not currently included in this document: material on mt1

§2.8: linear approx's and differentials

- linear approx: $Lf(x) - f(a) = f'(a) \cdot (x - a)$, for $f(x)$ differentiable at $x = a$
  - overapprox if concave down, underapprox if concave up
  - particular case: $L\sin(x) = x$ at $a = 0$
  - also written using "differentials" as $dy = f'(x)dx \leadsto \Delta y = f'(x)\Delta x$

§3.1: max and min values

- extreme value thm: if $f(x)$ cts on $[a, b]$, then it attains global extrema
- Fermat's thm: if $f(x)$ has local extremum at $x = c$, then $f'(c) = 0$ or $f'(c)$ DNE
- a *critical number* is $c \in [a, b]$ such that $f'(c) = 0$ or $f'(c)$ DNE; these and endpoints are the only possible locations of (local or global) extrema

§3.2: mean value thm

- Rolle's thm: if $f(x)$ cts on $[a, b]$ and diff'ble on (a, b) and $f(a) = f(b)$, then there is $c \in [a, b]$ such that $f'(c) = 0$
- mean value thm: if $f(x)$ cts on $[a, b]$ and diff'ble on (a, b), then there is $c \in [a, b]$ such that $f'(c) = (f(b) - f(a))/(b - a)$
- if $f'(x) = g'(x)$ for $x \in (a, b)$, then $f(x) - g(x)$ is a constant function

§3.3: shapes of graphs

- I/D test: if $f'(x) > 0$ then $f(x)$ increasing, et sim.
- 1st derivative test: at a critical number c, if $f'(x)$ changes from pos to neg then c is a max, et sim.
  - note: inconclusive if $f(x)$ doesn't change sign
- concave up $= f''(x) > 0$, concave down $= f''(x) < 0$; graph of $f(x)$ lies above/below all tangent lines, resp.
- an *inflection point* is where $f''(x)$ changes sign (need not have $f'' = 0$ there)
- 2nd derivative test: if $f'(c) = 0$ and $f''(c) > 0$ then c a local min, et sim.

§3.4: curve sketching

- domain, x- and y-intercepts, symmetry (even, odd, periodic), asymptotes, intervals of I/D, local extrema, concavity
- slant asymptote: e.g. as $x \to \pm\infty$, the function $f(x) = \frac{x^2}{x+5} + 7$ approaches the line $L(x) = x + 7$

§3.5: optimization

- identify relevant variables, set up system of constraints and function to be optimized, use the former to reduce the latter to a function of one variable, do calculus

§3.7: antiderivatives

- if $F(x)$ is any one antiderivative of $f(x)$ on an interval (i.e. $F'(x) = f(x)$ there), then the most general form of an antiderivative is $F(x) + C$ for arbitrary constant C

§4.1: areas and distances

- area under a curve approximated by *Riemann sum*: $\int_a^b f(x)dx \approx \sum_{i=1}^n f(x_i^*)\Delta x$ for $[a, b]$ broken into n (generally equally spaced) intervals and x_i^* arbitrary sample points and $\Delta x = (b - a)/n$
  - standard choices for sample points are left endpts, right endpts, midpts

- useful formulas include:
  - $\sum_{i=1}^{n} i = n(n+1)/2$
  - $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$
  - $\sum_{i=1}^{n} i^3 = (\sum_{i=1}^{n} i)^2 = (n(n+1)/2)^2$
- similar methods to solve the *distance problem*: usually distance = velocity × time, but if rate is varying then need to use calculus: distance $= \int_{t_0}^{t_1} v(t)dt$

§4.2: definite integrals

- if Riemann sum converges as $n \to \infty$, it will be independent of choices of sample points, so get *Riemann integral* $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$: the true area under the curve
  - really, a *signed* area: area below the x-axis counts negatively, and also in $\int_b^a$ where $a \leq b$
- standard properties: pull out a constant, sums and differences of integrands, additivity over adjacent intervals
- most general comparison property: if $m \leq f(x) \leq M$ for all $x \in [a,b]$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

§4.3: the evaluation theorem (half of the FTC)

- if $f(x)$ cts on $[a,b]$ and $F(x)$ any antider, then $\int_a^b f(x)dx = F(b) - F(a)$

§4.4: (the rest of) the FTC

- if f cts on $[a,b]$, then $\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$ for all $x \in (a,b)$
- mean value thm for integrals: if f cts on $[a,b]$, then $f(c) = f_{\text{avg on }[a,b]} := \frac{1}{b-a}\int_a^b f(x)dx$ for some $c \in [a,b]$

§4.5: u-du substitution

- if $u = g(x)$ is diffble and $f(x)$ is cts, then $\int f(g(x))g'(x)dx = \int f(u)du$ (basically just the chain rule)
- version for definite integrals: $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$ (note change in bounds of integration)
- implies that for f even $\int_{-a}^{a} f(x)dx = \int_0^a f(x)dx$ and for f odd that $\int_{-a}^{a} f(x)dx = 0$

