The Ballad of the Gallant Hero and the Malicious Adversary

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This 5-act play illustrates how to prove *that a limit exists. The idea is that whenever we (the Gallant Hero) claim something is true, the Malicious Adversary tries to poke holes in our argument. Our job is to convince this Malicious Adversary of our claim beyond a shadow of a doubt.*

As you follow along, sketch each graph in question at the bottom of the page and fill in the blanks as they appear.

Act I.

Gallant Hero: My, but $y = \log_{10}(x)$ certainly does approach ∞ as x approaches ∞!

Malicious Adversary: What? No way.

Gallant Hero: It's true! I swear. How can I convince you?

Malicious Adversary: Well, is this function even always positive when x gets really big?

Gallant Hero: Heck yes. Just make sure that x is greater than ____. *Say a number N so that for all $x > N$, $\log_{10}(x) > 0$.*

Malicious Adversary: Okay, fine. But then is $\log(x)$ eventually always greater than 100?

Gallant Hero: Again yes. Sure, as long as x is past ____. *Say a number N so that when $x > N$, $\log_{10}(x) > 100$.*

Malicious Adversary: Whatever. How about a million? A billion?

Gallant Hero: Well then just make sure that $x >$ ________ or $x >$ ________, respectively.

Malicious Adversary: So it sounds like no matter what number I say, you're going to have an answer. Is that true?

Gallant Hero: Yup. That's why I made the claim in the first place.

Malicious Adversary: Maaaaaybe. But I'm calling your bluff. What if I say some number m (for malicious)?

Gallant Hero: Well then I'll just respond with ____________________. *Give an expression $N(m)$ for N in terms of m so that when $x > N$, $\log_{10}(x) > m$.*

Malicious Adversary: Okay, I give up. You win....*this* round.

What does it mean to "approach infinity"? More precisely, what do we consider a "neighborhood of infinity"? Note that in this act, we're looking at neighborhoods of infinity in both our domain and our range.

Act II.

Gallant Hero: Wanna know something nifty? As $x \to \infty$, $f(x) = 2 - \frac{1}{x} \to 2$.

Malicious Adversary: Wait, what?? But it never actually equals ____!

Gallant Hero: Yeah. That's why I said "approaches".

Malicious Adversary: Oh. Right. But I'm still skeptical. Is the distance between $f(x)$ and 2 eventually always ________________ 1?

Gallant Hero: Sure thing, just require that ____________.

Malicious Adversary: Well okay fine, but I was going easy on you; 1 is a pretty big margin. How about 0.01? Is it eventually always true that $|f(x) - 2| < 0.01$?

Gallant Hero: Yes it is, as soon as ____________.

Malicious Adversary: Buuuuh. Let me guess. This is gonna be the same thing as before; if I say some number ε, you're going to have an answer.

Gallant Hero: Well sure, as long as ε is greater than zero. Otherwise it wouldn't be fair.

Freeze frame. Why wouldn't it be fair?

____

____

Malicious Adversary: Despite being a cold-hearted villain, I am nevertheless inexorably compelled by your immutable logic.

Gallant Hero: Duh. And so anyways, no matter what ε you say, as long as it's positive, I'm just going to respond with _____________. *Give an expression $N(\varepsilon)$ for N in terms of ε so that when $x > N$, $|f(x) - 2| < \varepsilon$.*

Malicious Adversary: ________________________________! *Choose an expletive. Bonus points if you're colorful.*

What changed from the previous scene? What didn't? What do we mean by a "neighborhood of 2"? Note that in this act we're still looking at a neighborhoods of infinity in our domain, but now we're looking at neighborhoods of 2 in our range.

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Act III.

Gallant Hero: Hey there friend. $\lim_{x\to 0}\left(-\frac{1}{x^2}\right) = -\infty$.

Malicious Adversary: Nuh-uh. Is $f(x) = -\frac{1}{x^2}$ lower than -100 when x gets near zero?

Gallant Hero: Jajaja. As long as x is closer than ________ to 0.

Malicious Adversary: Well how about 10000? Is $f(x) < -10000$ when x gets near zero?

Gallant Hero: For that, you just need to demand that $0 < |x| <$ ________.

Malicious Adversary: Let's just cut to the chase. Suppose I say $-m$ (where m is a really big positive number). Then what do you say?

Gallant Hero: ____. *This time, give an expression $\delta(m)$ for δ in terms of m so that when $0 < |x| < \delta$, $f(x) < m$.*

Malicious Adversary: Grrrr. You're getting good at this.

What do we mean by a "punctured neighborhood of 0"? Why do we want it to be punctured? Note that in our domain we're looking at punctured neighborhoods of 0, while in our range we're looking at neighborhoods of $-\infty$.

Act IV.

Gallant Hero: Dear sir/madam. Please be apprised that $\lim_{x\to 0}\left(\frac{x}{|x|}\right) = 1$.

Malicious Adversary: Naw bro. Think about when x is negative. No way that's true.

Gallant Hero: Oh. Good point.

Malicious Adversary: BOOYA!

You can't always get what you want.
- *The Rolling Stones*

Act V.

Gallant Hero: Let me have another shot. At least I'm sure that $\lim_{x\to 0^+}\left(\frac{x}{|x|}\right) = 1$.

Malicious Adversary: Sounds reasonable, but I still kinda doubt it. When x gets close to 0 from the right, is $f(x) = \frac{x}{|x|}$ eventually within 0.5 of 1?

Gallant Hero: Yes, it's *always* within 0.5 of 1.

Malicious Adversary: I don't get it because I'm a doofus and I want you to spell out everything explicitly.

Gallant Hero: Okay, fine. You can let $0 < x <$ ____. Then $|f(x) - 1| < 0.5$. There, are you happy?!??

Malicious Adversary: Never.

Gallant Hero: Okay, I'll just say what I have to say: for any ____________ number ε you throw at me, if we require $0 < x <$ ____, then ____________ $< \varepsilon$.

Malicious Adversary: *[Sobs uncontrollably.]*

The camera pans out across the grass-covered plains. Crows fly overhead and bison graze in the distance as the Gallant Hero dropkicks the Malicious Adversary and then rides a horse off into the setting sun.

$\mathcal{FIN}$.

The moral.

To say that $\lim_{x\to a} f(x) = L$ is to say that **for any neighborhood V of L, there is a *punctured* neighborhood U of a which f takes entirely into V.** This works for any $-\infty \le a, L \le \infty$ (although since we don't consider ∞ to be an actual number, *every* neighborhood of ∞ is punctured, and the same goes for $-\infty$). One-sided limits are the same, except that we only require U to be a "one-sided neighborhood" (still not containing a, of course). In symbols, assuming a and L are both finite:

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0 : (0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon)$$