Triangulated Categories #2

Bregje Pauwels

Background

The standard example of a triangulated category that we should keep in mind is the derived category of an abelian category: $\mathcal{A} \to K(\mathcal{A}) \to D(\mathcal{A})$. (Here, $\mathcal{A}$ is abelian, $K(\mathcal{A})$ is its homotopy category (which is triangulated), and $D(\mathcal{A})$ is obtained by inverting quasi-isomorphisms (and is triangulated).)

Definition 1. A functor $f : \mathcal{T} \to \mathcal{A}$ is *cohomological* if it turns exact triangles into exact sequences.

Example. Cohomology is cohomological. More generally, representable functors $\mathcal{T} \to \mathbf{Ab}$ given by $X \mapsto \mathrm{Hom}_{\mathcal{T}}(X, -)$ (and $\mathcal{T}^{op} \to \mathbf{Ab}$ given by $X \mapsto \mathrm{Hom}_{\mathcal{T}}(-, X)$) are cohomological.

Definition 2. An *exact functor* $F : \mathcal{T} \to \mathcal{T}'$ between triangulated categories is a functor that commutes with suspension and sends exact triangles to exact triangles.

Definition 3. We say that a multiplicative system S of morphisms in a triangulated category $\mathcal{T}$ is *compatible with the triangulisation* if:

1. whenever $\alpha \in S$, then $\Sigma^n \alpha \in S$ for all $n \in \mathbb{Z}$, and
2. given two exact triangles $X \to Y \to Z$ and $X' \to Y' \to Z'$ and maps $\alpha : X \to X'$ and $\beta : Y \to Y'$ such that $\alpha, \beta \in S$, then we can complete to a morphism (α, β, γ) of exact triangles.

Proposition. *When we are in this case, $\mathcal{T}[S^{-1}]$ is triangulated and the localization functor $Q : \mathcal{T} \to \mathcal{T}[S^{-1}]$ is exact.*

Example. If H is a cohomological functor, we can let S be the class of maps α such that $H(\Sigma^n \alpha)$ is an isomorphism for all $n \in \mathbb{Z}$. In particular, for $n = 0$ this gets us a quasi-isomorphism.

Definition 4. A *triangulated subcategory* is a full subcategory $\mathcal{C} \subseteq \mathcal{T}$ of a triangulated category such that

1. $\Sigma^n \mathcal{C} = \mathcal{C}$, and
2. whenever $X \to Y \to Z \to \Sigma X$ is an exact triangle in $\mathcal{T}$ such that two of the three objects are in $\mathcal{C}$, then the third object is in $\mathcal{C}$ as well.

Such a triangulated subcategory is called *thick* if it contains all direct summands of its objects.

Example. In $K(\mathcal{A})$, take $\mathcal{C}$ to be the full subcategory of acyclic objects. This is a thick triangulated subcategory.

Verdier localisation

Given a triangulated subcategory $\mathcal{C} \subseteq \mathcal{T}$, we want to find a multiplicative system S that the localization $Q : \mathcal{T} \to \mathcal{T}[S^{-1}]$ annihilates all objects of $\mathcal{C}$. We might say that we are searching for "$\mathcal{T}/\mathcal{C}$".

Lemma. *Let S be a multiplicative system compatible with the triangulisation, and write $Q : \mathcal{T} \to \mathcal{T}[S^{-1}]$. Then the following are equivalent:*

1. *$\alpha \in \mathrm{mor}(\mathcal{T})$ is annihilated by Q.*
2. *α factors through the cone of a map in S.*

Example. Take $K(\mathcal{A}) \to D(\mathcal{A})$. Then α is a quasi-isomorphism (i.e. it gets inverted in the localization) if and only if *cone*(α) is an acyclic complex.

Given a triangulated subcategory $\mathcal{C} \subseteq \mathcal{T}$, let us write $\mathrm{mor}_{\mathcal{C}}\mathcal{T}$ for the class of morphisms whose cone is in $\mathcal{C}$.

Lemma. $\mathrm{mor}_{\mathcal{C}}\mathcal{T}$ *is compatible with the triangulisation.*

Definition 5. The *Verdier localisation* of $\mathcal{T}$ at $\mathcal{C}$ is $\mathcal{T}/\mathcal{S} := \mathcal{T}[(\mathrm{mor}_{\mathcal{C}}\mathcal{T})^{-1}]$.

This enjoys the following properties.

