Introduction to Triangulated Categories

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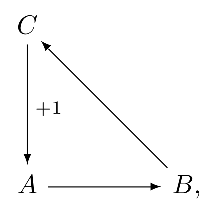
Pretriangulated and triangulated categories

Triangulated categories go back to Verdier, who as Grothendieck's student was tasked with developing the necessary homological algebra for Grothendieck's program. Given an abelian category $\mathcal{A}$ we can obtain the category of chain complexes $Ch(\mathcal{A})$, and from this Verdier defines the *derived category* $D(A) := Ch(\mathcal{A})[\text{quasi-iso's}^{-1}]$. Meanwhile, Puppe was working in topology to find properties that a good stable homotopy category should possess, and he came up with strikingly similar results.

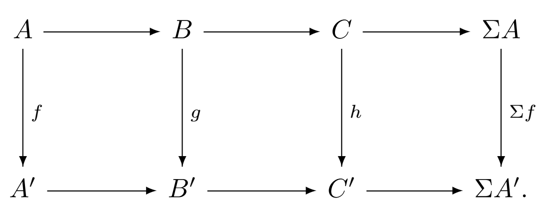
Definition 1. A *triangle* in a category $\mathcal{C}$ with respect to an endofunctor $\Sigma : \mathcal{C} \to \mathcal{C}$ (which we'll usually call "suspension") is a diagram in $\mathcal{C}$ of the form

$$A \overset{u}{\to} B \overset{v}{\to} C \overset{w}{\to} \Sigma A.$$

(This can be visualized as a triangle by writing



where the +1 indicates a degree shift.) A morphism of triangles is just a commutative diagram



Definition 2. A *pretriangulated category* is:

1. an additive category $\mathcal{A}$;

2. an automorphism $\Sigma : \mathcal{A} \to \mathcal{A}$;

3. a distinguished class ocf triangles, called "exact triangles", subject to the following axioms:

TR0 For every object $A \in \mathcal{A}$, the triangle $A \overset{\text{id}}{\to} A \to 0 \to \Sigma A$ is exact, and any triangle isomorphic to an exact triangle is also exact.

TR1 if $A \overset{u}{\to} B \overset{v}{\to} C \overset{w}{\to} \Sigma A$ is an exact triangle, then its two "rotations"

$$B \overset{v}{\to} C \overset{w}{\to} \Sigma A \overset{-\Sigma u}{\to} \Sigma B$$
$$\Sigma^{-1} \overset{-\Sigma^{-1} w}{\to} A \overset{u}{\to} B \overset{v}{\to} C$$

(left and right, resp.) are exact.

TR2 Every morphism $A \overset{u}{\to} B$ is the first morphism of an exact triangle.

TR3 Every commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \Sigma A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & \Sigma A' \end{array}$$

of exact triangles extends to a morphism

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \Sigma A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & \Sigma A' \end{array}$$

of exact triangles.

In order to understand these beasts, we will prove a few basic lemmas.
Suppose that $A \overset{u}{\to} B \overset{v}{\to} C \overset{w}{\to} \Sigma A$ is an exact triangle. Then we obtain the morphism

$$\begin{array}{ccccccc} A & \xrightarrow{\text{id}} & A & \longrightarrow & 0 & \longrightarrow & \Sigma A \\ \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle u} & & \downarrow & & \downarrow{\scriptstyle \text{id}} \\ A & \xrightarrow[u]{} & B & \xrightarrow[v]{} & C & \xrightarrow[w]{} & \Sigma A. \end{array}$$

We have the following version of the 5-lemma.

Lemma. *A morphism (f, g, h) of exact triangles is an isomorphism if f and g are isomorphisms.*

Corollary. *An exact triangle is determined up to isomorphism by any one of its maps.*

For example, given a map $A \overset{u}{\to} B$, the object C in the exact triangle $A \overset{u}{\to} B \to C \to \Sigma A$ is determined up to isomorphism by u. It is called "the" cone of u. Thus, we can (and perhaps should) think of an exact triangle just as a morphism and a choice of cone.

Proposition. *Given an exact triangle $A \overset{u}{\to} B \overset{v}{\to} cone(u) \overset{w}{\to} \Sigma A$, the morphism $B \overset{v}{\to} con(u)$ is a weak cokernel for u and the morphism $\Sigma^{-1}cone(u) \overset{\Sigma^{-1}w}{\to} A$ is a weak kernel for u. (Here, "weak" means that we've lost the uniqueness of the map from $cone(u)$ or to $\Sigma^{-1}cone(u)$.)*

Thus, saying that every map embeds into an exact triangle can be understand as saying that every map has a cone. The axiom TR0 says that every identity map has trivial cone.

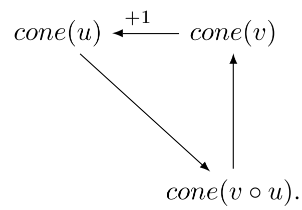
Note that even though cones are always isomorphic, there's no reason for them to be canonically isomorphic. As a result, there's no functorial way to associate morphisms to their cones. (This spurs some folks to look for souped-up versions of triangulated categories.) In any case, we now make our final definition.

Definition 3. A *triangulated category* is a pretriangulated category that satisfies the following axiom:

TR4 Suppose that we have two composable morphisms $X \overset{u}{\to} Y \overset{v}{\to} Z$ in $\mathcal{A}$. We can embed these individually into exact triangles, and we can also embed their composite into an exact triangle:

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ & {\scriptstyle \times 1}\nwarrow & \downarrow \\ & & cone(u) \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{v} & Z \\ & {\scriptstyle \times 1}\nwarrow & \downarrow \\ & & cone(v) \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{v \circ u} & Z \\ & {\scriptstyle \times 1}\nwarrow & \downarrow \\ & & cone(v \circ u). \end{array}$$

The “octahedral axiom” states that the cones make an exact triangle too:



(There’s also a hidden commutativity condition that we’re going to ignore.)

Thus, although cones aren’t functorial, there’s at least some sort of correspondence between the formation of cones and composition. However, nobody knows of a pretriangulated category that isn’t triangulated! On the other hand, there exist higher axioms which deal with more composable morphisms

Limits and colimits in triangulated categories

Limits and colimits rarely exist in triangulated categories. More precisely, products and coproducts often exist, but kernels and cokernels very rarely do. (Thus, our weak (co)kernels are very often the best we can do.) This leads to the theory of homotopy (co)limits, which don’t always exist but do for certain kinds of diagrams.

Why is it that kernels and cokernels rarely exist? The following result gives some indication.

Proposition. *In a pretriangulated category, every monomorphism and every epimorphism is split.*

In particular, every cokernel has a section, and similarly every kernel has a retract. This gums up the whole sitch. For example, this implies that any triangulated abelian category is semisimple (i.e. every exact sequence splits).

References:

- Neeman – *Triangulated Categories*
- Gelfand & Manin – *Methods of Homological Algebra*
- Verdier – *SGA 4.5*