Introduction to Equivariant Homotopy Theory

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G-(pre)spectra

The general goal of equivariant algebraic topology is to reproduce algebraic topology with groups acting everywhere. The key starting point is the following basic observation. Suppose X is a space and Y is a G-space. We can look at homotopy classes of maps $[X,Y]_G$ that are G-equivariant (with the trivial G-action on X). Then $[X,Y]_G \cong [X, Y^G]$, where Y^G denotes the fixed points of Y. In particular, mapping a fixed sphere will only ever capture the fixed points of our G-space. Thus, we'd better have spheres with G-actions if we want to detect anything useful. There are a few ways to think about these; we will look at *representation spheres*.

Definition 1. Let G be a compact Lie group and V be a finite-dimensional G-representation. Then S^V, the one-point compactification of V, admits a G-action with a fixed basepoint. (Nonequivariantly, $S^V \cong S^{|V|}$.)

So now we will start to suspend by representation spheres, rather than by normal spheres. But it's not as clear how to do this, because representation spheres don't all just sit inside each other nicely like a bunch of Russian dolls.

Definition 2. A *G-universe* $\mathcal{U}$ is a countably infinite representation of G that contains the trivial representation and contains countably many copies of each irreducible representation that it contains.

This could be just the fixed $\mathbb{R}^\infty$, or it could be infinitely many copies of the regular representation (assuming the group is finite).

Definition 3. A G-universe is called *trivial* if it only contains the trivial representation, and *complete* if it contains all finite-dimensional irreducible representations (irreps) of G.

We can now construct spectra indexed on universes exactly as before. Fix a G-universe $\mathcal{U}$. (We'll ignore the indexing set A.) For a finite-dimensional subrep $V \subseteq \mathcal{U}$, we write $\Sigma^V(-) = S^V \wedge (-)$.

Definition 4. A *G-prespectrum* indexed on $\mathcal{U}$ is a collection of G-spaces $\{EV\}$, one for each finite dimensional subrep $V \subseteq \mathcal{U}$, with equivariant maps $\sigma_{V,W} : \Sigma^{W-V} EV \to EW$ (that satisfy appropriate compatibility relations). A *G-spectrum* is a G-prespectrum E where the adjoint maps $\tilde{\sigma}_{V,W} : EV \to \Omega^{W-V} EW$ are homeomorphisms.

Example. Let $G = \{1\}, \mathcal{U} = \mathbb{R}^\infty$. Then we get normal (pre)spectra.

Example. Let X be a G-space. Then defining $EV = S^V \wedge X$ gives us a *suspension prespectrum.*

Definition 5. A G-spectrum indexed on a complete universe is called *genuine*. One indexed on a trivial universe is called *naive*.

Here are some facts:

1. There is a spectrification functor which is adjoint to the inclusion of spectra into prespectra.
2. Naive G-spectra are just sequence of G-spaces with suspension maps $\Sigma E_n \to E_{n+1}$ G-maps.
3. A map $D \to E$ of G-spectra is a weak equivalence if each every $DV \to EV$ is a weak equivalence.
4. One can get a nice smash product. (Just ask Rolf!)
5. The suspension spectrum functor Σ_G^∞ has a right adjoint Ω_G^∞, which takes a G-spectrum E to $E\{0\}$.
6. We can suspend and desuspend by any representation $V \subset \mathcal{U}$; moreover, these induce inverse equivalences of categories.

7. Given a (complete) G-universe $\mathcal{U}$, we can take G-fixed points $\mathcal{U}^G$. (Since we required there to be an infinite number of copies of the trivial rep, we know that at the very least we get $\mathbb{R}^\infty$.) The inclusion $i : \mathcal{U}^G \to \mathcal{U}$ gives adjoint "change-of-universe" functors, $G\mathcal{SU}(i_* D, E) \cong G\mathcal{SU}^G(D, i^* E)$ (for a genuine G-spectrum E and a naive G-spectrum D). Here, $(i^*E)(V) = E(i(V))$, and $(i_*D)(W) = DV \wedge S^{W-V}$ for $V = i^{-1}(W)$.

This is fundamentally different from nonequivariant spectra indexed on universes, because we can only play with the representations that we have in our universe, so e.g. there's a huge difference between genuine G-spectra and naive G-spectra.

Cohomology theories

Write $RO(G)$ for the real representation ring. Genuine G-spectra give $RO(G)$-indexed (co)homology theories. For any virtual representation $\nu = W - V$, we get a genuine G-spectrum $S^\nu = \Sigma^W S^{-V}$. Then for a genuine G-spectrum E and a G-space/spectrum X, we define

$$E^G_\nu(X) = [S^\nu, E \wedge X]_G$$
$$E^\nu_G(X) = [S^{-\nu} \wedge X, E]_G.$$

Fixed points

Given a space X and a G-space Y, recall that $Maps_G(X, Y) \cong Maps(X, Y^G)$. Since in spectra we don't have actual points to play around with, we'll use this adjunction to categorically define the "fixed points" of a G-spectrum.

We begin with the naive case. For a naive G-spectrum D and a nonequivariant spectrum C, we want to define D^G so that we get an adjunction $G\mathcal{SU}^G(C, D) \cong G\mathcal{SU}^G(C, D^G)$. The construction is to take $(D^G)(V) = (DV)^G$.

For a general G-spectrum indexed on $\mathcal{U}$, we take fixed points of the underlying naive spectrum; that is, we apply i^* (for $i : \mathcal{U}^G \to \mathcal{U}$ and then do the same construction as before: $E^G = (i^*E)^G$. This gives us the same adjunction as before.

Though this construction behaves well categorically, it's not very nice geometrically:

- There always exists a map $E^G \wedge E'^G \to (E \wedge E')^G$, but in general it's not an equivalence. This is in constrast with the situation in spaces.
- $(\Sigma^\infty_G X)^G \neq \Sigma^\infty(X^G)$.

So instead, we have another notion of fixed points (which we won't define), called *geometric fixed points*. This is a functor $\Phi^G : G\mathcal{SU} \to \mathcal{SU}^{\mathcal{G}}$ with properties:

1. $\Sigma^\infty(X^G) \simeq \Phi^G(\Sigma^\infty_G X)$, and
2. $\Phi^G(E) \wedge \Phi^G(E') \simeq \Phi^G(E \wedge E')$.