathcal{L}(1) \times \mathcal{L}(1)$ acts on $\mathcal{L}(2)$ on the right (via the operad map) and on $E \wedge F$ via the $\mathbb{L}$-spectrum structure

$$(\mathcal{L}(1) \times \mathcal{L}(1)) \ltimes (E \wedge F) \cong (\mathcal{L}(1) \ltimes E) \wedge (\mathcal{L}(1) \ltimes F) \to E \wedge F.$$

We can now analogize with algebra and form

$$E \wedge_{\mathcal{L}} F = \mathcal{L}(2) \ltimes_{\mathcal{L}(1)\times\mathcal{L}(1)} (E \wedge F) = \mathrm{Coeq}\left((\mathcal{L}(2) \times \mathcal{L}(1) \times \mathcal{L}(1)) \ltimes (E \wedge F) \rightrightarrows \mathcal{L}(2) \ltimes (E \wedge F) \right).$$

This gives $\mathbb{L}$-spectra an associative pairing.

Since S has the structure of an $\mathbb{L}$-spectrum, we have a natural map $S \wedge_{\mathcal{L}} M \to M$. Unfortunately, this is in general only a weak equivalence. If this is an isomorphism, we say that M is an S-*module*. This gives us the desired symmetric monoidal structure.

The final picture is as follows:

$$\mathbf{Top} \leftrightarrows \mathcal{PU} \leftrightarrows \mathcal{SU} \leftrightarrows \mathcal{SU}[\mathbb{L}] \leftrightarrows S - \mathbf{mod}.$$

Except for the first pair, these are all Quillen equivalences.