§5.1: inverse fxns

- for f to have an inverse fxn, need for it to be *one-to-one*: if $f(a) = f(b)$ then $a = b$ (check by horiz line test)
- for fxns that aren't one-to-one, can restrict domain so it is (e.g. $[0,+\infty)$ for x^2, $(-\frac{\pi}{2},\frac{\pi}{2})$ for $\tan(x)$)
- if f is one-to-one, then its *inverse* fxn f^{-1} is defined by $f^{-1}(y) = x$ exactly when $y = f(x)$ (so $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$)
- f and f^{-1} swap domain and range
- graph of f^{-1} obtained by reflecting over the line $y = x$, so slopes at corresponding points get inverted: if $f(x) = y$ then $(f^{-1})'(y) = \frac{1}{f'(x)}$

§5.2: ln

- by definition, $\ln(x) = \int_1^x \frac{1}{t}dt$ for $x > 0$
- so by FTC, $\frac{d}{dx}\ln(x) = \frac{1}{x}$; in particular, $\ln(x)$ is strictly increasing for all $x > 0$
- this gives us the one exception to the power rule $\int x^a dx = \frac{1}{a+1}x^{a+1} + C$, namely $\int x^{-1}dx = \ln(|x|) + C$
- "makes things one level simpler": $\ln(xy) = \ln(x) + \ln(y)$ and $\ln(x^r) = r\ln(x)$ for rational r (so also $\ln(\frac{x}{y}) = \ln(x) - \ln(y)$)
- $\lim_{x\to+\infty} \ln(x) = +\infty$ and $\lim_{x\to 0^+} \ln(x) = -\infty$

- by definition, e is the unique number with $\ln(e) = 1$
- logarithmic differentiation: given a function consisting of many terms multiplied and divided (e.g. $y = \frac{\cos(x)\sqrt[7]{x}}{(x+7)^3(x^2-1)^3}$), take ln of both sides so right side simplifies, then apply $\frac{d}{dx}$ and solve for $\frac{dy}{dx}$

§5.3: exp

- since $\ln(x)$ is strictly increasing it is one-to-one, so define $\exp = \ln^{-1}$ to be its inverse: $\exp(y) = x$ exactly when $y = \ln(x)$, and so $\exp(\ln(x)) = x$ and $\ln(\exp(y)) = y$
- in particular, $e = \exp(1)$
- for rational r we have $\ln(e^r) = r\ln(e) = r$, so $e^r = \exp(\ln(e^r)) = \exp(r)$
- thus, we *define* irrational exponentiation of e to be $e^x = \exp(x)$ for any real number x
- $\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to+\infty} e^x = +\infty$
- laws of exponents (follow from laws of logarithms): $e^{x+y} = e^x e^y$ and $(e^x)^r = e^{rx}$ for r rational (so also $e^{x-y} = \frac{e^x}{e^y}$)
- $\frac{d}{dx}e^x = e^x$, by logarithmic differentiation ($y = e^x$ means $\ln(y) = x$, so $\frac{y'}{y} = 1$, so $y' = y$)

§5.4: general logs and exps

- for base $a > 0$ and rational exponent r, $a^r = (e^{\ln(a)})^r = e^{r\ln(a)}$
- thus, we *define* irrational exponentiation of a to be $a^x = e^{x\ln(a)}$ for any real number x
- analogous laws of exponents ("makes things one level more complicated", e.g. $a^{x+y} = a^x a^y$)
- $\frac{d}{dx}a^x = a^x\ln(a)$, so $\int a^x dx = \frac{1}{\ln(a)}a^x + C$ for any positive $a \neq 1$
- since $f(x) = a^x$ is one-to-one (for positive $a \neq 1$), can define its inverse $\log_a$
- for positive $a \neq 1$, $\log_a(x) = \frac{\ln(x)}{\ln(a)} = \frac{\log_b(x)}{\log_b(a)}$ for any positive $b \neq 1$
- $\frac{d}{dx}\log_a(x) = \frac{1}{\ln(a)} \cdot \frac{1}{x}$
- $e = \lim_{x\to0}(1+x)^{1/x} = \lim_{n\to+\infty}(1+\frac{1}{n})^n$; these can be used to evaluate other more complicated limits

§5.5: exp growth/decay

- exp growth/decay: by definition, satisfies diff eq $\frac{dy}{dt} = ky$ (assuming $y \geq 0$, growth if $k > 0$ and decay if $k < 0$)
- all solutions of exp growth/decay are of the form $y(t) = Ce^{kt}$
- Newton's law of heating/cooling: $\frac{dT}{dt} = k(T - T_s)$ for a negative constant k and a constant surrounding temperature T_s
- all solutions to Newton's law of heating/cooling are of the form $T(t) = Ce^{kt} + T_s$ (to solve, make substitution $y(t) = T(t) - T_s$ to reduce to previous)
- compound interest: at rate r compounded n times per year, growth factor is $(1 + \frac{r}{n})^{nt}$; as $n \to \infty$ this approaches e^{rt}