1. Writing $Q : \mathcal{T} \to \mathcal{T}/\mathcal{C}$, the kernel $\ker(Q)$ of Q is the thick subcategory generated by $\mathcal{C}$.
2. This is universal, on two levels. Let $F : \mathcal{T} \to \mathcal{A}$ be cohomological or $F : \mathcal{T} \to \mathcal{T}'$ be exact such that $\ker(F) \supset \mathcal{C}$. Then F factors uniquely through Q.

Abelianisation

Recall that we had the construction $\mathcal{A} \to D(\mathcal{A})$ which embeds an abelian category into its derived category, which is triangulated. On the other hand, every triangulated category embeds into an abelian category, and this can be done in a universal way. We will be searching for a functor $F : \mathcal{T}^{op} \to \mathbf{Ab}$.

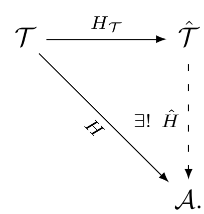
Definition 6. We say that F is a *coherent* functor whenever there exists an exact sequence of the form

$$\mathrm{Hom}_{\mathcal{T}}(-, X) \to \mathrm{Hom}_{\mathcal{T}}(-, Y) \to F \to 0.$$

We call this a *representation* of the functor F. The category of coherent functors is an additive category, which we will denote $\hat{\mathcal{T}}$. Moreover, whenever $\mathcal{T}$ has weak kernels (e.g. when it is triangulated), $\hat{\mathcal{T}}$ is abelian.

We have seen that the Yoneda functor $\mathcal{T} \to \hat{\mathcal{T}}$ given by $X \mapsto \mathrm{Hom}_{\mathcal{T}}(-, X)$ is cohomological.

Theorem. *The Yoneda functor $H_{\mathcal{T}}$ is the universal cohomological functor. More precisely, if $H : \mathcal{T} \to \mathcal{A}$ is cohomological, then we have*



Proof sketch. Given $F \in \hat{\mathcal{T}}$, take a representation $\mathrm{Hom}_{\mathcal{T}}(-, X) \to \mathrm{Hom}_{\mathcal{T}}(-, Y) \to F \to 0$. The first map can be written as $\mathrm{Hom}_{\mathcal{T}}(-, \phi)$ for some $\phi : X \to Y$. Then set $\hat{H}(F) = \mathrm{coker}(H\phi)$. $\square$

This can be used to construct idempotent completions. It can also be used to prove the Brown representability theorem.

Idempotent completion

It is not hard to see that $\hat{\mathcal{T}}$ is a *Frobenius* category: that is, it has enough projectives and enough injectives and they coincide. These will be exactly the representables. (This is again just a Yoneda-type argument.)

Explicitly, write $\tilde{\mathcal{T}}$ for the full subcategory of projective objects in $\hat{\mathcal{T}}$. Then the Yoneda functor $\mathcal{T} \to \tilde{\mathcal{T}}$ will have *split idempotents*. (Recall that this means that every idempotent morphism ϕ (i.e. $\phi^2 = \phi$) has a kernel. This is very useful for many proofs. Triangulated categories often have split idempotents, but not always. For example, $D^b(\mathcal{A})$ will have split idempotents, but $D(\mathcal{A})$ may not.)

This idempotent completion was constructed by Balmer-Schlichting. The point is not just that we can make this completion, but that there's still a triangulation on it.

Brown representability

Under certain conditions, Brown representability gives us a precise condition that detects representable functors.

Definition 7. An object S is called a *perfect generator* for $\mathcal{T}$ if

1. there is no proper full subcategory of $\mathcal{T}$ which contains S and is closed under coproducts, and
2. whenever we have a countable number of maps $X_i \to Y_i$ in $\mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{T}}(S, X_i) \to \mathrm{Hom}_{\mathcal{T}}(S, Y_i)$ is surjective, then the induced map $\mathrm{Hom}_{\mathcal{T}}(S, \coprod_i X_i) \to \mathrm{Hom}_{\mathcal{T}}(S, \coprod_i Y_i)$ is also surjective.

Theorem (Brown representability)**.** *Suppose $\mathcal{T}$ is a triangulated category with arbitrary coproducts and with a perfect generator S. Then the following are equivalent:*

1. *A functor $F : \mathcal{T}^{op} \to \mathbf{Ab}$ is cohomological and sends coproducts in $\mathcal{T}$ to products in $\mathbf{Ab}$.*
2. *F is representable.*

References:

- Verdier's thesis
- Neeman's *Triangulated Categories*
- Balmer-Schlichting 2001
- Krause's paper on coherent functors (proving Brown representability)