The Geometry of Tensored Triangulated Categories

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This talk is structured like a Tarantino movie: we start towards the end, go back to the beginning, and then jump to the finale.

Pretty much all of this theory is due to Paul Balmer.

Towards the end.

Definition 1. A *tensored triangulated category* (or TT category) is a triangulated category together with a tensor-structure which is compatible with the triangulation; that is, tensoring with exact triangles should preserve exactness and tensoring should commute with suspension. We will generally denote this by K.

Example. The stable homotopy category, the derived category of a commutative ring, singularity categories (e.g. the stable module category $stmod(kG)$ associated to a group ring) are all tensored triangulated categories. These also arise in many other places, including noncommutative geometry and the motivic stable $\mathbb{A}^1$-homotopy category.

We should keep in mind the analogy between a TT category and a ring.

Definition 2. A $\otimes$-ideal $J \subseteq K$ of a TT category is a triangulated subcategory such that if $a \in J$, $b \in K$, $a \otimes b \in J$. A $\otimes$-ideal P is a *proper thick $\otimes$-ideal* if whenever $a \otimes b \in P$, then $a \in P$ or $b \in P$.

From now on, we will assume that K is essentially small; that is, it has a skeleton whose objects form a *set*. (In our examples, we can take SH^{fin}, $D^b(R)$, et al.)

Definition 3. We define the topological space $Spc(K)$ which as a set is $\{P \subsetneq K : P \text{ is prime}\}$, and we define a basis for the topology by for each $a \in K$ looking at the *support* $supp(a) = \{P : a \notin P\}$ of a. (Primeness of an ideal is an isomorphism invariant, so this makes sense.) This does indeed define a topology, because of the following properties of the support:

- $supp(0) = \emptyset$, $supp(1) = Spc(K)$;
- $supp(a \oplus b) = supp(a) \cup supp(b)$;
- $supp(\Sigma a) = supp(a)$;
- given a distinguished triangle $a \to b \to c \to \Sigma a$ then $supp(c) \subset supp(a) \cup supp(b)$;
- $supp(a \otimes b) = supp(a) \cap supp(b)$.

(To explain the terminology, the "support" is precisely the primes in $supp(a)$ are exactly those for which the image of a isn't trivial in K/P.)

This construction enjoys the following nice properties.

1. If $K \neq 0$, then $Spc(K) \neq \emptyset$. (This wasn't true in previous attempts at definitions.)
2. If $F : K \to L$ is a TT functor, then we get a continuous map $Spc(F) : Spc(L) \to Spc(K)$. Thus Spc is a contravariant functor.
3. If $J \to K \to K/J$ is a localization, then $spc(K/J) \cong \{P \in Spc(K) : P \supset J\}$.
4. $Spc(K)$ is always quasi-compact and quasi-separated.

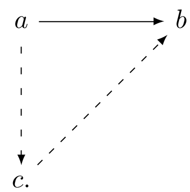
Exercise 1. An open subset $U \subseteq Spc(K)$ is quasi-compact iff $U = U(a) := supp(a)^c$ for some object $a \in K$.

Corollary. *$Spc(K)$ is Noetherian (as a space) iff every closed subset is realized as $supp(a)$.*

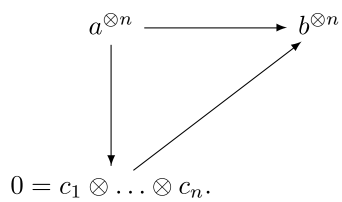
Definition 4. For a function $f : a \to b$, we define the *support* of f by $supp(f) = \{P : f \notin P\}$.

Lemma. *An object $a \in K$ maps to 0 in K/P for all P iff $a^{\otimes n} = 0$ (for some n). A morphism $f \in K$ maps to 0 in K/P for all P iff $f^{\otimes n} = 0$.*

Proof. If $f \in K$ maps to 0 in K/P, then there is some $c_P \in P$ such that we have a factorization



Then $Spc(K) = \bigcup_P U(c_P)$, and by quasi-compactness there are finitely many of these – say $c_1, \ldots, c_n$ – such that $c_1 \otimes \ldots \otimes c_n = 0$ and



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The beginning: chromatic homotopy theory.

The idea of chromatic homotopy theory is that MU sees a lot of stuff in SH^{fin}. Of particular importance is Ravenel's *nilpotence conjecture*: If $f : a \to b$ is a map of finite spectra and $MU_* f = 0$, then f is $\otimes$-nilpotent.

Here's a time-saving lie. Let's localize everything at a prime p. There are these things called *Morava K-theories* $K(n)$ (for $1 \leq n \leq \infty$), and we can equivalently say that if $K(n)_* f = 0$ for all n then f is $\otimes$-nilpotent. This was proved by Devinatz-Hopkins-Smith. If we look at the space $\mathbb{N} \cup \infty$ with the topology that the closure of n is $\{m : m \geq n\} \cup \infty$, then we can write $supp(a) = \{n : K(n)_*(a) = 0\}$, and then the nilpotence theorem is really nothing more than our lemma from before.

Hopkins-Smith described associated categories $\mathcal{C}_0 \supsetneq mcC_1 \supsetneq \cdots \supsetneq \mathcal{C}_\infty = 0$, the thick subcategories of kernels of the Morava K-theories. They found that these are all the thick subcategories of $SH^{fin}_{(p)}$.

So somehow even though we can't compute anything, we can nevertheless say something about the "global" structure of the stable homotopy category!

This is very useful for studying *generic* properties (i.e. those that are closed under taking direct summands, cofibers, etc.), because the complexes that satisfy such properties must for a thick subcategory.

Taking a step back, to classify $D^b(R)$, we should just look at $Spec(R)$. We define $supp(M_\bullet) = \{p \in Spec(R) : H_*(M)_p \neq 0\}$, the *(homological) support.*

Theorem (Thomason)**.** *If X is a quasi-compact, quasi-separated scheme, $D^{perf}(X) \subseteq D_{qcoh}(X)$ (which we should think of as complexes of vector bundles over X – this agrees with the above in the case $X = Spec(R)$), then*

$$\{\text{radical thick } \otimes\text{-ideals}\} \simeq \{\text{“Thomason-closed” subsets of } X\}.$$

(Y is Thomason-closed if $Y = \cup Y_i$ such that each $X \backslash Y_i$ is quasi-compact.)

Finale.

Definition 5. If (X, σ) is a pair where X is a space and $\sigma : \mathrm{ob}(K) \to \{\text{closed subsets of } X\}$ satisfying the axioms of support (above), we say that (X, σ) is a *support datum* for K.

Proposition. *$(Spc(K), supp)$ is the universal support datum for K.*

Theorem. *For any essentially small TT category K, Thomason's bijection is given by*

$$\begin{array}{rcl} \{\text{radical thick } \otimes\text{-ideals}\} & \simeq & \{\text{“Thomason-closed” subsets of } X\} \\ J & \mapsto & supp(J) = \cup_{a \in J} supp(a) \\ K_Y := \{a \in K : supp(a) \subseteq Y\} & \leftarrow & Y. \end{array}$$

This is the beginning of a new field called “tensored triangulated geometry”, wherein one tries to do algebraic geometry on TT categories. One puts a sheaf of rings on $Spc(K)$ by to any open U associating $(K/K_Z)^\natural$ (taking the nilpotent completion) and sheafifying.